Math. J. Okayama Univ. **52** (2010), 111–122

ON GENERALIZED EPI-PROJECTIVE MODULES

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Abstract. A module M is said to be *generalized N-projective* (or *N-dual ojective*) if, for any epimorphism $g : N \longrightarrow X$ and any homomorphism $f : M \longrightarrow X$, there exist decompositions $M = M_1 \oplus M_2$, $N = N_1 \oplus N_2$, a homomorphism $h_1 : M_1 \longrightarrow N_1$ and an epimorphism $h_2 : N_2 \longrightarrow M_2$ such that $g \circ h_1 = f|_{M_1}$ and $f \circ h_2 = g|_{N_2}$. This relative projectivity is very useful for the study on direct sums of lifting modules (cf. [5], [7]). In the definition, it should be noted that we may often consider the case when f to be an epimorphism. By this reason, in this paper we define relative (strongly) generalized epi-projective modules and show several results on this generalized epi-projectivity. We apply our results to the known problem when finite direct sums $M_1 \oplus \cdots \oplus M_n$ of lifting modules M_i $(i = 1, \cdots, n)$ is lifting.

1. Preliminaries

Throughout this paper R is a ring with identity and all modules considered are unitary right R-modules.

A submodule S of a module M is called a *small* submodule, if $M \neq K + S$ for any proper submodule K of M. In this case we write $S \ll M$. Let M be a module and let N and K be submodules of M with $K \subseteq N$. K is called a *co-essential* submodule of N in M if $N/K \ll M/K$ and we write $K \subseteq_c N$ in M. Let X be a submodule of M. X is called a *co-closed* submodule in M if X does not have a proper co-essential submodule in M. X' is called a *co-closure* of X in M if X' is a co-closed submodule of M with $X' \subseteq_c X$ in M. $K <_\oplus N$ means that K is a direct summand of N. Let $M = M_1 \oplus M_2$ and let $\varphi : M_1 \to M_2$ be a homomorphism. Put $\langle M_1 \xrightarrow{\varphi} M_2 \rangle = \{m_1 - \varphi(m_1) \mid m_1 \in M_1\}$. Then this is a submodule of M which is called *the graph* with respect to $M_1 \xrightarrow{\varphi} M_2$. Note that $M = M_1 \oplus M_2 = \langle M_1 \xrightarrow{\varphi} M_2 \rangle \oplus M_2$.

A module M is said to be *lifting* if, for any submodule X, there exists a direct summand X^* of M such that $X^* \subseteq_c X$ in M.

Let $\{M_i \mid i \in I\}$ be a family of modules. The direct sum decomposition $M = \oplus_I M_i$ is said to be *exchangeable* if, for any direct summand X of M, there exists $\overline{M_i} \subseteq M_i$ $(i \in I)$ such that $M = X \oplus (\oplus_I \overline{M_i})$. A module M is said to have the *(finite) internal exchange property* if, any (finite) direct sum decomposition $M = \oplus_I M_i$ is exchangeable.

Mathematics Subject Classification. Primary 16D10; Secondary 16D99.

Key words and phrases. (strongly) generalized epi-projective module, lifting module.

Let X be a submodule of a module M. A submodule Y of M is called a *supplement* of X in M if $M = X + Y$ and $X \cap Y \ll Y$. Note that a supplement Y of X in M is co-closed in M. A module M is *supplemented* (*$\oplus$-supplemented*) if, for any submodule X of M, there exists a submodule (direct summand) Y of M such that Y is a supplement of X in M. A module M is called *amply supplemented* if, X contains a supplement of Y in M whenever $M = X + Y$. We see that M is an amply supplemented module if and only if M is a supplemented module and any submodule of M has a co-closure in M (cf. [4, Lemma 1.7]).

Let M and N be modules. M is called *im-small N-projective* if, for any submodule A of N, any homomorphism $f : M \longrightarrow N/A$ with $f(M) \ll N/A$ can be lifted to a homomorphism $g : M \longrightarrow N$. M is called *epi-N-projective* if, for any submodule A of N, every epimorphism $f : M \longrightarrow N/A$ can be lifted to a homomorphism $g : M \longrightarrow N$.

Let M be any module. Consider the following conditions:

(D_2) If $A \leq M$ such that M/A is isomorphic to a direct summand of M, then A is a direct summand of M.

(D_3) If M_1 and M_2 are direct summands of M with $M = M_1 + M_2$, then $M_1 \cap M_2$ is a direct summand of M.

Then the module M is called *discrete* if it is lifting and satisfies the condition (D_2) and it is called *quasi-discrete* if it is lifting and satisfies the condition (D_3). Since (D_2) implies (D_3), every discrete module is quasi-discrete.

In this paper, we show the following:

(1) Let M and N be lifting modules with the finite internal exchange property. Then M is generalized N-projective if and only if M is strongly generalized epi-N-projective and im-small N-projective.

