ON ALMOST $N$-SIMPLE-PROJECTIVES

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Abstract. The concept of almost $N$-projectivity is defined in [5] by M. Harada and A. Tazaki to translate the concept “lifting module” in terms of homomorphisms. In [6, Theorem 1] M. Harada defined a little weaker condition “almost $N$-simple-projective” and gave the following relationship between them:

For a semiperfect ring $R$ and $R$-modules $M$ and $N$ of finite length,

$M$ is almost $N$-projective if and only if $M$ is almost $N$-simple-projective.

We remove the assumption “of finite length” and give the result in Theorem 5 as follows:

For a semiperfect ring $R$, a finitely generated right $R$-module $M$ and an indecomposable right $R$-module $N$ of finite Loewy length,

$M$ is almost $N$-projective if and only if $M$ is almost $N$-simple-projective.

We also see that, for a semiperfect ring $R$, a finitely generated $R$-module $M$ and an $R$-module $N$ of finite Loewy length, $M$ is $N$-simple-projective if and only if $M$ is $N$-projective.

Throughout this paper, we let $R$ be a semiperfect ring unless otherwise stated and $R$-modules unitary. For an $R$-module $M$, we denote the Loewy length and the composition length of $M$ by $L(M)$ and $|M|$, respectively.

Let $M$ and $N$ be $R$-modules. We say that $M$ is $N$-projective if, for any submodule $L$ of $N$ and an $R$-homomorphism $\varphi : M \to N/L$, there exists an $R$-homomorphism $\tilde{\varphi} : M \to N$ with $\nu \tilde{\varphi} = \varphi$, where $\nu : N \to N/L$ is the natural epimorphism. If, in this definition, we only consider the $R$-homomorphisms $\varphi$ with simple images, $M$ is said to be $N$-simple-projective.

First we give a lemma in which $N$-simple-projectivity is investigated for an $R$-homomorphism with its image semisimple artinian.

Lemma 1. Let $R$ be a ring, $M$ and $N$ $R$-modules, $L$ a submodule of $N$ and $\varphi : M \to N/L$ an $R$-homomorphism with $\operatorname{Im} \varphi$ semisimple artinian. If $M$ is $N$-simple-projective, then there exists an $R$-homomorphism $\tilde{\varphi} : M \to N$ with $\nu \tilde{\varphi} = \varphi$, where $\nu : N \to N/L$ is the natural epimorphism.

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Proof. Let \( \text{Im} \varphi = N'/L \), where \( N' \) is a submodule of \( N \). Then \( M \) is \( N' \)-simple-projective by assumption. So the statement follows from [6, Lemma 1]. \( \square \)

Using Lemma 1 we obtain the following result which is a generalization of [6, Lemma 1]. And we also note that, in [2, Proposition 2], Baba and Oshiro gave the dual result which played an important role to characterize Fuller’s theorem for injective modules.

**Proposition 2.** Let \( M \) be a finitely generated right \( R \)-module and \( N \) a right \( R \)-module with \( L(N) < \infty \). If \( M \) is \( N \)-simple-projective, then \( M \) is \( N \)-projective.

Proof. Let \( L \) be a submodule of \( N \), \( \varphi : M \to N/L \) an \( R \)-homomorphism and \( \nu : N \to N/L \) the natural epimorphism. Since \( L(N) < \infty \), there exists \( n_1 \in \mathbb{N} \) such that \( \text{Im} \varphi \subseteq (N/L)J^{n_1-1} \) but \( \text{Im} \varphi \not\subseteq (N/L)J^{n_1} \). Then (\( \text{Im} \varphi + (N/L)J^{n_1})/(N/L)J^{n_1} \) is semisimple artinian since \( R \) is semiperfect and \( M \) is finitely generated. So, by Lemma 1, there exists an \( R \)-homomorphism \( \tilde{\varphi}_1 : M \to N \) with \( \nu_1 \nu \tilde{\varphi}_1 = \nu_1 \varphi \), where \( \nu_1 : N/L \to (N/L)/(N/L)J^{n_1} \) is the natural epimorphism.

