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SOME APPLICATIONS OF DIFFERENTIAL SUBORDINATION FOR A GENERAL CLASS OF MULTIVALENTLY ANALYTIC FUNCTIONS INVOLVING A CONVOLUTION STRUCTURE

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Abstract. In the present paper we investigate a class of multivalently analytic functions which essentially involves a Hadamard product of two multivalent functions. We apply the techniques of differential subordination and derive some useful characteristics of this function class. The applications to generalized hypergeometric functions and various consequences of the main results exhibiting also relevant connections with some of the known (and new) results (including also an improved version of a known result) are also pointed out.

1. Introduction and Definitions

Let $\mathcal{A}_p$ denote the class of functions of the form

$$f(z) = z^p + \sum_{k=p+1}^{\infty} a_k z^k \qquad (p \in \mathbb{N} = \{1, 2, ...\}), \tag{1.1}$$

which are analytic and $p-$valent in the open unit disk $\mathbb{U} = \{z : |z| < 1\}$. For the functions f and g analytic in $\mathbb{U}$, we say f is subordinate to g in $\mathbb{U}$, and write $f \prec g$, if there exists a function $w(z)$ analytic in $\mathbb{U}$ such that $|w(z)| < 1,\ z \in \mathbb{U}$, and $w(0) = 0$ with $f(z) = g(w(z))$ in $\mathbb{U}$. If f is univalent in $\mathbb{U}$, then $f \prec g$ is equivalent to $f(0) = g(0)$ and $f(\mathbb{U}) \subset g(\mathbb{U})$.

Let $\mathcal{P}(\gamma)$ denote the class of functions $\phi(z)$ of the form

$$\phi(z) = 1 + c_1 z + c_2 z^2 + ... \tag{1.2}$$

which are analytic in $\mathbb{U}$ and satisfy the following inequality:

$$\Re(\phi(z)) > \gamma \qquad (0 \leq \gamma < 1;\ z \in \mathbb{U}).$$

If $f \in \mathcal{A}_p$ is given by (1.1) and $g \in \mathcal{A}_p$ given by

$$g(z) = z^p + \sum_{k=p+1}^{\infty} b_k z^k, \tag{1.3}$$

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then the Hadamard product (or convolution) $f * g$ of f and g is defined (as usual) by

$$(1.4) \qquad (f * g)(z) := z^p + \sum_{k=p+1}^{\infty} a_k b_k z^k.$$

For a given function $g(z) \in \mathcal{A}_p$ (defined by (1.3)), we introduce here a new function class $\mathcal{J}_p(g; \alpha, A, B)$ consisting of functions $f(z)$ of the form (1.1) if and only if

$$(1.5) \qquad (1-\alpha)\,\frac{(f * g)(z)}{z^p} + \frac{\alpha}{p}\,\frac{(f * g)'(z)}{z^{p-1}} \prec \frac{1+Az}{1+Bz}$$

$$(z \in \mathbb{U};\ \alpha > 0;\ -1 \leq B < A \leq 1).$$

For simplicity, we put $\mathcal{J}_p(g; 1, A, B) = \mathcal{M}_p(g; A, B)$.

In the present paper we establish some interesting characteristics of the function class $\mathcal{J}_p(g; \alpha, A, B)$ defined in terms of a convolution structure by invoking the subordination principle. The usefulness of considering the convolution structure in defining the above class lies in the fact that one can select suitably the arbitrary coefficients b_k in (1.4) to deduce various other related classes some of which are mentioned in the concluding section. Applications involving generalized hypergeometric functions and some important consequences of the main results, and their relevant connections with various known and new results, are also pointed out.

We require the following lemmas to investigate the function class $\mathcal{J}_p(g; \alpha, A, B)$ (defined above).