(2) Let $M_1, \cdots, M_n$ be lifting modules with the finite internal exchange property and put $M = M_1 \oplus \cdots \oplus M_n$. Then M is lifting with the finite internal exchange property if and only if M_i is generalized $\oplus_{j\neq i} M_j$-projective ($\oplus_{j\neq i} M_j$ is generalized M_i-projective) for any $i \in \{1, \cdots, n\}$ if and only if M_i is strongly generalized epi-$\oplus_{j\neq i} M_j$-projective ($\oplus_{j\neq i} M_j$ is strongly generalized epi-M_i-projective) for any $i \in \{1, \cdots, n\}$ and M_k is im-small M_l-projective for any $k \neq l \in \{1, \cdots, n\}$.

Especially, in the case of $n = 2$, we obtain the following:

Let M_1 and M_2 be lifting modules with the finite internal exchange property and put $M = M_1 \oplus M_2$. Then M is lifting with the finite internal exchange property if and only if M_1 is generalized M_2-projective and M_2 is im-small M_1-projective (M_2 is generalized M_1-projective and M_1 is im-small M_2-projective).

As a corollary of the result (2), we obtain

(3) Let $M_1, \cdots, M_n$ be quasi-discrete and put $M = M_1 \oplus \cdots \oplus M_n$. Then M is lifting with the (finite) internal exchange property if and only if M_i is generalized M_j-projective for any $i \neq j \in \{1, \cdots, n\}$ if and only if M_i is (strongly) generalized epi-M_j-projective and im-small M_j-projective for any $i \neq j \in \{1, \cdots, n\}$.

We emphasize the assumption "with finite internal exchange property" is quite natural.

For undefined terminologies, the reader is referred to [2] and [8].

Lemma 1.1. *Let $X' \subseteq X \subseteq M$. Then*

(1) If $M = X' + Y$ and $X \cap Y \ll M$, then $X' \subseteq_c X$ in M.

(2) If $X' \ll M$ and X is co-closed in M then $X' \ll X$.

Proof. By [5, Lemma 1.4] and [3, Lemma 2.5]. □

Lemma 1.2. *(cf.* [6, Lemmas 1.7 and 1.8]*) Let $f : M \to N$ be an epimorphism with* $\ker f \ll M$*. Then*

(1) If X is co-closed in M, then $f(X)$ is co-closed in N.

(2) If $M = X \oplus Y$, then $f(X) \cap f(Y) \ll N$.

(3) If $S \ll N$, then $f^{-1}(S) \ll M$.

Lemma 1.3. *Let M be a module and let N be a $\oplus$-supplemented module. Then M is im-small N-projective if and only if for any small submodule X of N and any homomorphism $f : M \to N/X$ with $Im\ f \ll N/X$, there exists a homomorphism $h : M \to N$ such that $\pi \circ h = f$, where $\pi : N \to N/X$ is the canonical epimorphism.*

Proof. "Only if" part is clear.

"If" part : Let $\pi : N \to N/X$ be the canonical epimorphism and let $f : M \to N/X$ be a homomorphism with $\mathrm{Im}\ f \ll N/X$. Since N is $\oplus$-supplemented, there exists a direct summand N^* of N such that $N = X + N^*$ and $X \cap N^* \ll N^*$. Then $\pi|_{N^*} : N^* \to N/X$ is an epimorphism with $\ker(\pi|_{N^*}) \ll N^*$. Put $N = N^* \oplus N^{**}$ and define $g : N = N^* \oplus N^{**} \to N/X \oplus N^{**}$ by $g(n^* + n^{**}) = \pi(n^*) + n^{**}$, where $n^* \in N^*$ and $n^{**} \in N^{**}$. Then g is a small epimorphism and so there exists a homomorphism $h : M \to N$ such that $g \circ h = f$. Hence $\pi \circ h = f$. □

In the proof of the following proposition, we use the idea described in Y. Baba and M. Harada [1, pp. 54-56].

Proposition 1.4. *(1) Let $M' <_\oplus M$ and $N' <_\oplus N$. If M is im-small N-projective then M' is im-small N'-projective.*

(2) Let $M_1, \cdots, M_n$ be modules and put $M = M_1 \oplus \cdots \oplus M_n$. If M_i is im-small N-projective ($i = 1, \cdots, n$) then M is im-small N-projective.

(3) Let $N_1, \cdots, N_t$ be $\oplus$-supplemented modules and put $N = N_1 \oplus \cdots \oplus N_t$. If M is im-small N_i-projective ($i = 1, \cdots, t$) then M is im-small N-projective.

Proof. (1) and (2) are clear. (3) : Since finite direct sums of $\oplus$-supplemented modules are $\oplus$-supplemented, it is enough to prove the case of $N = N_1 \oplus N_2$.

Let $\pi : N \to N/X$ be a canonical epimorphism and let $f : M \to N/X$ be a homomorphism with $\mathrm{Im}\, f \ll N/X$. By Lemma 1.3, we can assume that $X = \ker \pi \ll N$. Let $p_i : N = N_1 \oplus N_2 \to N_i$ be the projection ($i = 1, 2$), let $\alpha : N/X \to N/(p_1(X) \oplus p_2(X))$ be the canonical epimorphism, $\beta : N/(p_1(X) \oplus p_2(X)) \to N_1/p_1(X) \oplus N_2/p_2(X)$ be the canonical isomorphism and put $\nu = \beta \circ \alpha$. Let $q_i : N_1/p_1(X) \oplus N_2/p_2(X) \to N_i/p_i(X)$ be the projection and let $\pi_i : N_i \to N_i/p_i(X)$ be the canonical epimorphism ($i = 1, 2$).

