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SOME PROPERTIES OF EF-EXTENDING RINGS

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ABSTRACT. In [16], Thuyet and Wisbauer considered the extending property for the class of (essentially) finitely generated submodules. A module M is called ef-extending if every closed submodule which contains essentially a finitely generated submodule is a direct summand of M. A ring R is called right ef-extending if R_R is an ef-extending module. We show that a ring R is right ef-extending and the R-dual of every simple left R-module is simple if and only if R is semiperfect right continuous with $S_l = S_l \leq^e R_R$. We also prove that a ring R is a QF-ring if and only if R is left Kasch and $R_R^{(\omega)}$ is ef-extending if and only if R is right AGP-injective satisfying DCC on right (or left) annihilators and $(R \oplus R)_R$ is ef-extending.

1. INTRODUCTION AND DEFINITIONS

Throughout the paper, R represents an associative ring with identity $1 \neq 0$ and all modules are unitary R-modules. We write M_R (resp., ${}_RM$) to indicate that M is a right (resp., left) R-module. We also write J (resp., Z_r, S_r) for the Jacobson radical (resp., the right singular ideal, the right socle of R) and $E(M_R)$ for the injective hull of M_R. If X is a subset of R, the right (resp., left) annihilator of X in R is denoted by $r_R(X)$ (resp., $l_R(X)$) or simply $r(X)$ (resp., $l(X)$) if no confusion appears. If N is a submodule of M (resp., proper submodule) we denote by $N \leq M$ (resp., $N < M$). Moreover, we write $N \leq^e M$, $N \leq^{\oplus} M$ and $N \leq^{max} M$ to indicate that N is an essential submodule, a direct summand and a maximal submodule of M, respectively. A module M is called uniform if $M \neq 0$ and every non-zero submodule of M is essential in M. It is called that a module M has *finite uniform dimension* if M does not contain an infinite direct sum of non-zero submodules. Let M, N be R-modules. M is said to be N-injective if, for any submodule H of N, every R-homomorphism $f : H \longrightarrow M$ can be extended to an R-homomorphism $\bar{f} : N \longrightarrow M$. Let M be any module. A submodule K of M is *closed* (in M), if $K \leq^e L \leq M$, then $K = L$. A ring R is called *right AGP-injective* if for each $0 \neq a \in R$, there exists $n \in \mathbb{N}$ such that $a^n \neq 0$ and $lr(a^n) = Ra^n \oplus X_a$ with $X_a \leq {}_RR$ (see [17]). A ring R is called QF if it is right (or left) artinian and right (or left) self-injective. A ring R is said to be right PF if R_R is an injective cogenerator in the category

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of right R-modules. A ring R is called *semiregular* if R/J is von Neumann regular and idempotents lift modulo J. A ring R is called right *Kasch* if every simple right R-module is embeded in R.

We consider the following conditions on a module M_R:

- C1: Every submodule of M is essential in a direct summand of M.
- C2: Every submodule of M that is isomorphic to a direct summand of M is itself a direct summand of M.
- C3: $M_1 \oplus M_2$ is a direct summand of M for any two direct summand $M_1,\ M_2$ of M with $M_1 \cap M_2 = 0$.

Module M_R is called extending (or CS) (resp., continuous) if it satisfies C1 (resp., both C1 and C2). R is called right extending (resp., continuous) if R_R is an extending module (resp., continuous). Module M_R is called quasi-continuous if it satisfies C1 and C3. A module M is called *uniform-extending* if every uniform submodule is essential in a direct summand of M.

Recently, the theory of extending modules has been developed. Some results for extending modules contribute to plentiful theory of ring and module, particularly decomposition into direct sum of indecomposable (or uniform) modules and application to theory of QF-rings. In [11], Oshiro considered a ring R such that every projective R-module is an extending module (i.e., R is co-H) and in [12] he proved that rings with this property are (left) Artinian. Also in [3, 7] it is proved that $R^{(I)}$ is an extending module for any index set I if and only if every projective R-module is an extending module.

In [3], Dung, Huynh, Smith and Wisbauer studied QF-rings via extending modules. In [6], Pardo and Asensio proved that R is right PF if and only if R is right cogenerator right extending. This result generalizes Osofsky's result in [13]. In [18], Yousif proved that R is right PF if and only if $R \oplus R$ is extending as a right R-module and the R-dual of every simple left R-module is simple.

