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THE BELYI FUNCTIONS AND DESSIN D'ENFANTS CORRESPONDING TO THE NON-NORMAL INCLUSIONS OF TRIANGLE GROUPS

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Abstract. We present the Belyi functions, dessin d'enfants, and monodromy permutations corresponding to the non-normal inclusions of triangle groups.

Contents

1. Introduction

Let X be a non-singular complete algebraic curve over $\mathbb{C}$. A morphism $\beta : X \to \mathbb{P}^1$ is called a Belyi function if it is unramified over $\mathbb{P}^1 \backslash \{0, 1, \infty\}$. The pair (X, β), called a Belyi pair, determines a dessin D by $D = \beta^{-1}([0, 1])$ as a topological graph illustrated on the Riemann surface $X(\mathbb{C})$. It is known that the mapping $(X, \beta) \to D$ leads to a bijection between the isomorphism classes of Belyi pairs and the (topological) isomorphism classes of dessins. In this paper, we focus on the Belyi functions β arising from the non-normal inclusions of triangle groups classified by D.Singerman [S]. These β satisfy the property:

(*) Exactly three cusps on X do not achieve the least common multiples of ramification indices of those cusps lying over the same image in $\{0, 1, \infty\} \subset \mathbb{P}^1$.

We shall complete a list of the Belyi functions that satisfy the above condition (*). In fact, computations of the non-compact cases (where triangle groups admit ∞ as their indices) have been given by H.Nakamura [N] and M.Wood [W] in processes of deriving equations for $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ in the

Grothendieck-Teichmuller group and new Galois invariants of Grothendieck dessins. We fill the remained computations in the compact cases.

Organization of this paper is as follows. In section 2, we state our main results showing a table of Belyi functions for the non-normal inclusions of triangle groups, and in section 3, we review some basic definitions concerning monodromy groups. In section 4, we discuss the cases A and C in relation with the Klein quartic. In section 5, we give the Belyi functions in the other compact cases, together with associated dessins and monodromy generators. In section 6, we review the non-compact cases treated in [N] for the sake of completeness of our table. In Appendix with H.Nakamura, we calculate a Belyi function of degree 7 in genus one that appears in subcoverings of the Klein quartic.

2. Main results

Definition 2.1. *The triangle group* $\Delta(e_1, e_2, e_3)$ *is defined by*

$$\Delta(e_1, e_2, e_3) := \langle x, y, z \mid xyz = x^{e_1} = y^{e_2} = z^{e_3} = 1\rangle,$$

where $e_1, e_2, e_3 \in \mathbb{N}$. *Especially, it is called of hyperbolic type if*

$$\frac{1}{e_1} + \frac{1}{e_2} + \frac{1}{e_3} < 1.$$

In [3], D. Singerman classified the non-normal inclusions of hyperbolic triangle groups with finite indices which are listed as follows:

	non-normal inclusion	index
A	$\Delta(7,7,7) \subset \Delta(2,3,7)$	24
B	$\Delta(2,7,7) \subset \Delta(2,3,7)$	9
C	$\Delta(3,3,7) \subset \Delta(2,3,7)$	8
D	$\Delta(4,8,8) \subset \Delta(2,3,8)$	12
E	$\Delta(3,8,8) \subset \Delta(2,3,8)$	10
F	$\Delta(9,9,9) \subset \Delta(2,3,9)$	12
G	$\Delta(4,4,5) \subset \Delta(2,4,5)$	6
H	$\Delta(n,4n,4n) \subset \Delta(2,3,4n)$	6
I	$\Delta(n,2n,2n) \subset \Delta(2,4,2n)$	4
J	$\Delta(3,n,3n) \subset \Delta(2,3,3n)$	4
K	$\Delta(2,n,2n) \subset \Delta(2,3,2n)$	3

Recall that, to each non-normal inclusion of triangle groups, a Belyi function is associated naturally.

