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TRIANGULAR MATRIX REPRESENTATIONS OF SKEW MONOID RINGS

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Abstract. Let R be a ring and S a u.p.-monoid. Assume that there is a monoid homomorphism $\alpha : S \longrightarrow Aut(R)$. Suppose that α is weakly rigid and $l_R(Ra)$ is pure as a left ideal of R for every element $a \in R$. Then the skew monoid ring $R{*}S$ induced by α has the same triangulating dimension as R. Furthermore, if R is a PWP ring, then so is $R * S$.

1. Introduction.

All rings considered here are associative with identity and R denotes such a ring. Recall from [1, 2] an idempotent $e \in R$ is *left* (resp. *right*) *semicentral* in R if $ere = re$ (resp. $ere = er$), for all $r \in R$. Equivalently, $e^2 = e \in R$ is left (resp. right) semicentral if eR (resp. Re) is an ideal of R. We use $\mathcal{S}_l(R)$ and $\mathcal{S}_r(R)$ for the sets of all left and all right semicentral idempotents of R, respectively. From [3], an idempotent e of R is called *semicentral reduced* if $\mathcal{S}_l(eRe) = \{0, e\}$. A ring R is called *semicentral reduced* [3, 4] if 1 is semicentral reduced. From [3] a ring R has a *generalized triangular matrix representation* if there exists a ring isomorphism

$$\theta : R \longrightarrow \begin{pmatrix} R_1 & R_{12} & \dots & R_{1n} \\ 0 & R_2 & \dots & R_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & R_n \end{pmatrix},$$

where each diagonal ring, R_i, is a ring with unity, R_{ij} is a left R_i- right R_j-bimodule for $i < j$, and the matrices obey the usual rules for matrix addition and multiplication. If each R_i is semicentral reduced, then R has a *complete generalized triangular matrix representation* with triangulating dimension n ([3, 5]).

Recall from [1, 3, 5] that a *piecewise prime ring* (simply, PWP ring) is a quasi-Baer ring with finite triangulating dimension. In [3, Corollary 4.13] it was shown that the class of PWP rings properly includes all piecewise domains which were introduced in [8] (hence all right hereditary rings which

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are semiprimary or right Noetherian). Every PWP ring has a complete generalized triangular matrix representation with prime diagonal rings, R_i, (see [3, Theorem 4.4]). It was observed in [8, p.554] that n-by-n matrix rings and polynomial rings over piecewise domains are again piecewise domains. In [5], G. F. Birkenmeier and J. K. Park showed that for a PWP ring R the following ring extensions are PWP rings: $R[G]$, the monoid ring of a u.p.-monoid G; $R[X]$ and $R[[X]]$, where X is a nonempty set of not necessarily commuting indeterminates; $R[x, x^{-1}]$ and $R[[x, x^{-1}]]$, the Laurent polynomial ring and Laurent series ring, respectively; $R[x; \alpha]$ and $R[[x; \alpha]]$, the skew polynomial and skew power series ring, respectively, where α is a particular type of ring automorphism of R; $T_n(R)$ and $Mat_n(R)$ the n-by-n upper triangular and full matrix rings over R, respectively. Also open problems were raised in [5] to enlarge the class of ring extensions of PWP rings which are also PWP rings and to enlarge the class of ring extensions of rings with finite triangulating dimension which also have finite triangulating dimension. In this paper we will show that for a left p.q.Baer ring R and a u.p.-monoid S the skew monoid ring $R*S$ induced by a weakly rigid monoid homomorphism $\alpha : S \longrightarrow Aut(R)$ has the same triangulating dimension as R. Furthermore, if R is a PWP ring, then so is $R*S$, and hence $R*S$ has a complete generalized triangular matrix representation with prime diagonal rings.

2. Quasi-Baerness.

Recall that a monoid M is called a *u.p.-monoid* (unique product monoid) if for any two nonempty finite subsets $A, B \subseteq M$ there exists an element $g \in M$ uniquely presented in the form ab where $a \in A$ and $b \in B$. The class of u.p.-monoids is quite large and important (see [5], [18] and [19]). For example, this class includes the right or left ordered monoids, submonoids of a free group, and torsion-free nilpotent groups. Every u.p.-monoid M has no non-unity element of finite order.

Let R be a ring and S a u.p.momoid. Assume that there is a monoid homomorphism $\alpha : S \longrightarrow Aut(R)$. For any $s \in S$, we denote the image of s under α by α_s. Then we can form a *skew monoid ring* $R * S$ (induced by the monoid homomorphism α) by taking its elements to be finite formal combinations $\sum_{s\in S} a_s s$, with multiplication induced by:

$$(a_s s)(b_t t) = a_s \alpha_s(b_t)(st).$$

In the following, μ will always stand for the identity of a monoid S.

A submodule N of a left R-module M is called a *pure submodule* if $L \otimes_R N \longrightarrow L \otimes_R M$ is a monomorphism for every right R-module L. An ideal I of R is said to be *right s-unital* if, for each $a \in I$ there exists an $x \in I$ such that $ax = a$. By [21, Proposition 11.3.13], an ideal I is pure as a left ideal of R if and only if R/I is flat as a left R-module if and only if I is right s-unital.

