ON SELF MAPS OF $\mathbb{HP}^n$ FOR $n = 4$ AND 5.

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Abstract. We determine the cardinality of the set of the homotopy classes of self maps of $\mathbb{HP}^4$ with degree 0. And we shall determine the nilpotency of $\mathbb{HP}^5$.

1. Main results

Let $\mathbb{HP}^n$ be the quaternionic projective space of dimension $n$. We shall denote by $j_n : S^4 \to \mathbb{HP}^n$ the inclusion. Especially, we put $j = j_\infty$.

We shall prove the following theorem:

Theorem 1. The cardinality of $\mathcal{H}_0(\mathbb{HP}^4)$ is 2.

Here, for each integer $k$, $\mathcal{H}_k(\mathbb{HP}^n)$ is the totalities of the homotopy classes $[f]$ of maps $f : \mathbb{HP}^n \to \mathbb{HP}^n$ such that $f \circ j_n : S^4 \to \mathbb{HP}^n$ has degree $k$.

For $n = 2, 3$, the cardinalities $K(n, k)$ of $\mathcal{H}_k(\mathbb{HP}^n)$, if it is not 0, are determined in [2] so as to $K(2, 2k) = 1$, $K(2, 2k + 1) = 2$, $K(3, 2k) = 2$ and $K(3, 2k + 1) = 4$.

Next, for all based spaces $X$, we shall denote by $Z_\infty(X)$ (c.f., [1]) the totalities of the homotopy classes $\alpha \in [X, X]$ with the property that $\pi_n(\alpha) = 0 : \pi_n(X) \to \pi_n(X)$ for all integer $n \geq 0$. Clearly, $Z_\infty(\mathbb{HP}^n) \subseteq \mathcal{H}_0(\mathbb{HP}^n)$ holds.

In [1], the nilpotency $t_\infty(X)$ of a based space $X$ is defined to be the least natural number $k$ such that $x_1 \circ \cdots \circ x_k = 0$ holds for all $x_1, \cdots, x_k \in Z_\infty(X)$, if such $k$ exists. If not, we put $t_\infty(X) = \infty$. It was proved in [1] that $t_\infty(\mathbb{HP}^i) = 1$ for $i = 1, 2, 3$ and $t_\infty(\mathbb{HP}^4) = 2$. We shall prove:

Theorem 2. $t_\infty(\mathbb{HP}^5) = 2$

2. Proof of the theorems

We shall consider in the category $\mathscr{T}_0$ of based topological spaces and based maps. We denote by $[f]$ the homotopy class of each map $f$ in $\mathscr{T}_0$. We denote by 0 the homotopy class of any trivial maps. We put $\Sigma^n X = S^n \wedge X$.

By $h_n : S^{4n+3} \to \mathbb{HP}^n$, denotes the Hopf fiber map, and by $q_n : \mathbb{HP}^n \to S^{4n}$ the canonical quotient map. We shall put $r_n = \Sigma^{n-4}(q_1 \circ h_1) : S^{n+3} \to S^n$. We shall put, for each $m, n, k$ with $0 < k \leq m \leq n \leq \infty$, $\mathbb{HP}_k := \mathbb{HP}^n / \mathbb{HP}^{k-1}$, $q_k^n : \mathbb{HP}^n \to \mathbb{HP}_k^n$ to be the quotient map and $i_k^{m,n} : \mathbb{HP}_k^m \to \mathbb{HP}_k^n$.

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$\mathbb{H}P^n_k$ induced from the inclusion $i^{m,n} : \mathbb{H}P^m \to \mathbb{H}P^n$. Especially, we shall put $i_n := i^{n,n+1}$.

For all spaces $X, Y \in \mathcal{R}_0$, the symbol $F(X, Y)$ denotes the totalities of the maps $X \to Y$ in $\mathcal{R}_0$, endowed with the compact open topology and with the trivial map as its base point. We shall denote by $F_0(\mathbb{H}P^n, BS^3)$ the totality of the maps $\mathbb{H}P^n \to BS^3$ with degree 0 endowed with the relative topology induced from $F(\mathbb{H}P^n, BS^3)$.

Proof of the theorem 1. The theorem 1 consists of following two lemmas:

**Lemma 1.** The cardinality of $Z_\infty(\mathbb{H}P^4)$ is 2 and the only non-trivial homotopy class of the set is $[j_4] \circ (\Sigma \nu') \circ \mu_7 \circ [q_4]$.