Theorem 2.2. *The Belyi functions led by the non-normal inclusions of hyperbolic triangle groups are given by the following table:*

	Belyi function $\beta(t)$
A	$\beta_A(t) = \dfrac{(t^6+229t^5+270t^4-1695t^3+1430t^2-235t+1)^3(t^2-t+1)^3}{1728t(t-1)(t^3-8t^2+5t+1)^7}$
B	$\beta_B(t) = -\dfrac{2(3^2\cdot 7^4t^4+2^2\cdot 7^3\cdot 43t^3+2\cdot 3\cdot 7^2\cdot 31\cdot 47t^2+2^2\cdot 3\cdot 43\cdot 757t+7^2\cdot 167\cdot 239)^2}{3^3(t^2+7)(7t-13)^7}$
C	$\beta_C(t) = \dfrac{(t^2+3)(9t^2+2^4\cdot 3\cdot 29t+41\cdot 83)^3}{2^7\cdot 3(3t-13)^7}$
D	$\beta_D(t) = \dfrac{(4t^2-4t-1)^2(16t^4-32t^3+152t^2-136t+1)^2}{2^4\cdot 3^3t(t-1)(2t-1)^8}$
E	$\beta_E(t) = \dfrac{(2^8t^5+2^8\cdot 5^2t^4+2^5\cdot 13\cdot 23t^3+2^4\cdot 5^2\cdot 71t^2+17783t+2^3\cdot 5\cdot 11\cdot 73)^2}{(t^2+2)(4t-7)^8}$
F	$\beta_F(t) = -\dfrac{(t^6-2\cdot 3^2\cdot 19t^5-3^3\cdot 211t^4-2^2 3^3 5\cdot 19t^3-3^5 139t^2-2\cdot 3^6 19t-3^6 71)^2}{2^7 3^2(t-3)^9(t^2+3)}$
G	$\beta_G(t) = -\dfrac{(t^2+1)(2t-11)^4}{(4t+3)^5}$
H	$\beta_H(t) = -\dfrac{(t^2-16t+16)^3}{108t^4(t-1)}$
I	$\beta_I(t) = -\dfrac{16t(t-1)}{(4t^2-4t-1)^2}$
J	$\beta_J(t) = \dfrac{t(9t-8)^3}{64(t-1)}$
K	$\beta_K(t) = \dfrac{(4t-3)^3}{27(t-1)}$

The following table shows ramification types of the above Belyi functions over 1, where $P := t^2 - t$ *and* $Q := t^3 - 8t^2 + 5t + 1$ *for the case* A.

	Factorization of $\beta(t)$ *over* 1
A	$1-\beta_A(t) = -\dfrac{(Q^4-2\cdot 5\cdot 7^2PQ^3-3^2 7^4P^2Q^2-2\cdot 7^6P^3Q-7^7P^4)^2}{1728t(t-1)(t^3-8t^2+5t+1)^7}$
B	$1-\beta_B(t) = \dfrac{7(3\cdot 7^2t^3+7^2\cdot 29t^2+7\cdot 199t+5\cdot 29\cdot 71)^3}{3^3(t^2+7)(7t-13)^7}$
C	$1-\beta_C(t) = -\dfrac{(3^3t^4-2^3\cdot 3^3\cdot 43t^3-2\cdot 3^2\cdot 7867t^2-2^3\cdot 43\cdot 409t-3^2\cdot 167\cdot 251)^2}{2^7\cdot 3(3t-13)^7}$
D	$1-\beta_D(t) = -\dfrac{(16t^4-32t^3-40t^2+56t+1)^3}{2^4\cdot 3^3t(t-1)(2t-1)^8}$
E	$1-\beta_E(t) = -\dfrac{2(2^7t^3+2^3\cdot 3\cdot 17t^2+2^3\cdot 3\cdot 13t+17\cdot 47)^3}{(t^2+2)(4t-7)^8}$
F	$1-\beta_F(t) = \dfrac{(t+3)^3(t^3+3^2\cdot 17t^2+3^3t+3^3\cdot 17)^3}{2^7 3^2(t-3)^9(t^2+3)}$
G	$1-\beta_G(t) = \dfrac{(4t^3+84t^2-37t+122)^2}{(4t+3)^5}$
H	$1-\beta_H(t) = \dfrac{(t-2)^2(t^2+32t-32)^2}{108t^4(t-1)}$
I	$1-\beta_I(t) = \dfrac{(2t-1)^4}{(4t^2-4t-1)^2}$
J	$1-\beta_J(t) = -\dfrac{(27t^2-36t+8)^2}{64(t-1)}$
K	$1-\beta_K(t) = -\dfrac{t(8t-9)^2}{27(t-1)}$