We assume that \( \varphi \neq \nu \tilde{\varphi}_1 \). Since \( \text{Im}(\varphi - \nu \tilde{\varphi}_1) \subseteq (N/L)J^{n_1} \), there exists \( n_2 \in \mathbb{N} \) with \( n_2 > n_1 \), \( \text{Im}(\varphi - \nu \tilde{\varphi}_1) \subseteq (N/L)J^{n_2-1} \) but \( \text{Im}(\varphi - \nu \tilde{\varphi}_1) \not\subseteq (N/L)J^{n_2} \). Then \( (\text{Im}(\varphi - \nu \tilde{\varphi}_1) + (N/L)J^{n_2})/(N/L)J^{n_2} \) is semisimple artinian. So, by Lemma 1, there exists an \( R \)-homomorphism \( \tilde{\varphi}_2 : M \to N \) with \( \nu_2 \nu \tilde{\varphi}_2 = \nu_2(\varphi - \nu \tilde{\varphi}_1) \), where \( \nu_2 : N/L \to (N/L)/(N/L)J^{n_2} \) is the natural epimorphism.

We assume that \( \varphi \neq \nu(\tilde{\varphi}_1 + \tilde{\varphi}_2) \). Since \( \text{Im}(\varphi - \nu(\tilde{\varphi}_1 + \tilde{\varphi}_2)) \subseteq (N/L)J^{n_2} \), we have \( n_3 \in \mathbb{N} \) such that \( n_3 > n_2 \), \( \text{Im}(\varphi - \nu(\tilde{\varphi}_1 + \tilde{\varphi}_2)) \subseteq (N/L)J^{n_3-1} \) but \( \text{Im}(\varphi - \nu(\tilde{\varphi}_1 + \tilde{\varphi}_2)) \not\subseteq (N/L)J^{n_3} \).

Continuing this argument, we have \( k \in \mathbb{N} \) with \( \varphi = \nu(\tilde{\varphi}_1 + \cdots + \tilde{\varphi}_k) \) because \( L(N) < \infty \). \( \square \)

Now we define “almost \( N \)-projective” and “almost \( N \)-simple-projective”. Let \( M \) and \( N \) be \( R \)-modules. We say that \( M \) is almost \( N \)-projective if, for any submodule \( L \) of \( N \) and an \( R \)-homomorphism \( \varphi : M \to N/L \), letting \( \nu : N \to N/L \) be the natural epimorphism, either the following (I) or (II) holds:

(I) There exists an \( R \)-homomorphism \( \tilde{\varphi} : M \to N \) with \( \nu \tilde{\varphi} = \varphi \).

(II) There exist a non-zero direct summand \( N' \) of \( N \) and an \( R \)-homomorphism \( \tilde{\psi} : N' \to M \) with \( \phi \tilde{\psi} = \nu|_{N'} \).

If, in this definition, we only consider the \( R \)-homomorphisms \( \varphi \) with simple images, \( M \) is said to be almost \( N \)-simple-projective.
We note that, in these definitions, if \( N \) is indecomposable, the condition (II) is as follows:

\[
(\text{II}') \quad \text{There exists an } R\text{-homomorphism } \tilde{\psi} : N \to M \text{ with } \phi \tilde{\psi} = \nu.
\]

In this paper, we consider the case that \( N \) is indecomposable.

The following in which almost \( N \)-simple projective is investigated for an \( R \)-homomorphism with its image semisimple artinian is the first step to prove Theorem 5.

**Lemma 3.** Let \( M \) be an \( R \)-module, \( N \) an indecomposable \( R \)-module, \( L \) a submodule of \( N \) and \( \varphi : M \to N/L \) an \( R \)-homomorphism with \( \text{Im} \varphi \) semisimple artinian. We consider the following three conditions:

1. \( \varphi \) is not epic.
2. \(| \text{Im} \varphi | \geq 2.\)
3. \( L \not\ll N.\)

If \( M \) is almost \( N \)-simple-projective and, at least, one of the above three conditions holds, then there exists an \( R \)-homomorphism \( \tilde{\varphi} : M \to N \) with \( \nu \tilde{\varphi} = \varphi \), where \( \nu : N \to N/L \) is the natural epimorphism.