Since N_i is $\oplus$-supplemented, there exists a direct summand N_i^* of N_i such that $N_i = p_i(X) + N_i^*$ and $p_i(X) \cap N_i^* \ll N_i^*$. So we see

$$\ker(\pi_i|_{N_i^*}) \ll N_i^* \overset{\pi_i|_{N_i^*}}{\to} N_i/p_i(X) \to 0 \quad \cdots \ (i).$$

As $q_i \nu f(M) \ll N_i/p_i(X) \ \cdots \ (ii)$, there exists a homomorphism $h_i : M \to N_i^*$ such that $q_i \circ \nu \circ f = (\pi_i|_{N_i^*}) \circ h_i$ and so $h_i(M) \subseteq (\pi_i|_{N_i^*})^{-1}(q_i \nu f(M))$. On the other hand, (i) and (ii) imply $(\pi_i|_{N_i^*})^{-1}(q_i \nu f(M)) \ll N_i^*$ by Lemma 1.2 (3). Hence $h_i(M) \ll N_i$.

Put $\varphi = \pi(h_1 + h_2) - f$ and then $\mathrm{Im}\, \varphi \ll N/X \ \cdots \ (iii)$. Let $m \in M$ and express $\nu f(m)$ in $N_1/p_1(X) \oplus N_2/p_2(X)$ as $\nu f(m) = \overline{n_1} + \overline{n_2}$ ($\overline{n_1} \in N_1/p_1(X)$, $\overline{n_2} \in N_2/p_2(X)$). Then $\overline{n_i} = q_i \nu f(m) = \pi_i h_i(m)$. As $\nu\pi|_{N_i} = \pi_i$, $\nu\varphi(m) = \nu\pi(h_1 + h_2)(m) - \nu f(m) = \nu\pi h_1(m) + \nu\pi h_2(m) - (\overline{n_1} + \overline{n_2}) = \pi_1 h_1(m) + \pi_2 h_2(m) - \pi_1 h_1(m) - \pi_2 h_2(m) = 0$. Thus $\varphi(M) \subseteq \ker \nu = (p_1(X) \oplus p_2(X))/X = (p_1(X) + X)/X \subseteq (N_1 + X)/X = \pi(N_1) \ \cdots \ (iv)$. By Lemma 1.2 (1), $\pi(N_1)$ is co-closed in N/X and so (iii) and (iv) imply $\mathrm{Im}\, \varphi \ll \pi(N_1)$. Since M is im-small N_1-projective, there exists a homomorphism $h^* : M \to N_1$ such that $(\pi|_{N_1}) \circ h^* = \varphi$. Put $\psi = h_1 + h_2 - h^*$. Then, for any $m \in M$, $\pi\psi(m) = \pi h_1(m) + \pi h_2(m) - \pi h^*(m) = \pi h_1(m) + \pi h_2(m) - (\pi h_1(m) + \pi h_2(m) - f(m)) = f(m)$. Therefore M is im-small N-projective. $\square$

In [6], we announced that Proposition 1.4 (3) holds for any module N_i without the assumption "$\oplus$-supplemented". However, we must correct the result in the present form.

A module M is said to be *generalized N-projective* (or *N-dual ojective*) if, for any epimorphism $g : N \longrightarrow X$ and any homomorphism $f : M \longrightarrow X$, there exist decompositions $M = M_1 \oplus M_2$, $N = N_1 \oplus N_2$, a homomorphism

$h_1 : M_1 \longrightarrow N_1$ and an epimorphism $h_2 : N_2 \longrightarrow M_2$ such that $g \circ h_1 = f|_{M_1}$ and $f \circ h_2 = g|_{N_2}$. Note that any N-projective module is generalized N-projective.

Proposition 1.5. *(cf.* [5], [7]*) Let M and N be modules. Then*

(1) If M is generalized N-projective, then M is generalized N^*-projective for any $N^* <_\oplus N$.

(2) If M is generalized N-projective with the finite internal exchange property, then M^* is generalized N-projective for any $M^* <_\oplus M$.

(3) Let N be a lifting module. If M is generalized N-projective, then M is im-small N-projective.

Lemma 1.6. *Let M be lifting and let Y be amply supplemented. Then for any homomorphism $f : M \to Y$, there exists a decomposition $M = M_1 \oplus M_2$ such that $f(M_1)$ is co-closed in Y and $f(M_2)$ is small in Y.*

Proof. Put $X = f(M)$. Since Y is amply supplemented, there exist a co-closure X' of X in Y and supplement T of X in Y. Then $X = X' + (X \cap T)$. Since X is amply supplemented, there exists a co-closure S of $X \cap T$ in X. As M is lifting, there exists a decomposition $M = M_1 \oplus M_2$ with $M_2 \subseteq_c f^{-1}(S)$ in M. So $f(M_2) \subseteq_c f(f^{-1}(S)) = S$ in $f(M) = X$. As S is co-closed in X, $f(M_2) = S$. Thus

$$X = f(M) = f(M_1) + f(M_2) = f(M_1) + S.$$

As $f^{-1}(S) \cap M_1 \ll M_1$, $S \cap f(M_1) = f(f^{-1}(S) \cap M_1) \ll f(M) = X \quad \cdots (*)$.