**Lemma 2.** $Z_\infty(\mathbb{H}P^4) = \mathcal{H}_0(\mathbb{H}P^4)$ holds.

Proof of Lemma 1. Consider the following exact sequence:

Let $i_* : [\mathbb{H}P^4, \mathbb{H}P^3] \to [\mathbb{H}P^3, \mathbb{H}P^4]$ be induced from $i_3$. From Lemma 5.9 and Theorem 5.10 of [1], the image of $Z_\infty(\mathbb{H}P^4)$ by the map $i_*^{-1} \circ i_3^* : [\mathbb{H}P^4, \mathbb{H}P^4] \to [\mathbb{H}P^3, \mathbb{H}P^3]$ is contained in $Z_\infty(\mathbb{H}P^3) = \{0\}$. Hence, $Z_\infty(\mathbb{H}P^4)$ is contained in the image of the map $q_4^* : \pi_{16}(\mathbb{H}P^4) \to [\mathbb{H}P^4, \mathbb{H}P^4]$.

Conversely, let $\alpha \in \pi_{16}(\mathbb{H}P^4)$. Then, since $\pi_{16}(\mathbb{H}P^4) = \mathbb{Z}_2 \{[j_4] \circ (\Sigma \nu') \circ \mu_7 \} \oplus \mathbb{Z}_2 \{[j_4] \circ (\Sigma \nu') \circ \eta_7 \circ \varepsilon_8\}$ holds by taking adjoint of $\pi_{15}(S^3)$ (cf. [6]), $\alpha \circ [q_4] \circ [h_4] = \alpha \circ (4[r_{16}]) = 0$ holds in the group $\pi_{19}(\mathbb{H}P^4)$, and clearly $\alpha \circ [q_4]^\ast = 0$. Hence, from the Proposition 5.4 of [1], $\alpha \circ [q_4] \in Z_\infty(\mathbb{H}P^4)$. Therefore, $Z_\infty(\mathbb{H}P^4) = q_4^*(\pi_{16}(\mathbb{H}P^4))$.

Next, it is well known by [4] that $[j_4] \circ (\Sigma \nu') \circ \mu_7 \circ [q_4] \neq 0$, in the set $[\mathbb{H}P^4, \mathbb{H}P^4]$. Hence $(|1|) \text{card}(Z_\infty(\mathbb{H}P^4)) \geq 2$.

We shall express the co-action (c.f., chapter III of [7]) by the element $\alpha$ of $\pi_{16}(\mathbb{H}P^4)$ on the element $x$ of $[\mathbb{H}P^4, \mathbb{H}P^4]$ by the symbol $x + \alpha$. Then, from the Puppe theorem, to prove Lemma 1, we have only to prove the following lemma:

**Lemma 3.** $[j_4] \circ (\Sigma \nu') \circ \eta_7 \circ \varepsilon_8 \circ [q_4] = 0$ holds in $[\mathbb{H}P^4, \mathbb{H}P^4]$.

**Proof.** Let $h_3' = q_3 \circ h_3 : S^{15} \to S^{12}$, and $q' : \mathbb{H}P_3^4 \to S^{16}$ the quotient map. Then, we obtain the following exact sequence:

$$
\pi_{13}(BS^3) \to (\Sigma h_3')^* \to \pi_{16}(BS^3) \to q'^* \to [\mathbb{H}P^4_3, BS^3]
$$

It is well known that $[h_3'] \equiv \pm 3[r_{12}] \equiv \pm 3\nu_{12} \mod \Sigma^9 \nu' (= 2\nu_{12})$ and from [6], $\pi_{12}(S^3) = \mathbb{Z}_2 \{\mu_3\} \oplus \mathbb{Z}_2 \{\eta_3 \circ \varepsilon_4\}$ holds. Now, since from [5], $\mu_3 \circ \nu_{12} = \nu' \circ \eta_6 \circ \varepsilon_7$ and $\eta_3 \circ \varepsilon_4 \circ \nu_{12} = \varepsilon_3 \circ \eta_{11} \circ \nu_{12} = \varepsilon_3 \circ (\Sigma^8 \nu') \circ \eta_{14} = \varepsilon_3 \circ (2\nu_{11}) \circ \eta_{14} = 0$.
holds. Therefore, \((\Sigma h_2)^*(\pi_{13}(BS^3)) = \{0, [j] \circ (\Sigma \nu') \circ \eta_7 \circ \varepsilon_8\}\) holds, hence \([j] \circ (\Sigma \nu') \circ \eta_7 \circ \varepsilon_8 \circ [q_4] = [j] \circ (\Sigma \nu') \circ \eta_7 \circ \varepsilon_8 \circ [q'] \circ [q_3] = 0\) holds in the set \([\mathbb{H}P^4, BS^3]\). \(\square\)