**Proof.** First we consider the case that either the condition (1) or (2) holds. Let \( \text{Im} \varphi = \bigoplus_{i=1}^{n} S_i \), where \( S_i \) is simple for any \( i = 1, \ldots, n \). Further, for each \( i = 1, \ldots, n \), we let \( \pi_i : \bigoplus_{j=1}^{n} S_j \to S_i \) be the projection and \( \iota_i : S_i \to N/L \) the injection. Then \( \text{Im} \iota_i \pi_i \varphi \) is simple and a proper submodule of \( N/L \) by the condition (1) or (2). So, because \( M \) is almost \( N \)-simple-projective, there exists an \( R \)-homomorphism \( \tilde{\varphi}_i : M \to N \) with \( \nu \tilde{\varphi}_i = \iota_i \pi_i \varphi \).

Put \( \tilde{\varphi} := \tilde{\varphi}_1 + \cdots + \tilde{\varphi}_n \). Then \( \nu \tilde{\varphi} = \varphi \).

Next we consider the case that the condition (3) holds. Since \( L \ll N \), there exists a proper submodule \( L' \) of \( N \) with \( L + L' = N \). We consider an \( R \)-isomorphism \( \eta : N/L = (L + L')/L \to L'/L \cap L' \) naturally. Let \( \nu' : N \to N/(L \cap L') \) be the natural epimorphism and \( \iota : L'/L \cap L' \to N/(L \cap L') \) the inclusion map. The condition (1) holds for \( \nu \eta \varphi \), and so there exists an \( R \)-homomorphism \( \tilde{\varphi}' : M \to N \) with \( \nu' \tilde{\varphi}' = \nu \eta \varphi \). Then \( \text{Im} \tilde{\varphi}' \subseteq L' \). Hence \( \nu \tilde{\varphi}' = \varphi \) since \( \nu|_{L'} = \eta^{-1} \nu'|_{L'} \).

Using Lemma 3, we obtain the following.

**Lemma 4.** Let \( M \) be a finitely generated right \( R \)-module, \( N \) an indecomposable right \( R \)-module with \( \text{L}(N) < \infty \), \( L \) a proper submodule of \( N \) and \( \varphi : M \to N/L \) an \( R \)-homomorphism. Suppose that \( M \) is almost \( N \)-simple-projective and let \( \nu : N \to N/L \) be the natural epimorphism.

1. If \( \varphi \) is not epic, then there exists an \( R \)-homomorphism \( \tilde{\varphi} : M \to N \) with \( \nu \tilde{\varphi} = \varphi \).
(2) Suppose that there exist a proper submodule $N'/L$ of $N/L$ and an $R$-homomorphism $\varphi' : M \to N$ with $(\nu/\nu)\varphi' = \nu'\varphi$, where $\nu' : N/L \to N/N'$ is the natural epimorphism. Then there exists an $R$-homomorphism $\varphi'' : M \to N$ with $\nu\varphi'' = \varphi$.