Lemma 1 (Miller and Mocanu [5]). *Let $h(z)$ be a convex (univalent) function in $\mathbb{U}$ with $h(0) = 1$, and let the function $\phi(z)$ be of the form (1.2) be analytic in $\mathbb{U}$. If*

$$(1.6) \qquad \phi(z) + \frac{1}{\gamma}\, z\phi'(z) \prec h(z) \qquad (\Re(\gamma) \geq 0\ \ (\gamma \neq 0);\ z \in \mathbb{U}),$$

then

$$(1.7) \qquad \phi(z) \prec \psi(z) := \frac{\gamma}{z^\gamma} \int_0^z t^{\gamma-1}\, h(t)\, dt \prec h(z) \qquad (z \in \mathbb{U})$$

and $\psi(z)$ is the best dominant.

The generalized hypergeometric function ${}_qF_s$ is defined by (cf., e. g. [1])

$$(1.8) \qquad {}_qF_s(z) \equiv {}_qF_s(\alpha_1, ..., \alpha_q; \beta_1, ..., \beta_s;\ z) = \sum_{n=0}^{\infty} \frac{(\alpha_1)_n \dots (\alpha_q)_n}{(\beta_1)_n \dots (\beta_s)_n}.\ \frac{z^n}{n!}$$

($z \in \mathbb{U};\ \alpha_j \in \mathbb{C}\ (j = 1, ..., q),\ \beta_j \in \mathbb{C}\backslash\{0, -1, -2, ...\}(j = 1, ..., s),\ q \leq s + 1;\ q, s \in \mathbb{N}_0$), where $(\alpha)_k$ is the Pochhammer symbol defined by

$$(\alpha)_0 = 1; \ (\alpha)_k = \alpha(\alpha+1)...(\alpha+k-1) \ (k \in \mathbb{N}).$$

The following identities asserted by Lemma 2 below are well known [1].

Lemma 2. *For real or complex numbers* a, b *and* c $(c \neq 0, -1, -2, ...)$:

$$\int_0^1 t^{b-1} (1-t)^{c-b-1} (1-zt)^{-a} \, dt = \frac{\Gamma(b)\,\Gamma(c-b)}{\Gamma(c)} \ {}_2F_1 \ (a, b; \ c; z) \tag{1.9}$$

$$(\Re(c) > \Re(b) > 0),$$

$${}_2F_1 \ (a, b; \ c; z) = (1-z)^{-a} \ {}_2F_1 \left(a, c-b; c; \frac{z}{z-1} \right), \tag{1.10}$$

$$(b+1) \ {}_2F_1 \ (1, b; \ b+1; z) = (b+1) \ + \ b \ z \ {}_2F_1 \ (1, b+1; \ b+2; \ z), \tag{1.11}$$

$${}_2F_1 \left(a, b; \ \frac{a+b+1}{2}; \frac{1}{2} \right) \ = \ \frac{\sqrt{\pi}\,\Gamma((a+b+1)/2)}{\Gamma((a+1)/2)\,\Gamma((b+1)/2)}, \tag{1.12}$$

$${}_2F_1 \left(1, 1; \ 3; \frac{az}{az+1} \right) = \frac{2(1+az)}{az} \left(1 - \frac{ln(1+az)}{az} \right). \tag{1.13}$$

2. Main Results and Applications

Our first main result is given by Theorem 1 below.

Theorem 1. *If* $f(z) \in \mathcal{J}_p(g; \alpha, A, B)$, *then*

$$\Re \left(\left(\frac{(f*g)(z)}{z^p} \right)^{\frac{1}{m}} \right) \ > \ \mathcal{X}^{\frac{1}{m}} \qquad (m \in \mathbb{N}; \ z \in \mathbb{U}), \tag{2.1}$$

where

$$\mathcal{X} = \begin{cases} \frac{A}{B} + \left(1 - \frac{A}{B}\right)(1-B)^{-1} \ {}_2F_1 \left(1, 1; \ \frac{p}{\alpha} + 1; \frac{B}{B-1}\right) & (B \neq 0); \\ 1 - \frac{p}{\alpha+p} \ A, & (B = 0). \end{cases} \tag{2.2}$$

The result is best possible.