Proof of Lemma 2. We shall use the notations in [2]: Let \(d_{4,0} : \pi_{12}(BS^3) \to \pi_{15}(BS^3)\) be the composition of \(j_{3,0} : \pi_{12}(BS^3) \to \pi_0(F_0(\mathbb{H}P^3, BS^3))\) and the map \(\partial_{4,0} : \pi_0(F_0(\mathbb{H}P^3, BS^3)) \to \pi_{15}(BS^3)\) induced from \(h_3\). As in the proof of Proposition 1.3 of [3], \(d_{4,0}(l) = \pm 3l \circ [r_{12}]\) holds for \(l \in \pi_{12}(BS^3)\). Since \(\pi_{12}(BS^3) = \mathbb{Z}_2\{j_4 \varepsilon_4\}\) ([6]), \(d_{4,0}(j_4 \varepsilon_4) = [j] \circ \varepsilon_4 \circ [r_{12}] = [j] \circ \varepsilon_4 \circ \nu_{12} = 0\) in the set \(\pi_{15}(BS^3)\).

Finally, from Theorem 2 of [2], the cardinality of the set \(\mathcal{H}_0(\mathbb{H}P^3)\) \((\cong \pi_0(F_0(\mathbb{H}P^3, BS^3)))\) is 2. Therefore, \(\partial_{4,0}\) is injection, hence \(i_5^*\) \((\mathcal{H}_0(\mathbb{H}P^4)) \subset \text{Ker}(\partial_{4,0}) = 0\). Hence \(\mathcal{H}_0(\mathbb{H}P^4) \subset q_4(\pi_{16}(\mathbb{H}P^4)) = Z_\infty(\mathbb{H}P^4)\). This completes the proof of Lemma 2 so that Theorem 1 holds.

Proof of Theorem 2. The non trivial class \(\xi := [j_4] \circ (\Sigma \nu') \circ \mu_7 \circ [q_4] \in Z_\infty(\mathbb{H}P^4)\) is represented by the restriction of a map \(f : \mathbb{H}P^5 \to \mathbb{H}P^5\) which represents a non trivial class \(\alpha \in [\mathbb{H}P^5, \mathbb{H}P^5]\), because of \(\xi \circ [h_4] = 0\) and of the fact that \(\mathbb{H}P^5\) is the mapping cone of the map \(h_4 : S^{19} \to \mathbb{H}P^4\). Let \(x+\gamma\) be the co-action on \(x \in [\mathbb{H}P^5, \mathbb{H}P^5]\) by \(\gamma \in \pi_{20}(\mathbb{H}P^5)\).

Take two elements \(x, y \in Z_\infty(\mathbb{H}P^5)\). Then \(x\) has the form \(x = 0 + \gamma = \gamma \circ q_5\) or the form \(x = \gamma + \gamma\) for some \(\gamma \in \pi_{20}(\mathbb{H}P^5)\). For the former case, it is trivial that \(y \circ x = 0\). For the latter case, \(y \circ x = y \circ \alpha + y \circ \gamma = y \circ \alpha\). Therefore, it is enough that we can take \(\alpha\) so as to be factored through \(S^{15}\) (or \(S^4\)).

By [5], \([j_5] \circ (\Sigma \nu') \circ \mu_7 = [j_5] \circ (\Sigma \mu') \circ \eta_{15}\) and \(\eta_{15} \circ [q_4] \circ [h_4] = \eta_{15} \circ (4i_{16}) = 0\). Therefore, there exists \(\alpha' : \mathbb{H}P^5 \to S^{15}\) such that \(\alpha' \circ [i_4^{4,5}] = \eta_{15}\). Hence \([j_5] \circ (\Sigma \mu') \circ \eta_{15} \circ [q_4] = [j_5] \circ (\Sigma \mu') \circ \alpha' \circ [i_4^{4,5}] \circ [q_4] = [j_5] \circ (\Sigma \mu') \circ \alpha' \circ [q_5^3] \circ [i_4].\) We can put \(\alpha = [j_5] \circ (\Sigma \mu') \circ \alpha' \circ [q_5^3]\).

References


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