From A to G are compact cases, and from H to K are non-compact cases.

Normalization: Our Belyi functions $\beta(t)$ given in the above table are normalized so that the critical values of β are to be $\{0, 1, \infty\}$ and that the distinguished three cusps in $t \in X$ are to be $\{0, 1, \infty\}$ or to be of the form $\{\pm\sqrt{d}, \infty\}$ with d a square free integer.

3. Monodromy group and its generators

Let $\phi : X \to \mathbb{P}^1_{\mathbb{C}}$ be a finite covering map of degree n, and let $B := \{b_0, \ldots, b_k\} \subset \mathbb{P}^1_{\mathbb{C}}$ be the set of ramification points of ϕ. Fix a base point $p \in \mathbb{P}^1_{\mathbb{C}} \backslash B$, and set the elements of the fiber $\Omega := \phi^{-1}(p)$ to be $\{p_1, \ldots, p_n\}$. One can then introduce the presentation of the fundamental group Π of $\mathbb{P}^1_{\mathbb{C}} \backslash B$ as

$$(\Pi :=)\, \pi_1(\mathbb{P}^1_{\mathbb{C}} \backslash B, p) = \langle \gamma_0, \ldots, \gamma_k | \prod_{i=0}^{k} \gamma_i = 1 \rangle$$

where γ_i represents a loop around b_i $(i = 0, \ldots, k)$. Each loop $\gamma \in \Pi$ lifts to n paths $\tilde{\gamma}_i$ on $X_{\mathbb{C}}$ starting from the $p_i \in \Omega$ $(i = 1, \ldots, n)$. We associate the permutation $\rho(\gamma)$ of Ω by defining $\rho(\gamma)(p_i) = p_j$ if and only if the lift$\tilde{\gamma}_i$ goes from p_i to p_j. Thus we obtain the monodromy representation into the symmetric group $S(\Omega)$ on Ω:

$$\rho : \Pi \to S(\Omega) \qquad \rho(\gamma_i) = \sigma_i.$$

The image of ρ is called the monodromy group of ϕ and written $G(\phi)$. If we identify $\Omega = \{1, \ldots, n\}$ and regard $S(\Omega)$ as the symmetric group of degree n, then the above monodromy representation is determined by the $(k+1)$-tuple $(\sigma_0, \ldots, \sigma_k)$ of the permutations $\sigma_i := \rho(\gamma_i)$ $(i = 0, \ldots, k)$ with $\sigma_0 \cdots \sigma_k = 1$. We shall call it the associated monodromy permutation. Note that the cycle type of each permutation σ_i explains the ramification type over the point b_i $(i = 0, \ldots, k)$.

Below, we shall consider the case ϕ is given as a (non-cyclic) Belyi function arising from non-normal inclusions of triangle groups, where $k = 2$ and $B = \{0, 1, \infty\}$. In this case, $b_0 = 0$, $b_1 = 1$ and $b_2 = \infty$, but for simplicity, we use the notation $(\sigma_0, \sigma_1, \sigma_\infty)$ to designate the associated monodromy permutation. In the sequel, for each case of Singerman's list A∼K, we shall give the monodromy permutation that explains the associated Grothendieck dessin. We also give the ramification chart that illustrates the branch types over the three points $0, 1, \infty \in \mathbb{P}^1_{\mathbb{C}}$. These will help understanding computations of the Belyi functions. The ramification chart also indicates the location of the critical cusps with their normalized coordinates.