There are some interesting generalization of extending modules. For example, in [15] Smith and Tercan studied weak extending module and C_{11} module and in [16], Thuyet and Wisbauer considered the extending property for the class of (essentially) finitely generated submodules and defined ef-extending module. A module M is called *ef-extending* if every closed submodule which contains essentially a finitely generated submodule is a direct summand of M. A ring R is called right ef-extending if R_R is an ef-extending module.

In this paper, we prove that R is QF if and only if R is left Kasch and $R_R^{(\omega)}$ is ef-extending. Moreover, we prove that R is QF if and only if R is right

AGP-injective satisfying DCC on right (or left) annihilators and $(R \oplus R)_R$ is ef-extending.

2. Main Results.

From the definition of ef-extending module and ring, we have:

i) A right extending ring is right ef-extending. But the converse is not true in general.

Example. Let K be a division ring and ${}_KV$ a left K-vector space with infinite dimension. Take $S = \mathrm{End}({}_KV)$, then it right ef-extending but not right extending.

ii) Every finitely generated submodule of an ef-extending module M is essential in a direct summand of M.

Lemma 2.1 ([16]). *Every direct summand of an ef-extending module is ef-extending.*

It is well-known M is an extending module if and only if every closed submodule of M is a direct summand (see [3]).

Lemma 2.2. *Let M be a module such that $Soc(M)$ is finitely generated and essential in M. Then M is an extending module if and only if M is an ef-extending module.*

Proof. Assume that M is ef-extending. Let N be a closed submodule of M. We have $Soc(N) \leq Soc(M)$, and so $Soc(N)$ is finitely generated by hypothesis.

On the other hand, $Soc(N) = N \cap Soc(M)$. For every $x \in N$ such that $xR \cap Soc(N) = 0$, $xR \cap (N \cap Soc(M)) = 0$ or $xR \cap Soc(M) = 0$. Since $Soc(M) \leq^e M$, $xR = 0$. It follows that $x = 0$. So $Soc(N) \leq^e N$. Hence N is a closed submodule which contains essentially a finitely generated submodule. Thus N is a direct summand of M. □

Corollary 2.3. *Let R be a left perfect ring. Then R is right extending if and only if R is right ef-extending.*

If ${}_RM$ is a left R-module, recall that the R-*dual* $M^* = \mathrm{Hom}_R({}_RM, R)$ of M is a right R-module via $(fr)(m) = f(rm)$ for all $r \in R$, $f \in M^*$, and $m \in M$.

Proposition 2.4. *The following statements are equivalent for a ring R.*

(1) *R is right ef-extending and the R-dual of every simple left R-module is simple.*
(2) *R is semiperfect right continuous ring with $S_r = S_l \leq^e R_R$.*

Proof. (1) $\Rightarrow$ (2). By [10, Theorem 4.8], we have R is semiperfect, left Kasch with $S_r = S_l \leq^e R_R$. Since R is right ef-extending, e_iR is ef-extending for all $i = 1, 2, \ldots, n$. It follows that e_iR is uniform (because e_iR is indecomposable) and so $Soc(e_iR)$ is simple and essential in e_iR for all $i = 1, 2, \ldots, n$. Therefore S_r is finitely generated and hence R is right extending by Lemma 2.2. Thus R is right continuous by [10, Theorem 4.10].
(2) $\Rightarrow$ (1) is clear. □

Theorem 2.5. *The following statements are equivalent for a ring R.*

(1) *R is right PF.*
(2) *$R \oplus R$ is ef-extending as a right R-module and the R-dual of every simple left R-module is simple.*

Proof. (1) $\Rightarrow$ (2) is clear.
(2) $\Rightarrow$ (1). Assume that $R \oplus R$ is ef-extending as a right R-module and the R-dual of every simple left R-module is simple. Then R is semiperfect right continuous ring with $S_r = S_l \leq^e R_R$ by Proposition 2.4. Since R is semiperfect, right ef-extending and $S_r \leq^e R_R$, then S_r is finitely generated. Hence $Soc(R \oplus R)_R$ is finitely generated and essential in $(R \oplus R)_R$. Therefore $(R \oplus R)_R$ is extending by Lemma 2.2. We have $J = Z_r$ (since R is right continuous) and R is semiregular (since R is semiperfect), then $(R \oplus R)_R$ satisfies the C2 by [10, Example 7.18], and so $(R \oplus R)_R$ is continuous. Thus R is right self-injective by [10, Theorem 1.35]. □

Corollary 2.6 ([18], Theorem 2)**.** *The following statements are equivalent for a ring R.*

(1) *R is right PF.*
(2) *$R \oplus R$ is extending as a right R-module and the R-dual of every simple left R-module is simple.*

We write $R_R^{(\omega)}$ to indicate a countable direct sum of copies of the right R-module R_R.