Lemma 2.1. *Let R be a ring such that $l_R(Ra)$ is pure as a left ideal of R for every element $a \in R$ and S a u.p.-monoid. Suppose that $\phi = a_1s_1 + a_2s_2 + \cdots + a_ns_n$, $\psi = b_1t_1 + b_2t_2 + \cdots + b_mt_m \in R * S$ are such that $\phi R\psi = 0$. Then $a_i\alpha_{s_i}(rb_j) = 0$ for any $r \in R$, $i = 1, 2, \cdots, n$, $j = 1, 2, \cdots, m$.*

Proof. Suppose that $c_1, c_2, \cdots, c_n \in R$ are such that $a_i = \alpha_{s_i}(c_i)$ for $i = 1, 2, \cdots, n$. We proceed by induction on m.

If $m = 1$, then $\psi = b_1t_1$. Thus $0 = (a_1s_1 + a_2s_2 + \cdots + a_ns_n)r(b_1t_1) = a_1\alpha_{s_1}(rb_1)s_1t_1 + a_2\alpha_{s_2}(rb_1)s_2t_1 + \cdots + a_n\alpha_{s_n}(rb_1)s_nt_1$ for every $r \in R$. By [5, Lemma 1.1], S is a cancellative monoid. Thus $s_it_1 \neq s_jt_1$ for $s_i \neq s_j$. Hence $a_i\alpha_{s_i}(rb_1) = 0$, $i = 1, 2, \cdots, n$.

Now suppose that $m \geq 2$. Since S is a u.p.-monoid, there exist p, q with $1 \leq p \leq n$ and $1 \leq q \leq m$ such that s_pt_q is uniquely presented by considering two subsets $\{s_1, s_2, \cdots, s_n\}$ and $\{t_1, t_2, \cdots, t_m\}$ of S. Thus from $\phi R\psi = 0$ it follows that $a_p\alpha_{s_p}(rb_q)s_pt_q = 0$ and so $a_p\alpha_{s_p}(rb_q) = 0$. Thus $\alpha_{s_p}(c_prb_q) = 0$, which implies that $c_prb_q = 0$ for every $r \in R$ since α_{s_p} is an automorphism. Hence $c_p \in l_R(Rb_q)$. Since $l_R(Rb_q)$ is pure as a left ideal of R, there exists an element $e_q \in l_R(Rb_q)$ such that $c_p = c_pe_q$. Thus for every $r \in R$, we have

$$\begin{aligned} 0 &= \phi e_qr\psi = (a_1s_1 + a_2s_2 + \cdots + a_ns_n)e_qr \\ &\quad \cdot(b_1t_1 + b_2t_2 + \cdots + b_{q-1}t_{q-1} + b_{q+1}t_{q+1} + \cdots + b_mt_m) \\ &\quad +(a_1s_1 + a_2s_2 + \cdots + a_ns_n)((e_qrb_q)t_q) \\ &= (a_1s_1 + a_2s_2 + \cdots + a_ns_n)e_qr \\ &\quad \cdot(b_1t_1 + b_2t_2 + \cdots + b_{q-1}t_{q-1} + b_{q+1}t_{q+1} + \cdots + b_mt_m) \\ &= (a_1\alpha_{s_1}(e_q)s_1 + a_2\alpha_{s_2}(e_q)s_2 + \cdots + a_n\alpha_{s_n}(e_q)s_n)r \\ &\quad \cdot(b_1t_1 + b_2t_2 + \cdots + b_{q-1}t_{q-1} + b_{q+1}t_{q+1} + \cdots + b_mt_m) \end{aligned}$$

By induction, it follows that $a_i\alpha_{s_i}(e_q)\alpha_{s_i}(rb_j) = 0$ for any $r \in R$, $i = 1, 2, \cdots, n$, $j = 1, 2, \cdots, q-1, q+1, \cdots, m$. Thus $\alpha_{s_i}(c_ie_qrb_j) = 0$, which implies that $c_ie_qrb_j = 0$. Hence $c_ie_q \in l_R(Rb_j)$ for any $i = 1, 2, \cdots, n$ and $j = 1, 2, \cdots, q-1, q+1, \cdots, m$. Therefore

$$c_p = c_pe_q \in \cap_{j=1}^m l_R(Rb_j).$$

Now $a_p\alpha_{s_p}(Rb_j) = \alpha_{s_p}(c_pRb_j) = 0$ for any $j = 1, 2, \cdots, m$. Thus from $\phi R\psi = 0$ it follows that

$$\begin{aligned} 0 &= (a_1s_1 + a_2s_2 + \cdots + a_{p-1}s_{p-1} + a_{p+1}s_{p+1} + \cdots + a_ns_n) \\ &\quad \cdot r(b_1t_1 + b_2t_2 + \cdots + b_mt_m). \end{aligned}$$

By using the previous method, there exists $k \in \{1, 2, \cdots, p-1, p+1, \cdots, n\}$ such that $c_k \in \cap_{j=1}^m l_R(Rb_j)$. Thus $a_k\alpha_{s_k}(Rb_j) = \alpha_{s_k}(c_kRb_j) = 0$ for any

$j = 1, 2, \cdots, m$. Hence $(a_1s_1 + a_2s_2 + \cdots + a_{p-1}s_{p-1} + a_{p+1}s_{p+1} + \cdots + a_{k-1}s_{k-1} + a_{k+1}s_{k+1} + \cdots + a_ns_n)r(b_1t_1 + b_2t_2 + \cdots + b_mt_m) = 0$. Continuing this procedure yields $c_1, c_2, \cdots, c_n \in \cap_{j=1}^m l_R(Rb_j)$, which implies that $a_i\alpha_{s_i}(Rb_j) = 0$ for any $i = 1, 2, \cdots, n$ and $j = 1, 2, \cdots, m$.