**Proof.** (1) Since $L(N) < \infty$, there exists $n_1 \in \mathbb{N}$ such that $\text{Im}(\varphi) \not\subseteq (NJ^{n_1} + L)/L$ but $\text{Im}(\varphi) \subseteq (NJ^{n_1-1} + L)/L$. Let $\nu_1 : N/L \to N/(NJ^{n_1} + L)$ be the natural epimorphism and let $\text{Im}(\varphi) = \nu_1/L$, where $L_0$ is a submodule of $N$. Then $\text{Im}(\nu_1\varphi) = (L_0 + NJ^{n_1} + L)/L$ and it is semisimple artinian because $M$ is finitely generated. Hence we claim that there exists an $R$-homomorphism $\tilde{\varphi}_1 : M \to N$ with $\nu_1\nu\tilde{\varphi}_1 = \nu_1\varphi$. If $\nu_1\varphi$ is not epic, then the condition (1) in Lemma 3 holds. Assume that $\nu_1\varphi$ is epic and, further, $NJ^{n_1} + L \ll N$, i.e., the condition (3) in Lemma 3 does not hold for $\nu_1\varphi$. Then $\text{Ker}(\nu_1) = (NJ^{n_1} + L)/L \ll N/L$. Since $\nu_1\varphi$ is epic, we see that $\varphi$ is also epic, a contradiction. In consequence, either the condition (1) or (3) in Lemma 3 holds for $\nu_1\varphi$ and we obtain the desired $\tilde{\varphi}_1$.

Assume that $\nu\tilde{\varphi}_1 \neq \varphi$. There exists $n_2 \in \mathbb{N}$ such that $n_2 > n_1$, $\text{Im}(\varphi - \nu\tilde{\varphi}_1) \not\subseteq (NJ^{n_2} + L)/L$ but $\text{Im}(\varphi - \nu\tilde{\varphi}_1) \subseteq (NJ^{n_2-1} + L)/L$. Let $\nu_2 : N/L \to N/(NJ^{n_2} + L)$ be the natural epimorphism. Then, since $\text{Im}(\varphi - \nu\tilde{\varphi}_1) \subseteq (NJ^{n_1} + L)/L \ll N/L$, there exists an $R$-homomorphism $\tilde{\varphi}_2 : M \to N$ with $\nu_2\nu\tilde{\varphi}_2 = \nu_2(\varphi - \nu\tilde{\varphi}_1)$ by Lemma 3.

Assume that $\nu(\tilde{\varphi}_1 + \tilde{\varphi}_2) \neq \varphi$. Then there exists $n_3 \in \mathbb{N}$ such that $n_3 > n_2$, $\text{Im}(\varphi - \nu(\tilde{\varphi}_1 + \tilde{\varphi}_2)) \not\subseteq (NJ^{n_3} + L)/L$ but $\text{Im}(\varphi - \nu(\tilde{\varphi}_1 + \tilde{\varphi}_2)) \subseteq (NJ^{n_3-1} + L)/L$. Using this procedure finite times, since $L(N) < \infty$, we have $m \in \mathbb{N}$ with $\nu(\tilde{\varphi}_1 + \cdots + \tilde{\varphi}_m) = \varphi$.

(2) $\nu'(\varphi - \nu\tilde{\varphi}') = 0$. So $\text{Im}(\varphi - \nu\tilde{\varphi}') \subseteq \text{Ker}(\nu') = N'/L \ll N/L$. Therefore, from (1) which we already show, there exists an $R$-homomorphism $\tilde{\varphi} : M \to N$ with $\nu\tilde{\varphi} = \varphi - \nu\tilde{\varphi}'$. Hence $\nu(\tilde{\varphi} + \tilde{\varphi}') = \varphi$. \hfill \Box

Now we give a theorem which is a generalization of [6, Theorem 1].

**Theorem 5.** Let $M$ be a finitely generated right $R$-module and $N$ an indecomposable right $R$-module with $L(N) < \infty$. Suppose that $M$ is almost $N$-simple-projective. Then $M$ is almost $N$-projective.

**Proof.** We consider the following diagram:

\[
\begin{array}{ccc}
M \\
\downarrow \varphi \\
N \xrightarrow{\nu} N/L \to 0,
\end{array}
\]