Theorem 2.7. *The following statements are equivalent for a ring R.*

(1) *R is QF.*
(2) *R is left Kasch and $R_R^{(\omega)}$ is ef-extending.*

Proof. We prove (2) $\Rightarrow$ (1). Let T be a maximal left ideal of R. Since R is left Kasch, $r(T) \neq 0$. There exists $0 \neq a \in r(T)$ or $T \leq l(a)$ which yields $T = l(a)$ by maximality of T and so $r(T) = rl(a)$. Since R is right ef-extending, $aR \leq^e eR$ for some $e^2 = e \in R$. On the other hand, $aR \leq rl(a) \leq eR$ and then $rl(a) \leq^e eR$. Hence $r(T) \leq^e eR$. It implies that R is semiperfect by [10, Lemma 4.1]. Thus $R = e_1R \oplus \cdots \oplus e_nR$, where $\{e_i\}_{i=1}^n$ is the complete set of orthogonal local idempotents. For every $i \neq j$

($i, j \in \{1, 2, \cdots, n\}$), let $f : e_iR \longrightarrow e_jR$ be a monomorphism. We have $e_iR \cong f(e_iR) \leq e_jR$. Since R satisfies the right C2 (because R is left Kasch), then $f(e_iR)$ is a direct summand of e_jR or $f(e_iR) = e_jR$ (because e_jR is indecomposable). Hence f is an isomorphism. Since R is right ef-extending, every uniform right ideal of R is essential in direct summand of R_R. Therefore for every $i_0 \in \{1, 2, \cdots, n\}$, $\bigoplus_{\{1,2,\cdots,n\}\backslash\{i_0\}} e_iR$ is $e_{i_0}R$-injective by [3, Corollary 8.9]. Since e_iR is also ef-extending, indecomposable and so e_iR is quasi-continuous. By [9, Theorem 2.13], R is right quasi-continuous. Thus R is right continuous.

By Utumi's Theorem (see [10, Theorem 1.26]), $J = Z_r$. By [10, Example 7.18], $(R \oplus R)_R$ satisfies the C2 and so $(R \oplus R)_R$ is continuous. Thus R is right self-injective by [10, Theorem 1.35]. It implies that e_iR is injective for every $i = 1, 2, \ldots, n$. On the other hand, $R_R^{(\omega)} = (e_1R \oplus \cdots \oplus e_nR)^{(\omega)} = \bigoplus_{\omega'} \varepsilon_iR$ is uniform-extending, for some countable set ω' and $\varepsilon_i \in \{e_1, e_2, \cdots, e_n\}$ for each $i \in \omega'$. By [3, Corollary 8.10], $R_R^{(\omega)}$ is injective. By a well-known result of Faith ([4]), R has ACC on right annihilators and hence R is QF. $\square$

Corollary 2.8 ([18], Theorem 3)**.** *The following statements are equivalent for a ring R.*

(1) *R is QF.*

(2) *R is left Kasch and $R_R^{(\omega)}$ is extending.*

Corollary 2.9. *The following statements are equivalent for a ring R.*

(1) *R is right PF.*

(2) *R is left, right Kasch and $(R \oplus R)_R$ is ef-extending.*

Proof. (1) $\Rightarrow$ (2). Assume that R is right PF. Then R is left, right Kasch and R is right self-injective by [5, Theorem 2.8]. Hence $(R \oplus R)_R$ is extending module [10, Theorem 1.35].

(2) $\Rightarrow$ (1). In the proof of Theorem 2.7, we showed that R is right PF. $\square$

Lemma 2.10. *Assume that $R_R = e_1R \oplus e_2R \oplus \cdots \oplus e_nR$, where each e_iR is uniform for all $i = 1, 2, \ldots, n$. If every monomorphism $R_R \longrightarrow R_R$ is an epimorphism, then R is semiperfect.*

Proof. By [10, Lemma 4.26]. $\square$

A ring R is called *I-finite* if R contains no infinite sets of orthogonal idempotents (see [10]).