By induction, the result follows. $\square$

Note that in the proof of [11, Theorem 3.9], it is shown that if $l_R(Ra)$ is pure as a left ideal of R for every element $a \in R$, then $(a_0 + a_1x + \cdots + a_mx^m)R(b_0 + b_1x + \cdots + b_nx^n) = 0$ in $R[x]$ with $a_i, b_j \in R$ implies that $a_iRb_j = 0$ for all i, j. Clearly this result follows directly from Lemma 2.1.

Recall that R is *Baer* if the right annihilator of every nonempty subset of R is generated, as a right ideal, by an idempotent. Clark in [7] defines a ring to be *quasi-Baer* if the left annihilator of every ideal is generated, as a left ideal, by an idempotent. Moreover, he shows the left-right symmetry of this condition by proving that R is quasi-Baer if and only if the right annihilator of every right ideal is generated, as a right ideal, by an idempotent. He then uses the quasi-Baer concept to characterize when a finite dimensional algebra with unity over an algebraically closed field is isomorphic to a twisted matrix units semigroup algebra. Every prime ring is quasi-Baer. In [20] Pollingher and Zaks show that the class of quasi-Baer rings is closed under $n \times n$ matrix rings and under $n \times n$ upper (or lower) triangular matrix rings. It is proved in [13, Theorem 21] that if σ is an automorphism of a ring R with $\sigma(e) = e$ for any $e^2 = e \in R$ and R is an σ-skew Armendariz ring, then R is a Baer ring if and only if $R[x; \sigma]$ is a Baer ring. G.F.Birkenmeier, J.Y.Kim and J.K.Park show in [2, Theorem 1.8] that R is quasi-Baer if and only if $R[X]$ is quasi-Baer if and only if $R[[X]]$ is quasi-Baer if and only if $R[x, x^{-1}]$ is quasi-Baer if and only if $R[[x, x^{-1}]]$ is quasi-Baer, where X is an arbitrary nonempty set of not necessarily commuting indeterminates. Also [2, Theorem 1.2] shows that if R is quasi-Baer, then so are $R[x; \sigma]$, $R[[x; \sigma]]$, $R[x, x^{-1}; \sigma]$ and $R[[x, x^{-1}; \sigma]]$. C.Y. Hong, N.K. Kim and T. K. Kwak show in [12, Corollaries 12 and 22] that if σ is a rigid endomorphism of R, then R is a quasi-Baer ring if and only if $R[x; \alpha, \delta]$ is a quasi-Baer ring if and only if $R[[x; \sigma]]$ is a quasi-Baer ring. If R is a ring and $(S, \leq)$ a strictly totally ordered monoid which satisfies the condition that $0 \leq s$ for every $s \in S$, then it is shown in [16] that R is a quasi-Baer ring if and only if the ring $[[R^{S,\leq}]]$ of generalized power series over R is a quasi-Baer ring. If S is an ordered monoid, then, it is proved in [10, Theorem 1] that $R[S]$ is quasi-Baer if and only if R is quasi-Baer. This result has been generalized by G.F.Birkenmeier and J.K.Park in [5, Theorem 1.2(ii)] by showing that if S is a u.p.-monoid, then $R[S]$ is quasi-Baer if and only if R is quasi-Baer. For skew monoid ring $R * S$ we have the following result.

Proposition 2.2. *Let S be a u.p.-monoid. If R is quasi-Baer, then the skew monoid ring $R*S$ induced by any monoid homomorphism α is quasi-Baer.*