Now we show that $f(M_1)$ is co-closed in Y. Let $A \subseteq_c f(M_1)$ in Y. As $S \subseteq X \cap T \subseteq T$, $Y = X + T = (f(M_1) + S) + T = f(M_1) + T = A + T$ and hence $X = f(M) = A + (f(M) \cap T) = A + S$. By $(*)$ and Lemma 1.1(1), $A \subseteq_c f(M_1)$ in X. Since $f(M_1)$ is co-closed in X, $A = f(M_1)$. Thus $f(M_1)$ is co-closed in Y.

On the other hand, $f(M_2) = S \subseteq X \cap T \ll T \subseteq Y$ and so $f(M_2) \ll Y$. □

The following is easily shown:

Lemma 1.7. *If $M = A \oplus B \oplus C = K \oplus C$ then $K = \langle A \to C \rangle \oplus \langle B \to C \rangle$.*

2. Generalized Epi-projective Modules

Now we define a new concept "(strongly) generalized epi-projectivity" as follows:

Definition A module M is said to be *(strongly) generalized epi-N-projective* if, for any epimorphism $g : N \longrightarrow X$ and any epimorphism $f : M \longrightarrow X$, there exist decompositions $M = M_1 \oplus M_2$, $N = N_1 \oplus N_2$, a homomorphism (an epimorphism) $h_1 : M_1 \longrightarrow N_1$ and an epimorphism $h_2 : N_2 \longrightarrow M_2$ such that $g \circ h_1 = f|_{M_1}$ and $f \circ h_2 = g|_{M_2}$.

Clearly we see the following:

(1) K is strongly generalized epi-L-projective $\Leftrightarrow$ L is strongly generalized epi-K-projective.

(2) If M is strongly generalized epi-N-projective $\Rightarrow$ M is generalized epi-N-projective.

Proposition 2.1. *Let M and N be modules. If M is epi-N-projective and N is lifting, then M is strongly generalized epi-N-projective.*

Proof. Let $f : M \to X$ and $g : N \to X$ be epimorphisms. Since N is lifting, there exists a decomposition $N = N_1 \oplus N_2$ such that N_2 is a co-essential submodule of $\ker g$ in N. As M is epi-N_1-projective, there exists a homomorphism $h : M \to N_1$ with $g \circ h = f$. Since $\ker(g|_{N_1})$ is small in N_1, h is an epimorphism. Now define an epimorphism $h'(= 0) : N_2 \to 0(M = M \oplus 0)$. Hence we see M is strongly generalized epi-N-projective. □

Proposition 2.2. *Let M be a module with the finite internal exchange property and let M^* be a direct summand of M. If M is (strongly) generalized epi-N-projective, then M^* is (strongly) generalized epi-N-projective.*

Proof. By the same argument as the proof of [5, Proposition 2.2]. □

Corollary 2.3. *Let N be a module with the finite internal exchange property and let N^* be a direct summand of N. If M is strongly generalized epi-N-projective, then M is strongly generalized epi-N^*-projective.*

Proposition 2.4. *Let M be lifting with the finite internal exchange property and let N be quasi-discrete. If M is generalized epi-N-projective, then M is generalized epi-N^*-projective for any $N^* <_\oplus N$.*

Proof. Let $N = N^* \oplus N^{**}$ and let $f : M \to X$ and $g^* : N^* \to X$ be epimorphisms. By Proposition 2.2, we may assume $\ker f \ll M$. Define $g : N = N^* \oplus N^{**} \to X$ by $g(n^* + n^{**}) = g^*(n^*)$, where $n^* \in N^*$ and $n^{**} \in N^{**}$. As N^* is lifting, there exists a decomposition $N^* = \overline{N^*} \oplus \overline{\overline{N^*}}$ such that $\overline{\overline{N^*}} \subseteq_c \ker g^*$ in N^*. Then $\overline{\overline{N^*}} \oplus N^{**} \subseteq_c \ker g$ in N. Since M is generalized epi-N-projective, there exist decompositions $M = M_1 \oplus M_2$, $N = N_1 \oplus N_2$, a homomorphism $\varphi_1 : M_1 \to N_2$ and an epimorphism $\varphi_2 : N_1 \to M_2$ such that $g \circ \varphi_1 = f|_{M_1}$ and $f \circ \varphi_2 = g|_{N_1}$.

