BEL YI FUNCTION ON $X_0(49)$ OF DEGREE 7

Appendix to: “The Belyi functions and dessin d’enfants corresponding to the non-normal inclusions of triangle groups” by K. Hoshino

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In [H] §4, we identified two types of the non-normal inclusions of triangle groups (type A and type C from Singerman’s list [S]) as those corresponding to subcovers of the Klein quartic. N. D. Elkies [E] closely studied those subcovers in view of modular curves, i.e., as subcovers under the elliptic modular curve $X_7 = X(7)$ identified with the Klein quartic defined by $X^3Y + Y^3Z + Z^3Y = 0$. Let $\mathbb{C}(X_7)$ be the function field of $X_7$ generated by $y := Y/X$ and $z := Z/X$ with a relation $y + y^3z + z^3 = 0$, and consider two automorphisms of $\mathbb{C}(X_7)$ defined by

(A1) $\sigma : \begin{cases} y \mapsto \zeta^3 y, \\ z \mapsto \zeta z, \end{cases} \quad \tau : \begin{cases} y \mapsto 1/z, \\ z \mapsto y/z, \end{cases}$

where $\zeta := e^{2\pi i/7}$. Then, $\sigma$ and $\tau$ are automorphisms respectively of order 7 and 3, and they form an automorphism group $H$ of $\mathbb{C}(X_7)$ of order 21, a subgroup of the full automorphism group $G$ of order 168 of the Klein quartic. In this respect, the genus zero covers of type A and type C of [S] are respectively $X_1(7) \rightarrow X(1)$ and $X_0(7) \rightarrow X(1)$ arising from the inclusion relations of $\langle \sigma \rangle \subset H \subset G$. Remarkably, Elkies [E] studied deep arithmetic properties of a genus one subcover $E$ fixed by $\langle \tau \rangle$ with showing that $E$ is $\mathbb{Q}$-isomorphic to $X_0(49)$. Especially, he explicitly presented its function field $\mathbb{C}(E)$ as $\mathbb{C}(E) = \mathbb{C}(u, v)$ with $v^2 = 4u^3 + 21u^2 + 28u$, where

(A2) $u = -\frac{(y + z + yz)^2}{(1 + y + z)yz}, \quad v = -\frac{(2 - y - z + 2y^2 - yz + 2z^2)(y + z + yz)}{yz(1 + y + z)}$

Recalling also from [E] that standard coordinates of $X_1(7) \cong \mathbb{P}_t^1$ and $X_0(7) \cong \mathbb{P}_s^1$ may be given as

$t := -y^2z, \quad s := t + \frac{1}{1-t} + \frac{t-1}{t},$

we would like to interpret the degree 7 cover $E \rightarrow X_0(7)$ arising from $\langle \tau \rangle \subset H$ by expressing $s$ by $u, v$ explicitly. In this note, we show

**Proposition A.** Notations being as above, the covering of $\mathbb{P}_t^1$ by the elliptic curve $E : v^2 = 4u^3 + 21u^2 + 28u$ is ramified only above $s = 3\rho, 3\rho^{-1}, \infty$ (where $\rho = e^{2\pi i/6}$), and the equation is given by

$s = \frac{1}{2}\left((u^2 + 7u + 7)v + (7u^3 + 35u^2 + 49u + 16)\right).$
Thus $\beta = \frac{s - 3\rho}{3\rho - 1 - 3\rho}$ gives a Belyi function (i.e., unramified outside $\beta = 0, 1, \infty$) of degree $7$ on $E$ with valency list $[331, 331, 7]$.