4. Subcoverings of Klein quartic

In this section, we consider the compact cases A and C of Singerman's list that correspond subcovers of the Klein quartic (N.D.Elkies [E]). The Klein quartic is a non-singular complete algebraic curve of genus 3 defined by

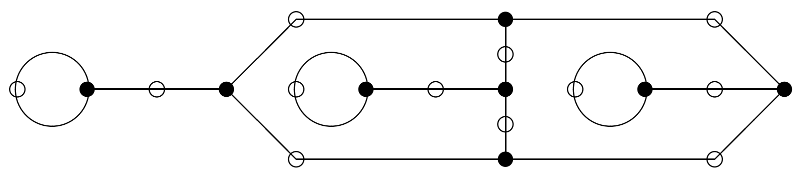
$$X = X(7) : x^3y + y^3z + z^3x = 0$$

in $\mathbb{P}^2(\mathbb{C})$. The covers $X_1(7) \to X(1)$ and $X_0(7) \to X(1)$ are subcoverings of X. The relation of these coverings is as follows:

$$\begin{tikzcd}
 & X = X(7) \arrow[dl, "7"'] \arrow[dr, "3"] & \\
X_1(7) \arrow[dr, "3"'] & & E \arrow[dl, "7"] \\
 & X_0(7) \arrow[d, "8"] & \\
 & X(1) &
\end{tikzcd}$$

where E is an elliptic curve isomorphic to $X_0(49)$ (See Appendix).

The case A : $\Delta(7,7,7) \subset \Delta(2,3,7)$, $index = 24$.
The corresponding dessin is illustrated as follows:



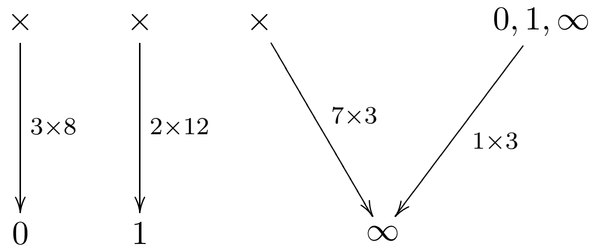
In this case, the monodromy permutation is given by

$$\begin{cases} \sigma_0 = (1\ 2\ 3)(4\ 5\ 6)(7\ 8\ 9)(10\ 11\ 12)(13\ 14\ 15)(16\ 17\ 18)(19\ 20\ 21)(22\ 23\ 24), \\ \sigma_1 = (1\ 2)(3\ 4)(5\ 7)(6\ 18)(8\ 19)(9\ 10)(11\ 16)(12\ 13)(14\ 15)(17\ 20)(21\ 22)(23\ 24), \\ \sigma_\infty = (1\ 4\ 18\ 20\ 8\ 5\ 3)(6\ 7\ 10\ 13\ 14\ 12\ 16)(9\ 19\ 22\ 23\ 21\ 17\ 11)(2)(15)(24). \end{cases}$$

This inclusion corresponds to the cover $X_1(7) \to X(1)$. From N.D.Elkies [E] p.88-89, we find:

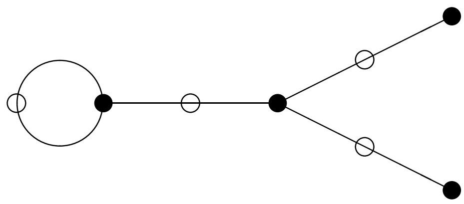
$$\begin{aligned} \beta_A(t) &= \frac{(\phi(t)^2 + 13\phi(t) + 49)(\phi(t)^2 + 245\phi(t) + 7^4)^3}{1728\phi(t)^7}, \\ 1 - \beta_A(t) &= -\frac{(\phi(t)^4 - 2\cdot 5\cdot 7^2\phi(t)^3 - 3^2 7^4\phi(t)^2 - 2\cdot 7^6\phi(t) - 7^7)^2}{1728\phi(t)^7}, \end{aligned}$$

where $\phi(t) := (t^3 - 8t^2 + 5t + 1)/(t^2 - t)$. The expressions of β_A and $1 - \beta_A$ in Theorem 2.2 follow from this. The ramification chart is the following:



Here and henceforce, '$a \times b$' in a ramification chart indicates that there are b cusps with multiplicity a on the t-line cover.