By Lemma 1.6, there exists a decomposition $M_1 = M_1' \oplus M_1''$ such that $\varphi_1(M_1')$ is co-closed in N_2 and $\varphi_1(M_1'')$ is small in N_2. So we see $f(M_1'') = g\varphi_1(M_1'') \ll g(N_2) \subseteq X$. By Lemma 1.2(1), $f(M_1'')$ is co-closed in X and so $f(M_1'') = 0$. Then $\ker f \ll M$ imply $M_1'' = 0$ and hence $\varphi_1(M_1) = \varphi_1(M_1')$ is co-closed in N_2. Thus there exists a decomposition $N_2 = \varphi_1(M_1) \oplus N_2'$. Since N_1 is lifting, there exists a decomposition $N_1 = N_1' \oplus N_1''$ with $N_1'' \subseteq_c \ker \varphi_2$ in N_1. By $N_1'' \subseteq \ker \varphi_2 \subseteq \ker(f \circ \varphi_2) = \ker(g|_{N_1})$ and $N_2' \subseteq \ker g + \varphi_1(M_1) + N_1'$, we see

$$N = N_1 \oplus N_2 = N_1' \oplus N_1'' \oplus \varphi_1(M_1) \oplus N_2' = (N_1' \oplus \varphi_1(M_1)) + \ker g \ \cdots \ (*).$$

As $(M_2 + \ker f) \cap M_1 \ll M_1$, $(N_1' + \ker g) \cap \varphi_1(M_1) \subseteq \varphi_1((M_2 + \ker f) \cap M_1) \ll \varphi_1(M_1)$. On the other hand, by Lemma 1.2(2), $g((\varphi_1(M_1) + \ker g) \cap N_1') \subseteq g(N_1') \cap g\varphi_1(M_1) = f(M_1) \cap f(M_2) \ll X$. Since $f(M_2)$ is co-closed in X, we see $g((\varphi(M_1) + \ker g) \cap N_1') \ll f(M_2)$ by Lemma 1.1(2). As $\ker(g|_{N_1'}) = \ker(f \circ (\varphi_2|_{N_1'}))$, by Lemma 1.2(3), $(\varphi_1(M_1) + \ker g) \cap N_1' \ll N_1'$. Since $(N_1' \oplus \varphi_1(M_1)) \cap \ker g \subseteq [(\varphi_1(M_1) + \ker g) \cap N_1'] + [(N_1' + \ker g) \cap \varphi_1(M_1)]$, we see

$$(N_1' \oplus \varphi_1(M_1)) \cap \ker g \ll N \ \cdots \ (**)$$

Since $\overline{\overline{N^*}} \oplus N^{**} \subseteq_c \ker g$ in N, by $(*)$ and $(**)$, we have

$$N = (N_1' \oplus \varphi_1(M_1)) + (\overline{\overline{N^*}} \oplus N^{**}) \text{ and } (N_1' \oplus \varphi_1(M_1)) \cap (\overline{\overline{N^*}} \oplus N^{**}) \ll N.$$

As N is quasi-discrete, we see

$$N = N_1' \oplus \varphi_1(M_1) \oplus \overline{\overline{N^*}} \oplus N^{**} = \overline{N^*} \oplus \overline{\overline{N^*}} \oplus N^{**}.$$

By Lemma 1.7, $\overline{N^*} = \langle N_1' \to \overline{\overline{N^*}} \oplus N^{**} \rangle \oplus \langle \varphi_1(M_1) \to \overline{\overline{N^*}} \oplus N^{**} \rangle$. Now we put $\psi_1 = (\varphi_2|_{N_1'}) \circ \epsilon_1 : \langle N_1' \to \overline{\overline{N^*}} \oplus N^{**} \rangle \to M_2$ and $\psi_2 = \epsilon_2 \circ \varphi_1 : M_1 \to \overline{\overline{N^*}} \oplus \langle \varphi_1(M_1) \to \overline{\overline{N^*}} \oplus N^{**} \rangle$, where $\epsilon_1 : \langle N_1' \to \overline{\overline{N^*}} \oplus N^{**} \rangle \to N_1'$ and $\epsilon_2 : \varphi_1(M_1) \to \langle \varphi_1(M_1) \to \overline{\overline{N^*}} \oplus N^{**} \rangle$ are canonical isomorphisms. Then we see

$$f \circ \psi_1 = g|_{\langle N_1' \to \overline{\overline{N^*}} \oplus N^{**} \rangle} \text{ and } g \circ \psi_2 = f|_{M_1}.$$

Therefore M is generalized epi-N^*-projective. □

Proposition 2.5. *Let M be a lifting module with the finite internal exchange property, let N be a lifting module and consider the following conditions:*

(1) M is generalized N-projective,

(2) M is strongly generalized epi-N-projective,

(3) M is generalized epi-N-projective.