where $L$ is a proper submodule of $N$ and $\nu$ is the natural epimorphism. If $\varphi$ is not epic, then, by Lemma 4 (1), there exists an $R$-homomorphism $\tilde{\varphi} : M \to N$ with $\nu\tilde{\varphi} = \varphi$. So we may assume that $\varphi$ is epic.
First we consider the case that $L \not\leq N$. Then there exists a proper submodule $N'$ of $N$ with $N = N' + L$. So we can define an $R$-isomorphism $\eta : N/(L \cap N') \to (N/L) \oplus (N/N')$ naturally. Further define an $R$-homomorphism $\phi' : M \to (N/L) \oplus (N/N')$ by $\phi(m) = (\varphi(m), \zeta)$ for any $m \in M$ and let $\nu_1 : N \to N/(L \cap N')$ be the natural epimorphism. Since $\phi'$ is not epic, by Lemma 4 (1), there exists an $R$-homomorphism $\tilde{\phi} : M \to N$ with $\eta \tilde{\phi} = \phi'$. Then, for any $m \in M$, $(\varphi(m), \zeta) = \phi'(m) = \eta \tilde{\phi}(m) = (\tilde{\phi}(m), \tilde{\phi}(m))$. So $\varphi(m) = \tilde{\phi}(m)$. Hence $\nu \tilde{\phi} = \varphi$.

Next we consider the case that $L \leq N$. Suppose that $N$ is not local. Then there exist proper submodules $N'$ and $N''$ of $N$ such that they contain $NJ$, $N'$ is a maximal submodule of $N$ and $N/NJ = (N'/NJ) \oplus (N''/NJ)$. Let $\nu' : N/L \to N/NJ$ be the natural epimorphism, $\pi : N/NJ \to N''/NJ$ the projection and $\iota : N'/NJ \to N/NJ$ the injection. Then $\iota \pi \nu' \varphi : M \to N/NJ$ and $\text{Im} \ i \pi \nu' \varphi$ is a simple proper submodule of $N/NJ$. So, by assumption, there exists an $R$-homomorphism $\varphi' : M \to N$ with $\nu' \varphi' = \iota \pi \nu' \varphi$. Then, letting $\nu'' : N/L \to N/N'$ be the natural epimorphism, $\nu'' \nu' \varphi' = \nu'' \varphi$. Hence, by Lemma 4 (2), there exists an $R$-homomorphism $\bar{\phi} : M \to N$ with $\nu \bar{\phi} = \varphi$.

Therefore suppose that $L \leq N$ and $N$ is local. We may assume that $N = eR/A$ and $N/L = eR/B$, where $e$ is a primitive idempotent in $R$ and $A$ and $B$ are submodules of $eR_R$ with $A < B$. Let $\nu_0 : eR/B \to eR/eJ$ be the natural epimorphism. By assumption either the following (I) or (II) holds.

(I) There exists an $R$-homomorphism $\bar{\phi}_1 : M \to eR/A$ with $\nu_0 \nu \bar{\phi}_1 = \nu_0 \varphi$.

(II) There exists an $R$-homomorphism $\bar{\psi} : eR/A \to M$ with $\nu_0 \varphi \bar{\psi} = \nu_0 \nu$.

In the case (I), we obtain an $R$-homomorphism $\bar{\phi} : M \to eR/A$ with $\nu \bar{\phi} = \varphi$ from Lemma 4 (2). So we consider the case (II). Put $m_1 := \bar{\psi}(\bar{e})$. Since $M$ is finitely generated, we have $m_2, \ldots, m_n \in M$ such that $M = m_1R + m_2R + \cdots + m_nR$ but $m_1 \not\subset m_2R + \cdots + m_nR$. Further we let $\varphi(m_1) = \pi$, where $u \in eRe$. Then $e - u \in eJe$ because $\nu_0 \nu = \nu_0 \varphi \bar{\psi}$. Therefore $u^{-1} - e \in eJe$. Let $u^{-1} = e + j$, where $j \in eJe$. Then the following claim holds.

Claim. There exists an $R$-homomorphism $\tilde{\zeta} : M \to eR/A$ with $\tilde{\zeta}(m_1) = \bar{j}$.