Proof. Let $f(z) \in \mathcal{J}_p(g; \alpha, A, B)$, and assume that

$$\frac{(f*g)(z)}{z^p} = q(z). \tag{2.3}$$

It is clear that $q(z)$ is of the form (1.2) and is analytic in $\mathbb{U}$ with $q(0) = 1$. Differenting (2.3) with respect to z, we get

$$\begin{aligned} (1-\alpha) \, \frac{(f*g)(z)}{z^p} + \frac{\alpha}{p} \, \frac{(f*g)'(z)}{z^{p-1}} \quad &= \quad q(z) + \frac{\alpha}{p} \, z \, q'(z) \\ &\prec \quad \frac{1+Az}{1+Bz} \qquad (z \in \mathbb{U}). \end{aligned}$$

Now, by using Lemma 1 for $\gamma = \frac{p}{\alpha}$ and the identities (1.9) to (1.11) of Lemma 2, we deduce that

$$\begin{aligned}(2.4)\qquad &\frac{(f*g)(z)}{z^p} \prec \mathcal{X}(z)\\ &= \tfrac{p}{\alpha} z^{-\frac{p}{\alpha}} \int_0^z t^{\frac{p}{\alpha}-1}\, \tfrac{1+At}{1+Bt} dt\\ &= \begin{cases} \frac{A}{B} + \left(1-\frac{A}{B}\right)(1+Bz)^{-1}_2F_1\left(1,1;\frac{p}{\alpha}+1;\frac{Bz}{Bz+1}\right), & (B \neq 0);\\ 1+\frac{p}{\alpha+p}Az, & (B=0).\end{cases}\end{aligned}$$

In view of $-1 \le B < A \le 1$ and $\alpha > 0$, it follows from (2.4) that

$$\Re\left(\frac{(f*g)(z)}{z^p}\right) = \frac{p}{\alpha}\int_0^1 u^{\frac{p}{\alpha}-1}\Re\left(\frac{1+A\,u\,w(z)}{1+B\,u\,w(z)}\right) du$$

$$(2.5)\qquad > \frac{p}{\alpha}\int_0^1 u^{\frac{p}{\alpha}-1}\,\frac{1-A\,u}{1-B\,u}\, du = \mathcal{X}(-1) = \mathcal{X},$$

and applying the elementary identity, viz.

$$\Re[w^{\frac{1}{m}}] \ge [\Re(w)]^{\frac{1}{m}} \quad (\Re(w) > 0; m \ge 1),$$

the result (2.1) follows directly from (2.5).

The sharpness of the result (2.1) can be established by considering the function $\mathcal{X}(z)$ defined by (2.4). It is sufficient to show that

$$(2.6)\qquad \inf_{|z|<1}\{\Re(\mathcal{X}(z))\} = \mathcal{X},$$

where $\mathcal{X}$ is given by (2.2). We observe from (2.4) that for $|z| \le r$ $(0 < r < 1)$:

$$\Re\left(\frac{(f*g)(z)}{z^p}\right) \ \ge\ \frac{p}{\alpha}\int_0^1 u^{\frac{p}{\alpha}-1}\,\Re\left(\frac{1+Aur}{1+Bur}\right) du \ \to \mathcal{X} \quad \text{as} \quad r \to 1-,$$

which establishes (2.6) and this completes the proof of Theorem 1.

If we set $p = m = 1$ and $\alpha = 1/2$ in Theorem 1 and apply (1.13) of Lemma 2, we get the following result.