Proof. Let I be an ideal of $R*S$ and let I_0 be the set of all coefficients in R of elements in I. For every $a \in I_0$, there exists $a_1s_1 + a_2s_2 + \cdots + a_ns_n \in I$ such that $a_1 = a$. Note that $\alpha_\mu = 1$. Thus, for every $r \in R$, $ra_1s_1 + ra_2s_2 + \cdots + ra_ns_n = (r\mu)(a_1s_1 + a_2s_2 + \cdots + a_ns_n) \in I$. It follows that $ra = ra_1 \in I_0$. Let J be the left ideal of R generated by I_0. Since R is quasi-Baer, there exists $e^2 = e \in R$ such that $l_R(J) = Re$. For any $\phi = a_1s_1 + a_2s_2 + \cdots + a_ns_n \in I$, $e\phi = ea_1s_1 + ea_2s_2 + \cdots + ea_ns_n = 0$. Thus $(R*S)e \subseteq l_{R*S}(I)$. In order to show that $l_{R*S}(I) \subseteq (R*S)e$, we take $\phi = a_1s_1 + a_2s_2 + \cdots + a_ns_n \in l_{R*S}(I)$. Let $a \in J$. Then there exist $w_1, w_2, \cdots, w_k \in I_0$ such that $a = w_1 + w_2 + \cdots + w_k$. For w_1, there exists $\psi = b_1t_1 + b_2t_2 + \cdots + b_mt_m \in I$ with $w_1 = b_1$. Since $\phi = a_1s_1 + a_2s_2 + \cdots + a_ns_n \in l_{R*S}(I)$, we have $(a_1s_1 + a_2s_2 + \cdots + a_ns_n)R(b_1t_1 + b_2t_2 + \cdots + b_mt_m) = 0$. Since R is quasi-Baer, $l_R(Rr)$ is pure as a left ideal of R for every element $r \in R$. By Lemma 2.1, it follows that $a_i\alpha_{s_i}(rb_j) = 0$ for any $r \in R$, $i = 1, 2, \cdots, n$, $j = 1, 2, \cdots, m$. Suppose that $c_1, c_2, \cdots, c_n \in R$ are such that $a_i = \alpha_{s_i}(c_i)$ for $i = 1, 2, \cdots, n$. Then $\alpha_{s_i}(c_irb_j) = a_i\alpha_{s_i}(rb_j) = 0$. Hence $c_irb_j = 0$, and so $c_i \in l_R(Rb_j)$. In particular, $c_i \in l_R(Rb_1)$. Thus $c_iw_1 = c_ib_1 = 0$, $i = 1, 2, \cdots, n$. Similarly, we can see $c_iw_l = 0$, $i = 1, 2, \cdots, n$, $l = 1, 2, \cdots, k$. Thus $c_ia = 0$. This means that $c_i \in l_R(J) = Re$ for any $i = 1, 2, \cdots, n$. Thus $c_i = c_ie$ and so $a_i = \alpha_{s_i}(c_i) = \alpha_{s_i}(c_ie) = a_i\alpha_{s_i}(e)$. This means that $a_1s_1 + a_2s_2 + \cdots + a_ns_n = (a_1s_1 + a_2s_2 + \cdots + a_ns_n)e \in (R*S)e$. Thus we have shown that $l_{R*S}(I) = (R*S)e$. Hence $R*S$ is quasi-Baer. ☐

Definition 2.3. *Let σ be an automorphism of a ring R. We define σ to be weakly rigid if $ab = 0$ implies $a\sigma(b) = \sigma(a)b = 0$ for any $a, b \in R$.*

A monoid homomorphism α from a monoid S into the group of automorphisms of R, $x \mapsto \alpha_x$, is called weakly rigid if $\alpha_x \in Aut(R)$ is weakly rigid for every $x \in S$.

Example 2.4. *(1). If for every $x \in S$, $\alpha_x = id$, then α is weakly rigid.*

(2). Let σ be an endomorphism of R. According to [12] and [15], σ is called a rigid endomorphism if $r\sigma(r) = 0$ implies $r = 0$ for $r \in R$. A ring R is said to be σ-rigid if there exists a rigid endomorphism σ of R. Clearly every rigid endomorphism is a monomorphism and every σ-rigid ring is reduced. Let σ be a rigid automorphism of R. It was shown in [12] that if $ab = 0$ then $a\sigma^n(b) = \sigma^n(a)b = 0$ for any positive integer n. Thus the map $\alpha : \mathbb{Z} \longrightarrow Aut(R) : \alpha(x) = \sigma^x$ is weakly rigid. Let β be a rigid automorphism of a ring R_0 and $S = \begin{pmatrix} \mathbb{Z} & \mathbb{Z} \\ 0 & \mathbb{Z} \end{pmatrix}$. Set $R_1 = R_0 \oplus S$, the direct sum of rings R_0 and S. Define an endomorphism σ of R_1 via

$\sigma(r,s) = (\beta(r), s)$. Then it is easy to see that σ is weakly rigid and, so the map $\alpha : \mathbb{Z} \longrightarrow Aut(R_1) : \alpha(x) = \sigma^x$ is weakly rigid but σ is not rigid.

(3). Let

$$R = \left\{ \begin{pmatrix} a & p \\ 0 & a \end{pmatrix} | a \in \mathbb{Z}, p \in \mathbb{Q} \right\},$$

where $\mathbb{Q}$ is the set of all rational numbers. Let $\sigma : R \longrightarrow R$ be an automorphism defined by

$$\sigma\left(\begin{pmatrix} a & p \\ 0 & a \end{pmatrix}\right) = \begin{pmatrix} a & p/2 \\ 0 & a \end{pmatrix}.$$

Since

$$\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \sigma\left(\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\right) = 0, \quad \text{but } \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \neq 0,$$

σ is not rigid. Suppose that $\begin{pmatrix} a & p \\ 0 & a \end{pmatrix}\begin{pmatrix} b & q \\ 0 & b \end{pmatrix} = 0$. Then $ab = 0$ and $aq+pb = 0$. Thus $aqa = 0$ and, so $aq/2 = 0$. This means that

$$\begin{pmatrix} a & p \\ 0 & a \end{pmatrix} \sigma\left(\begin{pmatrix} b & q \\ 0 & b \end{pmatrix}\right) = 0.$$

Similarly,

$$\sigma\left(\begin{pmatrix} a & p \\ 0 & a \end{pmatrix}\right)\begin{pmatrix} b & q \\ 0 & b \end{pmatrix} = 0.$$

Thus σ is weakly rigid and so the map $\alpha : \mathbb{Z} \longrightarrow Aut(R) : \alpha(x) = \sigma^x$ is weakly rigid.