Proof of Claim. When $j \in A$, $\tilde{\zeta} = 0$ is the desired map. So we assume that $j \not\in A$. Then we can define an $R$-homomorphism $\zeta_1 : M \to eR/(jJ + A)$ by $\zeta_1(m_1r_1 + m_2r_2 + \cdots + m_nr_n) = \bar{j}r_1$ since $m_1 \not\subset m_2R + \cdots + m_nR$ and $m_1e = m_1$. And $\text{Im} \ \zeta_1$ is a simple proper submodule of $eR/(jJ + A)$. So, by assumption, there exists an $R$-homomorphism $\tilde{\zeta}_1 : M \to eR/A$ with $\nu_1 \tilde{\zeta}_1 = \zeta_1$, where $\nu_1 : eR/A \to eR/(jJ + A)$ is the natural epimorphism. Let
\( \tilde{\zeta}_1(m_1) = j_1 \), where \( \tilde{j}_1 \in eRe \). Then \( j - \tilde{j}_1 \in jJ + A \) since \( \nu'_1 \tilde{\zeta}_1 = \zeta_1 \). Put \( j_2 + a_2 := j - \tilde{j}_1 \), where \( j_2 \in jJ \) and \( a_2 \in A \). Then we note that \( j_2 \in J^2 \).

If \( j_2 \in A \), then we put \( \tilde{\zeta} := \tilde{\zeta}_1 \), and this \( \tilde{\zeta} \) is the desired map. So assume that \( j_2 \notin A \). We define an \( R \)-homomorphism \( \zeta_2 : M \to eR/(j_2J + A) \) by \( \zeta_2(m_1r_1 + m_2r_2 + \cdots + m_nr_n) = j_2r_1 \). Then \( \text{Im} \zeta_2 \) is a simple proper submodule of \( eR/(j_2J + A) \). So, by assumption, there exists an \( R \)-homomorphism \( \tilde{\zeta}_2 : M \to eR/A \) with \( \nu'_2 \tilde{\zeta}_2 = \zeta_2 \), where \( \nu'_2 : eR/A \to eR/(j_2J + A) \) is the natural epimorphism. We let \( \tilde{\zeta}_2(m_1) = \tilde{j}_2 \), where \( \tilde{j}_2 \in eRe \). Then \( j_2 - \tilde{j}_2 \in j_2J + A \) since \( \zeta_2 = \nu'_2 \tilde{\zeta}_2 \). Put \( j_3 + a_3 := j_2 - \tilde{j}_2 \), where \( j_3 \in j_2J \) and \( a_3 \in A \). Then we note that \( j_3 \in J^3 \).

Since \( L(eR/A) < \infty \), this procedure finitely terminates and there exists \( s \in \mathbb{N} \) with \( j_s - \tilde{j}_s \in A \), i.e., we may let \( j_{s+1} = 0 \) and \( a_{s+1} = j_s - \tilde{j}_s \). Then we put \( \tilde{\zeta} := \tilde{\zeta}_1 + \cdots + \tilde{\zeta}_s \), and \( \tilde{\zeta}(m_1) = \tilde{\zeta}_1(m_1) + \tilde{\zeta}_2(m_1) + \cdots + \tilde{\zeta}_s(m_1) = \tilde{j}_1 + \tilde{j}_2 + \cdots + \tilde{j}_s = (j - j_2 - a_2) + (j_2 - j_3 - a_3) + \cdots + (j_s - j_{s+1} - a_{s+1}) = \tilde{j} \). Hence this \( \tilde{\zeta} \) is the desired map. Claim is shown.

Therefore we put \( \tilde{\psi} := (1_M + \tilde{\psi}' \tilde{\zeta}) \tilde{\psi}' : eR/A \to M \), and \( \varphi \tilde{\psi}(\tilde{e}) = \varphi(1_M + \tilde{\psi}' \tilde{\zeta}) \tilde{\psi}'(\tilde{e}) = \varphi(m_1 + \tilde{\psi}'(\tilde{j})) = \varphi(m_1 + m_1 j) = \varphi(m_1)(e + j) = \varphi(m_1)u^{-1} = uu^{-1} = \tilde{e} = \varphi(\tilde{e}) \). Hence \( \varphi \tilde{\psi} = \nu \). \( \square \)

We say that \( M \) has the lifting property of simple module modulo radical (abbreviated \( \text{LPSM} \)) if, for any simple submodule \( \overline{S} \) of \( M/\text{Rad}(M) \), there exists a decomposition \( M = M_1 \oplus M_2 \) such that \( (M_1 + \text{Rad}(M))/\text{Rad}(M) = \overline{S} \).