Corollary 1. *If* $f(z) \in \mathcal{A}$ $(\mathcal{A} := \mathcal{A}_1)$ *and*

$$(2.7)\qquad \frac{1}{2}\left(\frac{(f*g)(z)}{z} + (f*g)'(z)\right) \prec \frac{1+Az}{1+Bz} \qquad (z \in \mathbb{U}),$$

then

$$(2.8)\quad \Re\left(\frac{(f*g)(z)}{z}\right) > \begin{cases} \frac{A}{B} - \frac{2}{B^2}\left(1-\frac{A}{B}\right)(ln(1-B)+B) & (B \neq 0);\\ 1-\frac{2}{3}A & (B=0).\end{cases}$$

The result is sharp.

On the other hand, when $A = 1-2\,\beta$ $(0 \leq \beta < 1)$, $B = -1$ *and* $g(z) = z/(1-z)^2$, then Corollary 1 leads to a known result given in [6].

Next, if we set $A = 1-2\beta$ $(0 \leq \beta < 1)$, $B = -1$, $p = m = 1$ and $\alpha = 2$ in Theorem 1 and using (1.12) of Lemma 2, we get Corollary 2 below.

Corollary 2. *If* $f(z) \in \mathcal{A}$ *satisfies the inequality*

$$\Re\left(-\frac{(f*g)(z)}{z} + 2(f*g)'(z)\right) > \beta \qquad (0 \leq \beta < 1;\ z \in \mathbb{U}), \tag{2.9}$$

then

$$\Re\left(\frac{(f*g)(z)}{z}\right) > \beta + (1-\beta)\left(\frac{\pi}{2} - 1\right) \qquad (z \in \mathbb{U}). \tag{2.10}$$

The result is sharp.

Remark 1. (i). We observe from Corollary 1 that if

$$\frac{1}{2}\left(\frac{(f*g)(z)}{z} + (f*g)'(z)\right) \prec \frac{1+A_1 z}{1+Bz} \qquad (B \neq 0; z \in \mathbb{U}),$$

where A_1 is given by

$$A_1 = \frac{2B\,(B + ln(1-B))}{[2(B + ln(1-B)) + B^2]},$$

then

$$\Re\left(\frac{(f*g)(z)}{z}\right) > 0 \qquad (z \in \mathbb{U}).$$

In particular, if $B = -1$, we observe that

$$\Re\left(\frac{(f*g)(z)}{z} + (f*g)'(z)\right) > 2\left(\frac{4\,ln2 - 3}{4\,ln2 - 2}\right) = -0.5886,$$

which implies that

$$\Re\left(\frac{(f*g)(z)}{z}\right) > 0 \qquad (z \in \mathbb{U}).$$

(ii) From Corollary 2, we note that, if $f(z) \in \mathcal{A}$ satisfies the following inequality:

$$\Re\left(-\frac{(f*g)(z)}{z} + 2(f*g)'(z)\right) > \frac{2-\pi}{4-\pi} \qquad (z \in \mathbb{U}),$$

then

$$\Re\left(\frac{(f*g)(z)}{z}\right) > 0 \qquad (z \in \mathbb{U}).$$

The result is sharp.

We consider now an application of the function class $\mathcal{M}_p(g; A, B)$ involving the generalized hypergeometric function defined by (1.8). This result is contained in the following:

Theorem 2. *Let the function $\delta(z)$ defined by*

$$\delta(z) \;=\; z^p \; {}_{r+1}F_{s+1}\left(\alpha_1, ..., \alpha_r, 1+\lambda^{-1}; \beta_1, ..., \beta_s, \lambda^{-1};\; z\right) \tag{2.11}$$

$$(r \leq s+1;\; \lambda > 0;\; z \in \mathbb{U})$$

be in the class $\mathcal{M}_p(g; A, B)$. Then the function

$$\theta(z) \;=\; z^p \; {}_rF_s\left(\alpha_1, ..., \alpha_r; \beta_1, ..., \beta_s;\; z\right) \tag{2.12}$$

satisfies the condition

(2.13)

$$\Re\left(\frac{(\theta * g)'(z)}{p\; z^{p-1}}\right)$$
$$> \nu := \begin{cases} \frac{A}{B} + \left(1 - \frac{A}{B}\right)(1-B)^{-1}\; {}_2F_1\left(1, 1;\; 1+\lambda^{-1}; \frac{B}{B-1}\right) & (B \neq 0) \\ 1 - \frac{A}{\lambda+1} & (B = 0). \end{cases}$$

The result is best possible.