Then (1) $\Rightarrow$ (2) $\Rightarrow$ (3). *In particular, if N is quasi-discrete then* (2) $\iff$ (3) *holds.*

Proof. (1) $\Rightarrow$ (2) : Let $f : M \to X$ and $g : N \to X$ be epimorphisms. By Proposition 1.5, we can assume that $\ker f \ll M$ and $\ker g \ll N$. Since M is generalized N-projective, there exist decompositions $M = M_1 \oplus M_2$, $N = N_1 \oplus N_2$, a homomorphism $h_1 : M_1 \to N_1$ and an epimorphism $h_2 : N_2 \to M_2$ such that $g \circ h_1 = f|_{M_1}$ and $f \circ h_2 = g|_{N_2}$. By Lemma 1.6, there exists a decomposition $M_1 = \overline{M_1} \oplus \overline{\overline{M_1}}$ such that $h_1(\overline{M_1})$ is co-closed in N_1 and $h_1(\overline{\overline{M_1}}) \ll N_1$. So we see

$$f(\overline{\overline{M_1}}) = gh_1(\overline{\overline{M_1}}) \ll X.$$

By Lemma 1.2(1), $f(\overline{\overline{M_1}})$ is co-closed in X and so $f(\overline{\overline{M_1}}) = 0$. As $\overline{\overline{M_1}} \subseteq \ker f \ll M$, $\overline{\overline{M_1}} = 0$. Since $h_1(M_1) = h_1(\overline{M_1})$ is co-closed in N_1 and N_1 is lifting, there exists a decomposition $N_1 = h_1(M_1) \oplus T$. Since f is an epimorphism, for any $t \in T$, there exists $m_i \in M_i$ $(i = 1, 2)$ with $g(t) = f(m_1+m_2) = f(m_1)+f(m_2)$. As h_2 is an epimorphism, there exists $n_2 \in N_2$ with $h_2(n_2) = m_2$. So we see

$$g(t) = f(m_1) + f(m_2) = gh_1(m_1) + fh_2(n_2) = gh_1(m_1) + g(n_2).$$

Thus $T \subseteq \ker g + h_1(M_1) + N_2$ and so $N = h_1(M_1) \oplus T \oplus N_2 = \ker g + (h_1(M_1) \oplus N_2) = h_1(M_1) \oplus N_2$. Thus h_1 is an epimorphism. Therefore M is strongly generalized epi-N-projective.

(2) $\Rightarrow$ (3) is clear.

Now we assume that N is quasi-discrete.

(3) $\Rightarrow$ (2) : Let $f : M \to X$ and $g : N \to X$ be epimorphisms. By Propositions 2.2 and 2.4, we can assume that $\ker f \ll M$ and $\ker g \ll N$. As M is generalized epi-N-projective, there exist decompositions $M = M_1 \oplus M_2$, $N = N_1 \oplus N_2$, a homomorphism $h_1 : M_1 \to N_1$ and an epimorphism $h_2 : N_2 \to M_2$ such that $g \circ h_1 = f|_{M_1}$ and $f \circ h_2 = g|_{N_2}$. By Lemma 1.6, there exists decomposition $M_1 = M_1' \oplus M_1''$ such that $h_1(M_1')$ is co-closed in N_1 and $h_1(M_1'')$ is small in N_1. By the same argument as the proof of (1) $\Rightarrow$ (2), we get $M_1 = M_1'$ and $N = h_1(M_1) \oplus N_2$. Thus h_1 is an epimorphism. □

Proposition 2.6. *Let M and N be lifting modules with the finite internal exchange property. Then M is strongly generalized epi-N-projective if and only if M is generalized epi-N^*-projective for any direct summand N^* of N.*

Proof. By the same argument as in the proof of Proposition 2.5. □

Proposition 2.7. *Let M and N be lifting modules with the finite internal exchange property. Then M is generalized N-projective if and only if M is strongly generalized epi-N-projective and im-small N-projective.*

Proof. "Only if" part is clear by Proposition 2.5 and Proposition 1.5(3).

"If" part: Let $g : N \to X$ be an epimorphism and let $f : M \to X$ be a homomorphism. By Proposition 2.2 and Corollary 2.3, we can assume that $\ker f \ll M$ and $\ker g \ll N$. By Lemma 1.6, there exists a decomposition $M = M_1 \oplus M_2$ such that $f(M_1)$ is co-closed in X and $f(M_2)$ is small in X. Since N is lifting and $f(M_1)$ is co-closed in X, there exists a decomposition

$N = N_1 \oplus N_2$ with $g(N_1) = f(M_1)$. Since M_1 is strongly generalized epi-N_1-projective, there exist decompositions $M_1 = M_1' \oplus M_1''$, $N_1 = N_1' \oplus N_1''$ and epimorphisms $\varphi_1 : M_1' \to N_1''$, $\varphi_2 : N_1' \to M_1''$ such that $g \circ \varphi_1 = f|_{M_1'}$ and $f \circ \varphi_2 = g|_{N_1'}$. On the other hand, as M_2 is im-small N-projective, there exists a homomorphism $\rho : M_2 \to N$ with $g \circ \rho = f|_{M_2}$. Let $p_{N_1'} : N = N_1' \oplus N_1'' \oplus N_2 \to N_1'$ be the projection and put $\alpha = \varphi_2 \circ p_{N_1'} \circ \rho$, $\rho^* = (1 - p_{N_1'}) \circ \rho \circ \varepsilon$, where $\varepsilon : \langle M_2 \xrightarrow{\alpha} M_1'' \rangle \to M_2$ is the canonical isomorphism. For any $m_2 - \alpha(m_2) \in \langle M_2 \xrightarrow{\alpha} M_1'' \rangle$, $\rho(m_2)$ is expressed in $N = N_1' \oplus N_1'' \oplus N_2$ as $\rho(m_2) = n_1' + n_1'' + n_2$. Then $f(m_2 - \alpha(m_2)) = g\rho(m_2) - f\varphi_2 p_{N_1'}\rho(m_2) = g(n_1' + n_1'' + n_2) - f\varphi_2(n_1') = g(n_1' + n_1'' + n_2) - g(n_1') = g(n_1'' + n_2) = g(1 - p_{N_1'})\rho(m_2) = g\rho^*(m_2 - \alpha(m_2))$.