Proposition 2.11. *The following statements are equivalent for a ring R.*

(i) *R is a semiperfect, right continuous ring.*

(ii) *R is a right ef-extending ring, $Z_r = J$, and R has DCC on principal projective right ideals.*
(iii) *R is a right ef-extending, right C2 ring, and R has DCC on principal projective right ideals.*
(iv) *R is a right ef-extending, right C2 and I-finite ring.*

Proof. $(i) \Rightarrow (ii) \Rightarrow (iii) \Rightarrow (iv)$ are clear.

$(iv) \Rightarrow (i)$. Since R is I-finite, write $1 = u_1 + \cdots + u_n$, where the u_i are orthogonal primitive idempotents. Hence $R = u_1R \oplus \cdots \oplus u_nR$, where each u_kR is uniform because it is an ef-extending module. On the other hand, R satisfies C2, every monomorphism $R_R \longrightarrow R_R$ is epimorphism. Thus R is semiperfect by Lemma 2.10. By a similar proof of Theorem 2.7, R is right continuous. □

Corollary 2.12. *The following statements on a ring R are equivalent:*

(1) *R is a semiperfect right self-injective ring.*
(2) *$(R \oplus R)_R$ is ef-extending, $Z_r = J$ and R has DCC on principal projective right ideals.*

Proof. (1) $\Rightarrow$ (2) is clear.

(2) $\Rightarrow$ (1). By Proposition 2.11, R is semiperfect. Thus by a similar proof of Theorem 2.7, R is right self-injective. □

Next, we consider properties of a ring which has DCC on right (left) annihilators such that $(R \oplus R)_R$ is ef-extending.

Theorem 2.13. *The following statements are equivalent for a ring R;*

(1) *R is QF.*
(2) *R is right AGP-injective satisfying DCC on right annihilators and $(R \oplus R)_R$ is ef-extending.*
(3) *R is right AGP-injective satisfying DCC on left annihilators and $(R \oplus R)_R$ is ef-extending.*

Proof. (2) $\Rightarrow$ (1). Assume that there is monomorphism $f : R \longrightarrow R$ which not epimorphism. Let $a = f(1)$. Then $r(a^n) = 0, \forall n \geq 1$. Assume that $aR \neq R$. Since R is right AGP-injective, there exist a positive integer $m \geq 1$ and $X_1 \leq {}_RR$ such that $a^m \neq 0$ and $lr(a^m) = Ra^m \oplus X_1$. It implies that $R = Ra^m \oplus X_1$ (since $r(a^m) = 0$) and so $Ra^m = Re$ for some $e^2 = e \in R$. Then

$$0 = r(a^m) = r(Ra^m) = r(Re) = r(e) = (1 - e)R,$$

and hence $e = 1$ or $Ra^m = R$. It implies that $R = Ra$, i.e., $ba = 1$ for some $b \in R$. If $ab \neq 1$, then by [8, Example 21.26], there are some $e_{ij} = a^ib^j - a^{i+1}b^{j+1} \in R$, $i, j \in \mathbb{N}$ such that $e_{ij}e_{kl} = \delta_{jk}e_{il}$ for all $i, j, k \in \mathbb{N}$, where δ_{jk} are the Kronecker deltas. Notice $e_{ij} \neq 0$ for all $i, j \in I\!N$, by construction. Set $e_i = e_{ii}$. Then $e_i e_j = \delta_{ij} e_i$, $\forall i, j \in I\!N$. Therefore we have

$$r(e_1) > r(\{e_1, e_2\}) > \cdots ,$$

this is a contradiction because R has DCC on right annihilators. Hence $ab = 1$ and so $aR = R$. This is a contradiction to our assumption. Thus f is an epimorphism.

On the other hand, R is I-finite, there exists an orthogonal set of primitive idempotents $\{e_i\}_{i=1}^{n}$ such that $R_R = e_1R \oplus e_2R \oplus ... \oplus e_nR$. Since R is right ef-extending, e_iR is ef-extending and so e_iR is uniform for every $i = 1, 2, \ldots, n$. Thus R is semiperfect by Lemma 2.10. But R is right AGP-injective, $J = Z_r$. Therefore R is right C2 by [10, Example 7.18]. Then R is QF by the later part of the proof in Theorem 2.7.