In effect, one can induce an isomorphism of covers

$$
X_0(49) \xrightarrow{\sim} E \\
\quad \downarrow \\
X_0(7) \xrightarrow{\sim} X_0(7)
$$

from the conjugacy by the ‘Fricke involution’ $\pm \frac{1}{\sqrt{7}}(0 \ 0 \ 1)$ between the modular group $\Gamma_0(49)$ and $\{ \pm (b \ c) \in \text{PSL}_2(\mathbb{Z}) : b \equiv c \equiv 0 \mod 7 \}$ in $\text{PSL}_2(\mathbb{R})$ (cf. [E] p.90). Therefore, the computation of $E \to X_0(7)$ may be reduced to combining classically well known equations that relate $X_0(7), X_0(49)$ with the $J$-line $X(1)$ found in, e.g., [F] pp.395–403. Here, however, we shall employ an alternative enjoyable discussion following the Elkies scheme:

$$
X_1(7) = \mathbb{P}^1_i \xleftarrow{f} \langle \sigma \rangle X_7 \xrightarrow{q} q
\\
\quad \downarrow \\
X_0(7) = \mathbb{P}^1_s \xleftarrow{p} E.
$$

First of all, since $\frac{ds}{dt} = \frac{(t^2 - t + 1)^2}{(t - 1)^2 t^2}$, the 3-cyclic cover $g$ is ramified only over $s = 3\rho, 3\rho^{-1}$ at $t = \rho, \rho^{-1}$, and the fiber over $s = \infty$ is formed by the three points $t = 0, 1, \infty$. We next chase the fibers of $f$ over these points and their images in $E$ by $q$. Noticing that $\mathbb{C}(X_7) = \mathbb{C}(y, z) = \mathbb{C}(y, t)$ with $y^7 = \frac{t^3}{1-t}$, we see that $f$ is totally ramified over $t = 0, 1, \infty$, and their images by $q$ coincide at the infinity point on $E : v^2 = 4u^3 + 21u^2 + 28u$. From this follows that $p : E \to \mathbb{P}^1_s$ is totally ramified at the infinity point of $E$ over $s = \infty$, hence $s$ is of the form $s = F(u) + G(u)v$ with $F, G \in \mathbb{C}[u], \deg(F) = 3, \deg(G) = 2$.

The fiber of $f$ over $t = \rho$ forms one orbit under the action of $\sigma (y \mapsto \zeta y)$ whose points are represented by the set of their $y$-coordinates $S_\rho := \{ \xi^2, \xi^5, \xi^8, \xi^{11}, \xi^{14}, \xi^{17}, \xi^{20} \}$, where $\xi := e^{2\pi i/21}$. The action of $\tau$ preserves $t = \rho$ and transforms those $y$-coordinates (over $t = \rho$) as $y \mapsto y^2 \xi^{14}$, hence decomposes $S_\rho$ into the three $\tau$-orbits $S^1_\rho := \{ \xi^2, \xi^{11}, \xi^8 \}, S^2_\rho := \{ \xi^5, \xi^{17}, \xi^{20} \},$ and $S^3_\rho := \{ \xi^4 = \rho^4 \}$ (This also explains the branch type of $p$ over $s = 3\rho$ is ‘331’). Then we compute the $u$-coordinates $u_1, u_2, u_3$ of the images of these orbits by $q$ after the formula (A2); it turns out that $u_1 = -\xi^2 - \xi^{11} - \xi^8 - 1$, $u_2 = -\xi^7 + \xi^2 + \xi^{11} + \xi^8 - 2$ and $u_3 = 3\rho - 1$. Eliminating $v$ from $F(u) +
BEL YI FUNCTION ON $X_0(49)$ OF DEGREE 7

$G(u)v = 3\rho$, we should then have an equation of the form

$$(4u^3 + 21u^2 + 28u)G(u)^2 - (F(u) - 3\rho)^2 = (u - u_1)^3(u - u_2)^3(u - u_3).$$

Using a symbolic computation software such as MAPLE to compare the coefficients of the both sides above (plus a slight more consideration of signs), we conclude $F(u) = \frac{1}{2}(7u^3 + 35u^2 + 49u + 16)$, $G(u) = \frac{1}{2}(u^2 + 7u + 7)$ as stated in Proposition A.

**Remark.** The monodromy representation associated with the above Belyi function $\beta$ is given by $x = (142)(356)(7)$, $y = (175)(346)(2)$, $z = (1234567)$. The following picture illustrates the uniformization of this Grothendieck dessin.

![Diagram of Grothendieck dessin]

**References**


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