Put $\varphi = \varphi_1 + \rho^*$ and $\psi = \varphi_2$. Then we see

$$g \circ \varphi = f|_{M_1' \oplus \langle M_2 \xrightarrow{\alpha} M_1'' \rangle} \text{ and } f \circ \psi = g|_{N_1'}$$

Thus M is generalized N-projective. □

Lemma 2.8. *(cf.* [5, Theorem 3.7]*) Let $M_1, \cdots, M_n$ be lifting modules with the finite internal exchange property and put $M = M_1 \oplus \cdots \oplus M_n$. Then the following are equivalent:*

(1) M is lifting with the finite internal exchange property,

(2) M is lifting and the decomposition $M = M_1 \oplus \cdots \oplus M_n$ is exchangeable,

(3) M_i and $\oplus_{j \neq i} M_j$ are mutually relative generalized projective.

Now, we are in a position to obtain the following results which are generalizations of [5, Theorem 3.7].

Theorem 2.9. *Let M_1 and M_2 be lifting modules with the finite internal exchange property and put $M = M_1 \oplus M_2$. Then the following are equivalent:*

(1) M is lifting with the finite internal exchange property,

(2) M is lifting and the decomposition $M = M_1 \oplus M_2$ is exchangeable,

(3) M_1 is generalized M_2-projective and M_2 is im-small M_1-projective,

(4) M_2 is generalized M_1-projective and M_1 is im-small M_2-projective,

(5) M_i is strongly generalized epi-M_j-projective and im-small M_j-projective $(i \neq j)$.

Proof. By Proposition 2.7 and Lemma 2.8. □

Theorem 2.10. *Let $M_1, \cdots, M_n$ be lifting modules with the finite internal exchange property and put $M = M_1 \oplus \cdots \oplus M_n$. Then the following are equivalent:*

(1) M is lifting with the finite internal exchange property,

(2) M is lifting and the decomposition $M = M_1 \oplus \cdots \oplus M_n$ is exchangeable,

(3) M_i is generalized $\oplus_{j\neq i}M_j$-projective ($\oplus_{j\neq i}M_j$ is generalized M_i-projective) for any $i \in \{1, \cdots, n\}$,

(4) M_i is strongly generalized epi-$\oplus_{j\neq i}M_j$-projective ($\oplus_{j\neq i}M_j$ is strongly generalized epi-M_i-projective) for any $i \in \{1, \cdots, n\}$ and M_k is im-small M_l-projective for any $k \neq l \in \{1, \cdots, n\}$,

(5) M_i is strongly generalized epi-$\oplus_{j\neq i}M_j$-projective ($\oplus_{j\neq i}M_j$ is generalized strongly epi-M_i-projective) for any $i \in \{1, \cdots, n\}$ and $M_k \oplus M_l$ is lifting with the finite internal exchange property (or this decomposition is exchangeable) for any $k \neq l \in \{1, \cdots, n\}$.

Proof. By induction, Propositions 1.4 and 1.5 and Theorem 2.9. □

Corollary 2.11. *Let A be a semisimple module and let B be a lifting module with the finite internal exchange property. If A is im-small B-projective then $M = A \oplus B$ is lifting with the finite internal exchange property.*

Proposition 2.12. *(cf.* [6]*) Let N be a quasi-discrete module, let $M = M_1 \oplus \cdots \oplus M_n$ be lifting with the finite internal exchange property. If M_i is strongly generalized epi-N-projective, then M is strongly generalized epi-N-projective.*