Proof. From (1.4) and (2.11), we get

$$\frac{(\delta * g)'(z)}{p\; z^{p-1}} = 1 + \sum_{k=p+1}^{\infty} [1 + \lambda(k-p)]\; b_k\; \frac{(\alpha_1)_{k-p} \cdots (\alpha_r)_{k-p}\; k}{(\beta_1)_{k-p} \cdots (\beta_s)_{k-p}\; p} \frac{z^{k-p}}{(k-p)!}$$

$$= w(z) + \lambda\; z\; w'(z), \tag{2.14}$$

where

$$w(z) \;=\; 1 + \sum_{k=p+1}^{\infty} b_k\; \frac{(\alpha_1)_{k-p} \cdots (\alpha_r)_{k-p}\; k}{(\beta_1)_{k-p} \cdots (\beta_s)_{k-p}\; p} \frac{z^{k-p}}{(k-p)!}$$
$$= \frac{(\theta * g)'(z)}{p\; z^{p-1}} \qquad (z \in \mathbb{U}).$$

By hypothesis $\delta(z) \in \mathcal{M}_p(g; A, B)$, the assertion (2.13) now follows from (2.14) by following the same procedure as adopted in the proof of Theorem 1.

Making use of a certain integral operator (defined below) in Theorem 1, we establish the following result.

Theorem 3. *Let $f(z) \in \mathcal{A}_p$ and*

$$\mathcal{F}_{\mu,p}(f)(z) = \frac{\mu + p}{z^{\mu}} \int_0^z t^{\mu-1} f(t)\; dt \qquad (\mu > -p;\; z \in \mathbb{U}). \tag{2.15}$$

If

$$(1-\alpha)\,\frac{(\mathcal{F}_{\mu,p}\,(f)*g)(z)}{z^p}+\alpha\,\frac{(f*g)(z)}{z^p}\prec\frac{1+Az}{1+Bz}\qquad(\alpha>0;\ z\in\mathbb{U}),\tag{2.16}$$

then

$$\Re\left(\left(\frac{(\mathcal{F}_{\mu,p}\,(f)*g)(z)}{z^p}\right)^{\frac{1}{m}}\right)>\tau^{\frac{1}{m}}\qquad(m\in\mathbb{N};\ z\in\mathbb{U}),\tag{2.17}$$

where

$$\tau=\begin{cases}\frac{A}{B}+\left(1-\frac{A}{B}\right)(1-B)^{-1}\ {}_2F_1\left(1,1;\ \frac{p+\mu}{\alpha}+1;\frac{B}{B-1}\right) & (B\neq 0)\\ 1-\frac{p+\mu}{p+\mu+\alpha}\ A & (B=0).\end{cases}\tag{2.18}$$

The result is sharp.

Proof. It follows from (2.15) that

$$z\,(\mathcal{F}_{\mu,p}\,(f)*g)'(z)\ =\ (\mu+p)\,(f*g)(z)\ -\ \mu\,(\mathcal{F}_{\mu,p}\,(f)*g)(z).\tag{2.19}$$

Let

$$\frac{(\mathcal{F}_{\mu,p}\,(f)*g)(z)}{z^p}\ =\ h(z),$$

then $h(z)$ is of the form (1.2). The remaining part of the proof is similar to that of Theorem 1, hence we omit the details.