Further, for \( R \)-modules \( M \) and \( N \) and an \( R \)-homomorphism \( \varphi : M \to N \), we represent a submodule \( \{ m + \varphi(m) \mid m \in M \} \) of \( M \oplus N \) by \( M(\varphi) \).

Relationship between almost \( N \)-projectivity and \( \text{LPSM} \) was given in [4, Proposition 2] by M. Harada and T. Mabuchi as follows:

For a semiperfect ring \( R \), a primitive idempotent \( e \) in \( R \) and submodules \( A \) and \( B \) of \( eR \) with either \( eRe/A \) or \( eR/B \) noetherian, \( eRe/A \) is almost \( eR/B \)-projective if and only if \( eR/A \oplus eR/B \) has \( \text{LPSM} \) and \( eJeA \leq B \).

Further in [7, Corollary 9.7] M. Harada showed the following:

Let \( e \) be a primitive idempotent in a ring \( R \) with \( eRe \) a local ring and let \( A \) and \( B \) be submodules of \( eR \) with \( |eR/A|, |eR/B| < \infty \).

Then the following are equivalent:

(a) \( eR/A \) is almost \( eR/B \)-projective.

(b) \( (i) \) \( eR/A \oplus eR/B \) has \( \text{LPSM} \).
(ii) \(eR/A\) is \(C/B\)-projective for any proper submodule \(C\) of \(eR_R\) with \(C > B\).

As an application of Proposition 2 and Theorem 5, last we give a corollary.

**Corollary 6.** Let \(e\) be a primitive idempotent in \(R\) and \(A\) and \(B\) submodules of \(eR_R\). If \(L(eR/B) < \infty\), then the following are equivalent.

(a) \(eR/A\) is almost \(eR/B\)-projective.
(b) \(eR/A\) is almost \(eR/B\)-simple-projective.
(c) (i) \(eR/A \oplus eR/B\) has LPSM.
    (ii) \(eR/A\) is \(eJ/B\)-projective.
(d) (i) \(eR/A \oplus eR/B\) has LPSM.
    (ii) \(eR/A\) is \(eJ/B\)-simple-projective.

**Proof.** (a) \(\Leftrightarrow\) (b) This follows from Theorem 5.
(c) \(\Leftrightarrow\) (d) This follows from Proposition 2.
(b) \(\Rightarrow\) (d) (i) Put \(M := (eR/A) \oplus (eR/B)\). Take any simple submodule \(S/(eJ \oplus eJ)\) of \((eR/eJ) \oplus (eR/eJ)\). If either \(S = eR \oplus eJ\) or \(S = eJ \oplus eR\), then \(M = (eR/A) \oplus (eR/B)\) is the desired direct decomposition. So we consider the remainder case. Then there exists \(\varphi \in \text{Aut}(eR/eJ)\) with \(S/(eJ \oplus eJ) = (eR/eJ)(\varphi)\). And we consider the following diagram:

\[
\begin{array}{ccc}
 eR/A & \downarrow \nu & eR/eJ \\
 & \Downarrow \varphi & \\
 eR/B & \xrightarrow{\nu'} & eR/eJ \rightarrow 0
\end{array}
\]

where \(\nu\) and \(\nu'\) are the natural epimorphisms. By assumption, either the following (I) or (II) holds.

(I) There exists an \(R\)-homomorphism \(\tilde{\varphi} : eR/A \rightarrow eR/B\) such that \(\nu' \tilde{\varphi} = \varphi \nu\).