Proof. It is enough to prove the case of $M = M_1 \oplus M_2$. Assume that $f : M \to X$ and $g : N \to X$ are epimorphisms. By Proposition 2.2 and Corollary 2.3, we can assume that $\ker f \ll M$ and $\ker g \ll N$. By Lemma 1.2(1), $f(M_1)$ and $f(M_2)$ are co-closed in X and $f(M_1) \cap f(M_2) \ll X$. Since N is lifting, there exists a decomposition $N = N_i \oplus N_i^*$ such that $N_i \subseteq_c g^{-1}f(M_i)$ in N $(i = 1, 2)$. By $g(N_i) \subseteq_c f(M_i)$ in X, $g(N_i) = f(M_i)$ and so $g(N) = X = f(M) = f(M_1)+f(M_2) = g(N_1)+g(N_2)$. As $\ker g \ll N$, $N = N_1 + N_2$. By Lemma 1.2(3), $g^{-1}(f(M_1) \cap f(M_2)) \ll N$. So we get $N_1 \cap N_2 \subseteq g^{-1}(f(M_1)) \cap g^{-1}(f(M_2)) = g^{-1}(f(M_1) \cap f(M_2)) \ll N$. Since N is quasi-discrete, $N = N_1 \oplus N_2$. By Corollary 2.3, M_i is strongly generalized epi-N_i-projective $(i = 1, 2)$. Hence there exist decompositions $M_i = M_i' \oplus M_i''$, $N_i = N_i' \oplus N_i''$ and epimorphisms $\alpha_i : M_i' \to N_i'$, $\beta_i : N_i'' \to M_i''$ such that $f \circ \beta_i = g|_{N_i''}$ and $g \circ \alpha_i = f|_{M_i'}$. Now define the epimorphisms $\varphi : M_1' \oplus M_2' \to N_1' \oplus N_2'$ and $\psi : N_1'' \oplus N_2'' \to M_1'' \oplus M_2''$ by $\varphi(m_1'+m_2') = \alpha_1(m_1')+\alpha_2(m_2')$, $\psi(n_1''+n_2'') = \beta_1(n_1'')+\beta_2(n_2'')$. Then for any $m_1'+m_2' \in M_1'\oplus M_2'$, $f(m_1'+m_2') = f(m_1')+f(m_2') = g\circ\alpha_1(m_1')+g\circ\alpha_2(m_2') = g(\alpha_1(m_1')+\alpha_2(m_2')) = g\circ\varphi(m_1'+m_2')$. Similarly, we see $g|_{N_1''\oplus N_2''} = f\circ\psi$. □

By the proposition above, we obtain the following:

Corollary 2.13. *Let M be a quasi-discrete module and let $N = N_1\oplus\cdots\oplus N_t$ be lifting with the finite internal exchange property. If M is strongly generalized epi-N_i-projective, then M is strongly generalized epi-N-projective.*

Corollary 2.14. *Let N be a quasi-discrete module and let $M = M_1 \oplus \cdots \oplus M_n$ be lifting with the finite internal exchange property. If M_i is generalized epi-N-projective, then $M = M_1 \oplus \cdots \oplus M_n$ is generalized epi-N-projective.*

Proof. By Propositions 2.5 and 2.12. □

Corollary 2.15. *Let M and $N = N_1 \oplus \cdots \oplus N_t$ be quasi-discrete. If M is generalized epi-N_i-projective, then M is strongly generalized epi-N-projective.*

Proof. By Proposition 2.5 and Corollary 2.13. □

Theorem 2.16. *Let $M_1, \cdots, M_n$ be quasi-discrete and put $M = M_1 \oplus \cdots \oplus M_n$. Then the following are equivalent:*

(1) M is lifting with the (finite) internal exchange property,

(2) M is lifting and the decomposition $M = M_1 \oplus \cdots \oplus M_n$ is exchangeable,

(3) M_i is generalized M_j-projective for any $i \neq j \in \{1, \cdots, n\}$,

(4) $M_i \oplus M_j$ is lifting with the finite internal exchange property for any $i \neq j \in \{1, \cdots, n\}$,

(5) M_i is strongly generalized epi-M_j-projective and im-small M_j-projective for any $i \neq j \in \{1, \cdots, n\}$,

(6) M_i is generalized epi-M_j-projective and im-small M_j-projective for any $i \neq j \in \{1, \cdots, n\}$.

Proof. (1)⇔(2)⇒(3)⇔(4) follows by Lemma 2.8 and Theorem 2.10.
(3)⇔(5)⇔(6) : By Propositions 2.5 and 2.7.
(5)⇒(1) : Let M_i be strongly generalized epi-M_j-projective and im-small M_j-projective ($i \neq j$). Then $\oplus_{i \neq j} M_i$ is im-small M_j-projective by Proposition 1.4. By Propositions 2.7 and 2.12 and Theorem 2.10, $M = M_1 \oplus \cdots \oplus M_n$ is lifting with the (finite) internal exchange property. □

Corollary 2.17. *Let $H_1, \cdots, H_n$ be hollow modules and put $M = H_1 \oplus \cdots \oplus H_n$. Then the following are equivalent:*

(1) M is lifting with the (finite) internal exchange property,

(2) M is lifting and the decomposition $M = H_1 \oplus \cdots \oplus H_n$ is exchangeable,

(3) H_i is generalized H_j-projective for any $i \neq j \in \{1, \cdots, n\}$,

(4) $H_i \oplus H_j$ is lifting with the finite internal exchange property for any $i \neq j \in \{1, \cdots, n\}$,

(5) H_i is strongly generalized epi-H_j-projective and im-small H_j-projective for any $i \neq j \in \{1, \cdots, n\}$,

(6) H_i is generalized epi-H_j-projective and im-small H_j-projective for any $i \neq j \in \{1, \cdots, n\}$.

Finally we raise the following question: Does there exist an example of a lifting module which does not satisfy the finite internal exchange property?

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(*Received March 15, 2009*)