(4). Let R be a reduced ring. Consider the ring

$$T = \left\{ \begin{pmatrix} a & b \\ 0 & a \end{pmatrix} | a, b \in R \right\}.$$

Let $\sigma : T \longrightarrow T$ be an automorphism defined by

$$\sigma\left(\begin{pmatrix} a & b \\ 0 & a \end{pmatrix}\right) = \begin{pmatrix} a & -b \\ 0 & a \end{pmatrix}.$$

By analogy with the proof of (3), we see σ is not rigid. Suppose that

$$\begin{pmatrix} a & b \\ 0 & a \end{pmatrix}\begin{pmatrix} c & d \\ 0 & c \end{pmatrix} = 0.$$

Then $ac = 0$ and $ad + bc = 0$. Since R is reduced, $ca = 0$. Thus $ada = (ad + bc)a = 0$. Hence $(ad)^2 = 0$, which implies that $ad = 0$ and, so $bc = 0$. Therefore

$$\begin{pmatrix} a & b \\ 0 & a \end{pmatrix} \sigma\left(\begin{pmatrix} c & d \\ 0 & c \end{pmatrix}\right) = 0, \quad \sigma\left(\begin{pmatrix} a & b \\ 0 & a \end{pmatrix}\right)\begin{pmatrix} c & d \\ 0 & c \end{pmatrix} = 0.$$

Thus σ is weakly rigid and so the map $\alpha : \mathbb{Z} \longrightarrow Aut(R) : \alpha(x) = \sigma^x$ is weakly rigid.

Proposition 2.5. *Let S be a u.p.-monoid and let α be weakly rigid. Then the skew monoid ring $R * S$ induced by α is quasi-Baer if and only if R is quasi-Baer.*

Proof. If R is quasi-Baer, then $R * S$ is quasi-Baer by Proposition 2.2. Suppose that $R * S$ is quasi-Baer. Let I be an ideal of R. Then there exists an idempotent f of $R * S$ such that

$$l_{R*S}((R * S)I) = (R * S)f.$$

Write $f = e_0\mu + e_1s_1 + e_2s_2 + \cdots + e_ns_n$. For any $a \in I$, from $fa = 0$ it follows that $e_0a\mu + e_1\alpha_{s_1}(a)s_1 + e_2\alpha_{s_2}(a)s_2 + \cdots + e_n\alpha_{s_n}(a)s_n = 0$. Thus $e_0a = 0$. This means that $Re_0 \subseteq l_R(I)$. Now let $b \in l_R(I)$. Then for any $\phi = b_1t_1 + b_2t_2 + \cdots + b_mt_m \in R * S$ and any $a \in I$,

$$b\phi a = bb_1\alpha_{t_1}(a)t_1 + bb_2\alpha_{t_2}(a)t_2 + \cdots + bb_m\alpha_{t_m}(a)t_m.$$

Since $bb_ia = 0$, it follows $bb_i\alpha_{t_i}(a) = 0$ by the weak rigidness of α. Thus $b\phi a = 0$. Now it is easy to see that $b \in l_{R*S}((R * S)I)$. Thus $b = bf = be_0\mu + be_1s_1 + be_2s_2 + \cdots + be_ns_n$, which implies that $b = be_0 \in Re_0$. Hence we have shown that $l_R(I) = Re_0$. This shows that R is quasi-Baer. □

Lemma 2.6. *([2, Lemma 1.9]) The following are equivalent:*
(1) R is an abelian Baer ring.
(2) R is a reduced quasi-Baer ring.

It was proved in [9, Theorem 2] and [5, Corollary 1.3(ii)] that if S is a u.p.-monoid then the monoid ring $R[S]$ is a reduced Baer ring if and only if R is a reduced Baer ring. By Lemma 2.6, for skew monoid rings we have the following result.

Corollary 2.7. *Let S be a u.p.-monoid and let α be weakly rigid. Then the skew monoid ring $R * S$ induced by α is a reduced Baer ring if and only if R is a reduced Baer ring.*

Corollary 2.8. *([5, Theorem 1.2(ii)]) Let S be a u.p.-monoid. Then $R[S]$ is quasi-Baer if and only if R is quasi-Baer.*

Note that from Corollary 2.8 it follows that if S is an ordered monoid, then $R[S]$ is quasi-Baer if and only if R is quasi-Baer([10, Theorem 1]).

Let σ be a weakly rigid automorphism of a ring R. For $S = \mathbb{Z}$, define $\alpha : S \longrightarrow Aut(R)$ as $\alpha(0) = 1$, and $\alpha(n) = \sigma^n$. Then α is weakly rigid.

Corollary 2.9. *Let σ be a weakly rigid automorphism of a ring R. Then the following conditions are equivalent:*

(1) R is quasi-Baer.
(2) $R[x;\sigma]$ is quasi-Baer.
(3) $R[x,x^{-1};\sigma]$ is quasi-Baer.