(3) $\Rightarrow$ (1). Assume that there is monomorphism $f : R \longrightarrow R$ which not epimorphism. The same argument of (2) $\Rightarrow$ (1), there exists a set of orthogonal idempotents $\{e_i \in R | e_i \neq 0, i \in I\!N\}$. Therefore we have

$$l(e_1) > l(\{e_1, e_2\}) > \cdots ,$$

this is a contradiction because R has DCC on left annihilators. Thus R is QF by the later part of the proof in (2) $\Rightarrow$ (1). $\square$

Theorem 2.14. *The following statements are equivalent.*

(1) *R is QF.*

(2) *$(R \oplus R)_R$ is ef-extending, R is right C2 and satisfies ACC on left annihilators.*

(3) *$(R \oplus R)_R$ is ef-extending, R is right C2 and satisfies ACC on right annihilators.*

Proof. (1) $\Rightarrow$ (2), (3) is clear.

(2) $\Rightarrow$ (1). Assume that there is monomorphism $f : R \longrightarrow R$ which not epimorphism. Let $a = f(1)$. If $aR \neq R$, then $a^nR > a^{n+1}R$ for all $n \geq 1$ (because $r(a) = 0$). Moreover, $a^nR \cong aR \cong R$ for all $n \geq 1$. Then for every $n \geq 1$, there exists $0 \neq e_n^2 = e_n \in R$ such that $a^nR = e_nR$ because R is right C2. Hence we have a strict ascending chain

$$l(e_1) < l(e_2) < \cdots ,$$

this is a contradiction. Thus $aR = R$ or f is epimorphism. It implies that R is semiperfect by Lemma 2.10. Then R is QF by the later part of the proof in Theorem 2.7.

(3) $\Rightarrow$ (1). By (2) $\Rightarrow$ (1), for every monomorphism $f : R \longrightarrow R$ which is not epimorphism and for every $n \geq 1$, there exists $0 \neq e_n^2 = e_n \in R$ such

that $a^n R = e_n R$ with $a = f(1)$. Hence we have

$$e_1 R > e_2 R > \cdots ,$$

this is a contradiction by [10, Lemma B.6]. Thus R is QF by the same argument of (2) ⇒ (1). □

From this theorem, we have the following proposition.

Proposition 2.15. *The following statements are equivalent for a ring R.*

(1) *R is QF.*
(2) *$(R \oplus R)_R$ is ef-extending, R satisfies ACC on right annihilators and $S_l \leq^e R_R$.*

Proof. We prove (2) ⇒ (1). By [14, Theorem 2.9], R is semiprimary. Therefore R is left Kasch by [10, Lemma 4.2]. Thus R is QF. □

In [1], the authors proved that R is QF if and only if R is right Artinian, $(R \oplus R)_R$ or ${}_R(R \oplus R)$ is extending and $S_r \leq S_l$. In this paper, we extend this result in case $(R \oplus R)_R$ or ${}_R(R \oplus R)$ is ef-extending.

Theorem 2.16. *Assume that R has ACC on right annihilators with $S_r \leq^e R_R$. Then the following statements are equivalent.*

(1) *R is QF.*
(2) (a) *$(R \oplus R)_R$ or ${}_R(R \oplus R)$ is ef-extending.*
    (b) *$S_r \leq S_l$.*

Proof. (1) ⇒ (2) is clear.

(2) ⇒ (1). Since R has ACC on right annihilators with $S_l \leq^e R_R$, R is semiprimary by [14, Theorem 2.9]. Hence R is left Kasch by [10, Lemma 4.2]. If $(R \oplus R)_R$ is ef-extending, then R is QF by Proposition 2.15.

If ${}_R(R \oplus R)$ is ef-extending, then R is left ef-extending. Therefore $Soc(Re)$ is simple for every local idempotent $e \in R$. It implies that $S_r e$ is simple or zero. Moreover, S_r is essential in R_R and so $Soc(eR) \neq 0$. It implies that R is right Kasch by [10, Theorem 3.12]. Therefore R is left C2. Thus R is QF by Theorem 2.14. □

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