Putting $m=\alpha=1$ in Theorem 3 and noting that

$$(\mathcal{F}_{\mu,p}(f)*g)(z)\ =\ \frac{\mu+p}{z^\mu}\int_0^z t^{\mu-1}(f*g)(t)\,dt\qquad(f\in\mathcal{A}_p;\ z\in\mathbb{U}),$$

we get the following:

Corollary 3. *If $f(z)\in\mathcal{A}_p$ such that*

$$\frac{(f*g)(z)}{z^p}\prec\frac{1+Az}{1+Bz}\qquad(z\in\mathbb{U}),\tag{2.20}$$

then

$$\Re\left(\frac{\mu+p}{z^{\mu+p}}\int_0^z t^{\mu-1}(f*g)(t)\,dt\right)\ >\ \xi\qquad(z\in\mathbb{U}),\tag{2.21}$$

where

$$\xi=\begin{cases}\frac{A}{B}+\left(1-\frac{A}{B}\right)(1-B)^{-1}\ {}_2F_1\left(1,1;\ p+\mu+1;\frac{B}{B-1}\right) & (B\neq 0);\\ 1-\frac{p+\mu}{p+\mu+1}\ A & (B=0).\end{cases}\tag{2.22}$$

The result is best possible.

A special case of Corollary 3, when $A=1-2\beta$ $(0\le\beta<1)$, $B=-1$, $p=1$ and $g(z)=z/(1-z)$, would immediately yield the following result.

Corollary 4. If $f(z) \in \mathcal{A}$ and

$$\Re\left(\frac{f(z)}{z}\right) > \beta \qquad (0 \leq \beta < 1;\ z \in \mathbb{U}), \tag{2.23}$$

then

$$\Re\left(\frac{\mu+1}{z^{\mu+1}}\int_0^z t^{\mu-1} f(t)dt\right) > \xi^*, \tag{2.24}$$

where

$$\xi^* = \beta + (1-\beta)\left({}_2F_1\left(1,1;\mu+2;\frac{1}{2}\right) - 1\right).$$

Remark 2 (i). In [6], it is proved that if $f(z) \in \mathcal{A}$ and

$$\Re\left(\frac{f(z)}{z}\right) > \beta \qquad (0 \leq \beta < 1;\ z \in \mathbb{U}),$$

then

$$\Re\left(\frac{\mu+1}{z^{\mu+1}}\int_0^z t^{\mu-1} f(t)dt\right) > \beta + \frac{1-\beta}{3+2\mu} \qquad (\mu > -1;\ z \in \mathbb{U}). \tag{2.25}$$

For $\mu = 1$, (2.25) gives

$$\Re\left(\frac{2}{z^2}\int_0^z f(t)dt\right) > \frac{4\beta+1}{5},$$

and for this value of $\mu (= 1)$ Corollary 4 in view of (1.13) shows that if $f(z) \in \mathcal{A}$ satisfies

$$\Re\left(\frac{f(z)}{z}\right) > \beta \qquad (0 \leq \beta < 1;\ z \in \mathbb{U}),$$

then

$$\Re\left(\frac{2}{z^2}\int_0^z f(t)dt\right) > (4\, ln2 - 2)\beta - 4\, ln2 + 3. \qquad (z \in \mathbb{U}).$$

Evidently,this shows that the above deduced result from Corollary 4 is an improvement of the result (2.25) given in [6].

(ii). For $p = \mu = 1$, we also note that Corollary 3 in view of (1.13) yields an assertion which we express as follows:

If

$$\frac{(f*g)(z)}{z} \prec \frac{1+A_2 z}{1+Bz} \quad (z \in \mathbb{U};\ f \in \mathcal{A};\ -1 \leq B < A_2 \leq 1;\ B \neq 0), \tag{2.26}$$

where A_2 is given by

$$A_2 = \frac{2B[B + ln(1-B)]}{2[B + ln(1-B)] + B^2},$$

then

$$\Re\left(\frac{1}{z^2}\int_0^z (f*g)(t)\,dt\right) > 0 \qquad (z \in \mathbb{U}). \tag{2.27}$$

Further, on choosing $B = -1$ in (2.26) and (2.27), and using the principle of subordination, we arrive at the following assertion:

If

$$\Re\left(\frac{(f*g)(z)}{z}\right) > \frac{4\,ln2-3}{4\,ln2-2} = -0.2943 \qquad (z \in \mathbb{U};\ f \in \mathcal{A}),$$

then

$$\Re\left(\frac{1}{z^2}\int_0^z (f*g)(t)\,dt\right) > 0 \qquad (z \in \mathbb{U}).$$

For $f \in \mathcal{A}_p$, we define a linear operator

$$\mathcal{S}^n f(z) : \mathcal{A}_p \to \mathcal{A}_p \ (n \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}; z \in \mathbb{U})$$

as follows:

$$\begin{aligned} \mathcal{S}^0 f(z) &= f(z) \\ \mathcal{S}^1 f(z) = \mathcal{S} f(z) &= \frac{1}{p+1}\,[f(z) + zf'(z)], \\ \mathcal{S}^2 f(z) &= \frac{1}{p+1}\,[\mathcal{S}f(z) + z(\mathcal{S}f)'(z)], \\ &\vdots \end{aligned}$$

$$\mathcal{S}^{n+1} f(z) = \frac{1}{p+1}\,[\mathcal{S}^n f(z) + z(\mathcal{S}^n f)'(z)] \qquad (z \in \mathbb{U}). \tag{2.28}$$

The explicit form of $\mathcal{S}^n$ is given by

$$\mathcal{S}^n f(z) = z^p + \sum_{k=p+1}^{\infty} \left(\frac{k+1}{p+1}\right)^n a_k z^k. \tag{2.29}$$

We now prove the following result (Theorem 4 below) involving the linear operator $\mathcal{S}^n f$.

Theorem 4. *Let $f(z) \in \mathcal{A}_p$, then $f(z) \in \mathcal{J}_p(g; \alpha, A, B)$ if and only if $\mathcal{F}_{1,p}(f) \in \mathcal{J}_p(\mathcal{S}g; \alpha, A, B)$.*

Proof. Let $f(z) \in \mathcal{A}_p$, then (2.15) readily gives

$$\mathcal{F}_{\mu,p}\,(f)(z) = z^p + \sum_{k=p+1}^{\infty} \left(\frac{p+\mu}{k+\mu}\right) a_k z^k.$$

For $\mu = 1$, we obtain the relationship that

$$(\mathcal{F}_{1,p}(f) * \mathcal{S}g)(z) = (f*g)(z), \tag{2.30}$$

and the assertion of Theorem 4 follows by appealing to the definition of the class $\mathcal{J}_p(g;\alpha,A,B)$.

The following result makes use of the generalized hypergeometric function (1.8) and Theorem 4.

Theorem 5. *Let*

$$\Theta(z) \;=\; {}_{m+1}F_m\,(p+1,...,p+1,1;p+2,...,p+2;\;z) \tag{2.31}$$

$$(m\in\mathbb{N};\; z\in\mathbb{U})$$

then $f(z)\in\mathcal{J}_p(g;\alpha,A,B)$ *if and only if* $(z^p\,\Theta * f)(z)\in\mathcal{J}_p(\mathcal{S}^m g;\alpha,A,B)$.

Proof. Let $f(z)\in\mathcal{A}_p$ be given by (1.1), then (2.15) in terms of the Gaussian hypergeometric function gives

$$\begin{aligned}
\mathcal{F}_{1,p}(f) &= z^p+\sum_{k=p+1}^{\infty}\left(\frac{p+1}{k+1}\right)a_k\,z^k\\
&= z^p\left(1+\sum_{k=1}^{\infty}\frac{(p+1)_k\;(1)_k}{(p+2)_k}\;\frac{z^k}{k!}\right)*f(z)\\
&= \{z^p\;{}_2F_1(p+1,1;p+2;z)\}*f(z).
\end{aligned}$$

In view of Theorem 4, we infer that $\{z^p\;{}_2F_1(p+1,1;p+2;z)\}*f(z)$ belongs to the class $\mathcal{J}_p(\mathcal{S}g;\alpha,A,B)$. Applying Theorem 4 m times, we get the desired assertion of Theorem 5.