The following example shows that the converse of Proposition 2.2 is not true in general. Thus the condition "α is weakly rigid" in Proposition 2.5 is not superfluous.

Example 2.10. *([10, Example 2]) Let $S = \mathbb{N} \cup \{0\}$. Then S is a u.p.-monoid. Let F be a field, let $A = F[s,t]$ be a commutative polynomial ring, and consider the ring $R = A/(st)$. Let $\bar{s} = s + (st)$ and $\bar{t} = t + (st)$ in R. Define an automorphism σ of R by $\sigma(\bar{s}) = \bar{t}$ and $\sigma(\bar{t}) = \bar{s}$. Then, by [10, Example 2], $R * S = R[x;\sigma]$ is quasi-Baer but R is not quasi-Baer. Clearly $\bar{s}\bar{t} = 0$, but $\bar{s}\sigma(\bar{t}) = \bar{s}\bar{s} \neq 0$.*

As a generalization of quasi-Baer rings, G. F. Birkenmeier, J. Y. Kim and J. K. Park in [1] introduce the concept of principally quasi-Baer rings. A ring R is called left principally quasi-Baer (or simply left p.q.Baer) if the left annihilator of a principal left ideal of R is generated by an idempotent. Similarly, right p.q.Baer rings can be defined. A ring is called p.q.Baer if it is both right and left p.q.Baer. Observe that every biregular ring and every quasi-Baer ring is a p.q.Baer ring. For more details and examples of right p.q.Baer rings, see [1, 6].

It was proved in [6, Theorem 2.1] that a ring R is right p.q.Baer if and only if $R[x]$ is right p.q.Baer. If R is an α-rigid ring, then it was shown in [12, Corollary 15] that R is a right p.q.Baer ring if and only if $R[x;\alpha,\delta]$ is a right p.q.Baer ring. Let S be a u.p.-monoid. Then from [5, Theorem 1.2(i)] it follows that R is left p.q.Baer if and only if $R[S]$ is left p.q.Baer. From these results and from Propositions 2.2 and 2.5, it is natural to conjecture that the results of Propositions 2.2 and 2.5 remain valid if the term "quasi-Baer" is substituted for "left p.q.Baer". However we have a negative answer to this situation by the following example.

Example 2.11. *([10, Example 1]) Let $S = (\mathbb{Z}, +)$. Then S is a u.p.-monoid. Let K be a field and $A = \prod_{i\in\mathbb{Z}} A_i$ with $A_i = K$ for each i. Consider the automorphism σ of A defined by $\sigma((a_i)_{i\in\mathbb{Z}}) = (a_{i+2})_{i\in\mathbb{Z}}$. Let $R = K1 + (\oplus_{i\in\mathbb{Z}} A_i)$. Then by [10, Example 1], R is a left p.q.Baer ring but $R * S = [x,x^{-1};\sigma]$ is not left p.q.Baer.*

But we have the following affirmative result.

Proposition 2.12. *Let S be a monoid and let α be weakly rigid. If the skew monoid ring $R * S$ induced by α is left p.q.Baer then R is left p.q.Baer.*

Proof. It follows from the proof of Proposition 2.5. □

Note that in Example 2.10, $l_R(R\bar{s}) = R\bar{t}$ is not generated by any idempotent of R. Hence R is not left p.q.Baer. This shows that the condition "α is weakly rigid"in Proposition 2.12 is not superfluous.

3. Triangulating dimensions and PWP-rings.

Lemma 3.1. *Let R be a ring such that $l_R(Ra)$ is pure as a left ideal of R for every element $a \in R$ and S a u.p.-monoid. Let $e \in R * S$ and e_0 the coefficient of μ in e. If e is a left semicentral idempotent of $R * S$, then e_0 is a left semicentral idempotent of R and $e(R * S) = e_0(R * S)$.*

Proof. Let $e = e_0\mu + e_1s_1 + e_2s_2 + \cdots + e_ns_n$ be a left semicentral idempotent of $R * S$. Then $(e-1)(R * S)e = 0$ and hence $(e-1)re = 0$ for every $r \in R$. Thus

$$((e_0 - 1)\mu + e_1s_1 + e_2s_2 + \cdots + e_ns_n)R(e_0\mu + e_1s_1 + e_2s_2 + \cdots + e_ns_n) = 0.$$

By Lemma 2.1, $(e_0 - 1)Re_i = 0$ for $i = 0, 1, \cdots, n$, and $e_i\alpha_{s_i}(e_0) = 0$ for $i = 1, 2, \cdots, n$. Thus e_0 is a left semicentral idempotent of R, $e_0e = e$ and $ee_0 = e_0\mu + e_1\alpha_{s_1}(e_0)s_1 + e_2\alpha_{s_2}(e_0)s_2 + \cdots + e_n\alpha_{s_n}(e_0)s_n = e_0$. Hence $e(R * S) = e_0(R * S)$. □

Lemma 3.2. *Let α be weakly rigid. Then, for any $s \in S$ and any $b^2 = b \in R$, $\alpha_s(b) = b$.*

Proof. By analogy with the proof of [17, Lemma 3.2], we can complete the proof. □

Recall from [3, 5] that an ordered set $\{b_1, \cdots, b_n\}$ of nonzero distinct idempotents in a ring R is called a set of *left triangulating idempotents* of R if all the following hold:

(i) $1 = b_1 + \cdots + b_n$;

(ii) $b_1 \in \mathcal{S}_l(R)$; and

(iii) $b_{k+1} \in \mathcal{S}_l(a_kRa_k)$, where $a_k = 1 - (b_1 + \cdots + b_k)$, for $1 \le k \le n-1$.