(II) There exists an \(R\)-homomorphism \(\tilde{\psi} : eR/B \rightarrow eR/A\) such that \(\varphi \nu \tilde{\psi} = \nu'\).

In the case (I), \(M = (eR/A)(\tilde{\varphi}) \oplus (eR/B)\). And let \(X/(A \oplus B) = (eR/A)(\tilde{\varphi})\), where \(X\) is a submodule of \(eR \oplus eR\). Then \((X + (eJ \oplus eJ))/(eJ \oplus eJ) = S/(eJ \oplus eJ)\).

In the case (II), by the similar argument, we see that \(M = (eR/A) \oplus (eR/B)(\tilde{\psi})\) is the desired direct decomposition.

Hence \(eR/A \oplus eR/B\) has LPSM.
(ii) We consider the following diagram:

\[
\begin{array}{ccc}
\frac{eR/A}{\varphi} & \mapright{\nu} & \frac{eJ/B}{\nu} \\
\end{array}
\]

where \( \text{Im} \varphi \) is simple, \( B' \) is a submodule of \( eJ \) with \( B' \geq B \) and \( \nu \) is the natural epimorphism. Let \( \nu' : eR/B \to eR/B' \) be the natural epimorphism. From (b), there exists an \( R \)-homomorphism \( \tilde{\varphi} : eR/A \to eR/B \) with \( \nu' \tilde{\varphi} = \varphi \). Then \( \text{Im} \tilde{\varphi} \subseteq eJ/B \) since \( \text{Im} \varphi \subseteq eJ/B' \). Hence \( \nu' \tilde{\varphi} = \varphi \).

(d) \( \Rightarrow \) (b) We consider a diagram:

\[
\begin{array}{ccc}
\frac{eR/A}{\varphi} & \mapright{\nu} & \frac{eJ/B}{\nu} \\
\end{array}
\]

where \( \text{Im} \varphi \) is simple, \( B' \) is a submodule of \( eR \) with \( B' \geq B \) and \( \nu \) is the natural epimorphism. When \( \varphi \) is not epic, there exists an \( R \)-homomorphism \( \tilde{\varphi} : eR/A \to eR/B \) from (d) (ii). So we assume that \( \varphi \) is epic. Then \( B' = eJ \). Put \( M := (eR/A) \oplus (eR/B) \). We consider a submodule

\[
N := \{ (x_1, x_2) \mid x_1 \in eR/A, x_2 \in eR/B, \varphi(x_1) = \nu(x_2) \}
\]

of \( M \). And we put \( M_1 := \{ (x_1, 0) \mid x_1 \in eR/A \} \) and \( M_2 := \{ (0, x_2) \in M \mid x_2 \in eR/B \} \). Then, by the internal exchange property, either the following (I) or (II) holds:

(I) \( M = N \oplus M_1 \).

(II) \( M = N \oplus M_2 \).

First we consider the case (II). Let \( \pi_2 : M = N \oplus M_2 \to M_2 \) be the projection and put \( \tilde{\varphi} := -\pi_2|_{M_1} : M_1 \to M_2 \). Then we claim that \( \nu \tilde{\varphi} = \varphi \). Take any \( x_1 \in eR/A \). There exist \((y_1, y_2) \in N \) and \((0, x_2) \in M_2 \) with \((x_1, 0) = (y_1, y_2) + (0, x_2) \). Then \( \overline{x}_1 = y_1, \overline{y}_2 = -x_2 \) and \( \varphi(y_1) = \nu(y_2) \). So \( \nu \tilde{\varphi}(x_1) = \nu(-\pi_2(x_1)) = \nu(0, -x_2) = \nu(0, \overline{y}_2) = \nu(y_1) = \varphi(x_1) \).

Next we consider the case (I). Let \( \pi_1^1 : M = N \oplus M_1 \to M_1 \) be the projection and put \( \tilde{\psi} := -\pi_1^1|_{M_2} : M_2 \to M_1 \). Then we see, by the same way as the case (II), that \( \varphi \tilde{\psi} = \nu \). \( \square \)

References


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