3. Some Observations and Concluding Remarks

(i). If the sequence b_k in (1.3) and the value of parameter α in (1.5) are, respectively, chosen as follows:

$$b_k=\frac{\Gamma(k+1)\;\Gamma(p+1-\lambda)}{\Gamma(p+1)\;\Gamma(k+1-\lambda)}\quad\text{and}\quad\alpha=\frac{\delta p}{p-\lambda}$$

$$(-\infty<\lambda<p+1;\;\delta\geq 0;\;p\in\mathbb{N})$$

and in the process making use of the identity [8, p.112, Eq. (1.10)] in (1.5), then Theorems 1 and 2 correspond, respectively to the results given recently by Patel and Mishra [8, p. 115, Theorems 1.8 and 1.9].

(ii). Next, if we set the coefficient b_k in (1.3) and the value of parameter α in (1.5), respectively, as follows:

$$g(z)=\frac{(\lambda+p)_{k-p}}{(k-p)!}\qquad\text{and}\qquad\alpha=\frac{\delta p}{\lambda+p}\qquad(\lambda>-p;\;\delta>0;\;p\in\mathbb{N}),$$

and in the process apply identity [2, p. 124, Eq.(4)] in (1.5), then the results of Theorem 1 and Theorem 2, reduces to the recently establish result due to Dinggong and Liu [2, p. 124, Theorem 1; p. 126, Theorem 2].

(iii). Also, if we choose the coefficients b_k in (1.3) and the value of the parameter α in (1.5), respectively, as follows:

$$b_k = \left(\frac{p+1}{k+1}\right)^{\sigma} \qquad \text{and} \qquad \alpha = \frac{\lambda}{(p+1)} \qquad (\sigma > 0;\ \lambda \geq 0;\ p \in \mathbb{N}),$$

and apply the following identity [7, p. 3, Eq. (1.17)] in (1.5), then the results contained in Theorem 1 yields the recently established results due to Ozkan [7].

(iv). Lastly, if we set the coefficients b_k in (1.3) and the value of parameter α in (1.5), respectively, as follows:

$$b_k = \frac{(\alpha_1)_{k-p} \dots (\alpha_q)_{k-p}}{(\beta_1)_{k-p} \dots (\beta_s)_{k-p}\,(k-p)!} \qquad \text{and} \qquad \alpha = \frac{p\lambda}{\alpha_i}$$

$$(\alpha_j > 0(j = 1, ..., q),\ \beta_j > 0(j = 1, ..., s),\ q \leq s+1;\ q, s \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}),$$

and also in the process making use of the identity [4, p. 2, Eq. (1.7)] in (1.5), then Corollary 1 correspond to the results given recently by Liu [4, p. 3, Theorem 2.4].

We conclude this paper by remarking that in view of the function class defined by the subordination relation (1.5) and expressed in terms of the convolution (1.4) involving arbitrary coefficients, the main results would lead to additional new results. In fact, by appropriately selecting the arbitrary sequences, the results presented in this paper would find further applications for the classes which incorparate generalized forms of linear operators [3](defined by means of the Hadamard product of the function (1.1) with the generalized Wright's generalized hypergeometric function [9]). The generalized form of linear operator of Dziok and Raina [3] contains such well known operators as the Dziok-Srivastava linear operator, Hohlov linear operator, Saitoh generalized linear operator, Carlson-Shaffer linear operator, Ruscheweyh derivative operator (as well as its generalized versions), Bernardi-Libera-Livingston operator, and Srivastava-Owa fractional derivative operator. Also, Theorems 2 and 5 would eventually lead further to new results for the classes of functions defined analogously by associating in the process the Wright's generalized hypergeometric function. These considerations can fruitfully be worked out and we skip the details in this regard.

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