Similarly we can define a set of right triangulating idempotents of R using (i), $b_1 \in \mathcal{S}_r(R)$, and $b_{k+1} \in \mathcal{S}_r(a_kRa_k)$.

Let B be a set of left triangulating idempotents of R and Γ a ring extension of R. From [5], we say Γ is *B-triangularly linked to* R if whenever $b \in B$ and $0 \neq a \in \mathcal{S}_l(b\Gamma b)$, then there exists $0 \neq a_0 \in \mathcal{S}_l(bRb)$ such that $a_0\Gamma \subseteq a\Gamma$. We say Γ is *B-triangularly compatible with* R if B is a set of left triangulating idempotents of Γ.

Lemma 3.3. *Let R be a ring such that $l_R(Ra)$ is pure as a left ideal of R for every element $a \in R$ and S a u.p.-monoid. Let α be weakly rigid. If B is a set of left triangulating idempotents of R, then $R * S$ is B-triangularly linked to R.*

Proof. Suppose that $b \in B$ and $0 \neq \phi \in \mathcal{S}_l(b(R*S)b)$. For every $a_1s_1 + a_2s_2 + \cdots + a_ns_n \in R*S$, it is easy to see that $b(a_1s_1 + a_2s_2 + \cdots + a_ns_n)b = ba_1\alpha_{s_1}(b)s_1 + ba_2\alpha_{s_2}(b)s_2 + \cdots + ba_n\alpha_{s_n}(b)s_n = ba_1bs_1 + ba_2bs_2 + \cdots + ba_nbs_n \in (bRb)*S$ by Lemma 3.2. Conversely if $g = c_1t_1 + c_2t_2 + \cdots + c_mt_m \in (bRb)*S$, then clearly $g = bc_1bt_1 + bc_2bt_2 + \cdots + bc_mbt_m = bc_1\alpha_{t_1}(b)t_1 + bc_2\alpha_{t_2}(b)t_2 + \cdots + bc_m\alpha_{t_m}(b)t_m = b(c_1t_1 + c_2t_2 + \cdots + c_mt_m)b \in b(R*S)b$. Thus $b(R*S)b = (bRb)*S$. For every $bdb \in bRb$, consider $l_{bRb}((bRb)(bdb))$. If $brb \in l_{bRb}((bRb)(bdb))$, then $(brb)(Rbdb) = 0$. Thus $brb \in l_R(Rbdb)$. Since $l_R(Rbdb)$ is pure as a left ideal of R, there exists an $x \in l_R(Rbdb)$ such that $brbx = brb$. Hence $brb = (brb)(bxb)$ and $bxb \in l_{bRb}((bRb)(bdb))$. This shows that $l_{bRb}((bRb)(bdb))$ is pure as a left ideal of bRb. For any $s \in S$, α_s induce an automorphism $\alpha_s|_{bRb}$ of $Aut(bRb)$ by Lemma 3.2. Clearly $\alpha_s|_{bRb}$ is weakly rigid for every $s \in S$. Now from Lemma 3.1, there exists $0 \neq a \in \mathcal{S}_l(bRb)$ such that $a((bRb)*S) = \phi((bRb)*S)$. Thus $a = \phi a$ and so $a(R*S) \subseteq \phi(R*S)$. □

Lemma 3.4. *Let Γ be a ring extension of R and $B = \{b_1, \cdots, b_n\}$ be a set of left triangulating idempotents of R. Assume that Γ is spanned, as a left R-module, by a set T. If $tb = btb$ for every $b \in B$ and for every $t \in T$, then Γ is B-triangularly compatible with R.*

Lemma 3.5. *Let S be a u.p.-monoid and let α be weakly rigid. If $B = \{b_1, b_2, \cdots, b_n\}$ is a set of left triangulating idempotents of R, then $R*S$ is B-triangularly compatible with R.*

Proof. Clearly $R*S$ is spanned, as a left R-module, by the set S. For every $s \in S$ and every $b \in B$, by Lemma 3.2, we have

$$bsb = b\alpha_s(b)s = bbs = bs = \alpha_s(b)s = sb.$$

Thus, by Lemma 3.4, $R*S$ is B-triangularly compatible with R. □

A set $\{b_1, \cdots, b_n\}$ of left (right) triangulating idempotents is said to be *complete* if each b_i is also semicentral reduced. Note that any complete set of primitive idempotents determines a complete set of left triangulating idempotents [3, Proposition 2.18].

Lemma 3.6. *([3, Proposition 1.3]) R has a (respectively, complete) set of left triangulating idempotents if and only if R has a (respectively, complete) generalized triangular matrix representation.*

From [3] the number of elements in a complete set of left triangulating idempotents is unique for a given ring R (which has such a set) and this is also the number of elements in any complete set of right triangulating idempotents of R. Thus it is natural to see that R has *triangulating dimension n*, written $\mathrm{Tdim}(R) = n$, if R has a complete set of left triangulating idempotents with exactly n elements. If R has no complete set of left triangulating idempotents, then we say R has infinite triangulating dimension, denoted $\mathrm{Tdim}(R) = \infty$. Note that R is semicentral reduced if and only if $\mathrm{Tdim}(R) = 1$.

Lemma 3.7. *([5, Proposition 4.3]) Let Γ be a ring extension of R. If Γ is B-triangularly linked to R and B-triangularly compatible with R for every set B of left triangulating idempotents of R, then $Tdim(R) = Tdim(\Gamma)$.*

If S is a u.p.-monoid and R is a right p.q.-Baer ring, or S is a free monoid, then it was shown in [5, Theorem 4.4] that the ring $R[S]$ has the same triangulating dimension as R. For skew monoid rings we have the following result.

Theorem 3.8. *Let R be a ring such that $l_R(Ra)$ is pure as a left ideal of R for every element $a \in R$ and S a u.p.-monoid. If α is weakly rigid, then the skew monoid ring $R * S$ induced by α has the same triangulating dimension as R.*

Proof. It follows from Lemmas 3.3, 3.5, 3.7. □

Theorem 3.9. *Let R be a PWP ring and S a u.p.-monoid. If α is weakly rigid, then the skew monoid ring $R * S$ induced by α is a PWP ring.*

Proof. It follows from Proposition 2.2 and Theorem 3.8. □

Thus if R is a quasi-Baer ring with a complete set of left triangulating idempotents $B = \{b_1, b_2, \cdots, b_n\}$, and S a u.p.-monoid, then the skew monoid ring $R * S$ induced by a weakly rigid monoid homomorphism α is a quasi-Baer ring with B determining a complete generalized triangular matrix representation for $R * S$ in which each diagonal ring, R_i, is a prime ring.

Let R be a quasi-Baer ring with a complete set of left triangulating idempotents $B = \{b_1, \cdots, b_n\}$. It was proved in [5, Theorem 4.8] that if $\Gamma = R[x, x^{-1}]$, or $\Gamma = R[x; \sigma]$, where σ is a ring automorphism such that $\sigma(bR) \subseteq bR$ for all $b \in B$, then Γ is a PWP-ring. Here we have the following results.

Corollary 3.10. *Let $\sigma \in Aut(R)$ be weakly rigid. If $l_R(Ra)$ is pure as a left ideal of R for every element $a \in R$, then the skew Laurent polynomial ring $R[x, x^{-1}; \sigma]$ has the same triangulating dimension as R. Furthermore, if R is a PWP ring, then so is $R[x, x^{-1}; \sigma]$.*

Let $S = \mathbb{Z}^n$ and $\sigma_1, \sigma_2, \cdots, \sigma_n \in Aut(R)$. Suppose that $\sigma_i\sigma_j = \sigma_j\sigma_i$ for all i, j. Define $\alpha : S \longrightarrow Aut(R)$ via $\alpha((k_1, k_2, \cdots, k_n)) = \sigma_1^{k_1}\sigma_2^{k_2}\cdots\sigma_n^{k_n}$. Then $R * S = R[x_1, x_2, \cdots, x_n, x_1^{-1}, x_2^{-1}, \cdots, x_n^{-1}; \sigma_1, \sigma_2, \cdots, \sigma_n]$.

Corollary 3.11. *Suppose that $\sigma_1, \sigma_2, \cdots, \sigma_n \in Aut(R)$ are weakly rigid. If R is a PWP ring, then so is the ring*

$$R[x_1, x_2, \cdots, x_n, x_1^{-1}, x_2^{-1}, \cdots, x_n^{-1}; \sigma_1, \sigma_2, \cdots, \sigma_n].$$

Proof. Note that $S = \mathbb{Z}^n$ is a u.p.-monoid. $\square$

Remark 3.12. *Let R be the complex field and $0 \neq q \in R$. Let σ be the R-automorphism on $R[x]$ determined by $\sigma(x) = qx$. Define $\alpha : \mathbb{N} \cup \{0\} \longrightarrow Aut(R[x])$ via $\alpha(0) = 1$, the identity map of $R[x]$, and $\alpha(k) = \sigma^k$ for any $k \in \mathbb{N}$. It is easy to see that α is weakly rigid. Thus the quantum plane $R[x][y; \sigma]$ (see [14]) is a PWP ring.*

Remark 3.13. *Let F be a field. Let σ be the F-automorphism of $F[x]$ sending x to $x - 1$. Define $\alpha : \mathbb{N} \cup \{0\} \longrightarrow Aut(F[x])$ via $\alpha(0) = 1$, the identity map of $F[x]$, and $\alpha(k) = \sigma^k$ for any $k \in \mathbb{N}$. If V is the binary space $Fe_1 \oplus Fe_2$ with a Lie algebra structure given by the Lie product $[e_1, e_2] = e_2$, then the universal enveloping algebra of $(V, [,])$ is $F[x][y; \sigma]$. Since F is an Armendariz ring, it is easy to see that α is weakly rigid. Thus $F[x][y; \sigma]$ is a PWP ring.*

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