The case C : $\Delta(3,3,7) \subset \Delta(2,3,7)$, *index* $= 8$.
The corresponding dessin is illustrated as follows:



The monodromy permutation is given by

$$\begin{cases} \sigma_0 & = & (1\ 2\ 3)(4\ 5\ 6)(7)(8), \\ \sigma_1 & = & (1\ 2)(3\ 4)(5\ 7)(6\ 8), \\ \sigma_\infty & = & (1\ 4\ 8\ 6\ 7\ 5\ 3)(2). \end{cases}$$

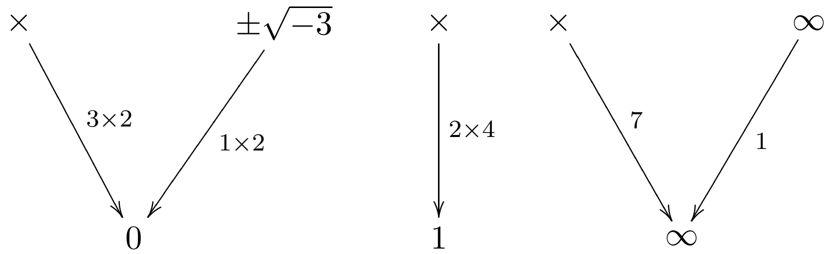
This is the cover $X_0(7) \to X(1)$. It follows from N.D.Elkies [E] p.88 that the Belyi function is

$$\frac{(t^2 + 13t + 49)(t^2 + 245t + 7^4)^3}{1728t}.$$

Substituting $(3t - 13)/2$ for t, we obtain the normalized Belyi function as follows:

$$\begin{aligned} \beta_C(t) &= \frac{(t^2+3)(9t^2 + 2^4 \cdot 3 \cdot 29t + 41 \cdot 83)^3}{2^7 \cdot 3(3t-13)^7}, \\ 1 - \beta_C(t) &= -\frac{(3^3t^4 - 2^3 \cdot 3^3 \cdot 43t^3 - 2 \cdot 3^2 \cdot 7867t^2 - 2^3 \cdot 43 \cdot 409t - 3^2 \cdot 167 \cdot 251)^2}{2^7 \cdot 3(3t-13)^7}. \end{aligned}$$

The ramification chart is given by



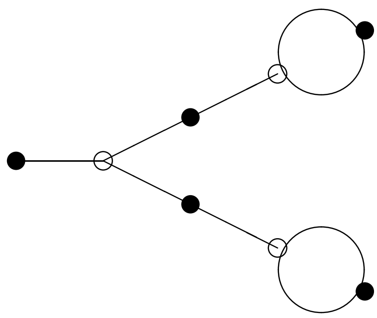
5. Other compact cases

In this section, we show the generators of monodromy groups and illustrate their dessins of the compact case B, D, E, F, and G.

The case B : $\Delta(2,7,7) \subset \Delta(2,3,7)$, *index* $= 9$.
In this case, the monodromy permutation is given by

$$\begin{cases} \sigma_0 & = & (1\ 2)(3\ 4)(5)(6\ 7)(8\ 9), \\ \sigma_1 & = & (1\ 2\ 3)(4\ 6\ 5)(7\ 8\ 9), \\ \sigma_\infty & = & (1)(2\ 3\ 5\ 6\ 9\ 7\ 4)(8). \end{cases}$$

The corresponding dessin is illustrated as follows:



To compute the Belyi function β_B, we may put

$$\frac{1}{\beta_B(t)} = \frac{t^7 f_2(t)}{f_4(t)^2},$$

where $f_n(t)$ denotes a polynomial with $\deg(f_n(t)) = n$. Note that we may suppose that the constant term of $f_4(t)$ equals 1. From the condition of ramification over 1, we derive that

$$(f_4(t))^2 - t^7(f_2(t)) = (f_3(t))^3. \tag{1}$$

Put $f_4(t) = a_4t^4 + a_3t^3 + a_2t^2 + a_1t + 1$. Here, without loss of generality, we may assume $a_1 = 0$ or $a_1 = 1$. Set $f_3(t) = b_3t^3 + b_2t^2 + b_1t + b_0$. Substituting these into the above equation (1), and comparing the coefficients of degrees $0, \ldots, 3$, we find

$$\begin{aligned}
0 &= 1 - b_0^2, \\
0 &= 2a_1 - 3b_0^2 b_1, \\
0 &= 2a_2 + a_1^2 - 3b_0^2 b_2 - 3b_0 b_1^2, \\
0 &= 2a_3 + 2a_1 a_2 - b_1^3 - 3b_0^2 b_3 - 6b_0 b_1 b_2.
\end{aligned}$$

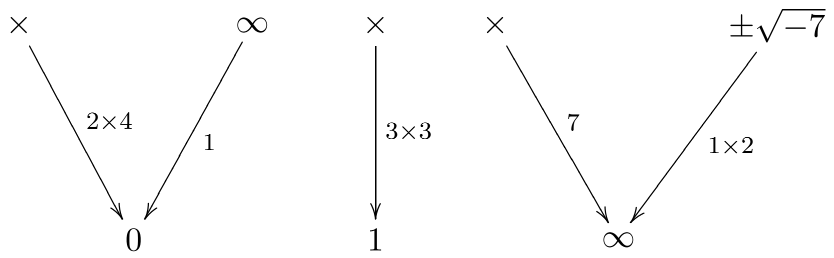
From this, we obtain expressions of b_0, b_1, b_2 in terms of a_1, a_2, a_3, a_4. Then, we substitute those expressions again into (1), then from degrees 4,5,6, we obtain

$$\begin{aligned}
0 &= -\frac{2}{3}a_1 a_3 + 2a_4 - \frac{1}{3}a_2^2 + \frac{4}{9}a_1^2 a_2 - \frac{7}{81}a_1^4, \\
0 &= -\frac{2}{3}a_2 a_3 + 2a_1 a_4 + \frac{20}{81}a_1^3 a_2 - \frac{14}{243}a_1^5 - \frac{4}{9}a_1^2 a_3, \\
0 &= 2a_2 a_4 - \frac{1}{3}a_3^2 + \frac{16}{27}a_1^2 a_2^2 - \frac{8}{9}a_1 a_2 a_3 - \frac{46}{243}a_1^4 a_2 + \frac{8}{81}a_1^3 a_3 + \frac{35}{2187}a_1^6 - \frac{8}{27}a_2^7.
\end{aligned}$$

If $a_1 = 0$, then $a_2 = a_3 = a_4 = 0$ leading to contradiction. So, we must have $a_1 = 1, a_2 = 7/12, a_3 = 35/216, a_4 = 7/288$. From this we can determine all coefficients. After normalization $t \mapsto (21t - 39)/32$, we conclude :

$$\begin{aligned}
\beta_B(t) &= -\frac{2(3^2 \cdot 7^4 t^4 + 2^2 \cdot 7^3 \cdot 43t^3 + 2 \cdot 3 \cdot 7^2 \cdot 31 \cdot 47t^2 + 2^2 \cdot 3 \cdot 43 \cdot 757t + 7^2 \cdot 167 \cdot 239)^2}{3^3(t^2+7)(7t-13)^7}, \\
1 - \beta_B(t) &= \frac{7(3 \cdot 7^2 t^3 + 7^2 \cdot 29t^2 + 7 \cdot 199t + 5 \cdot 29 \cdot 71)^3}{3^3(t^2+7)(7t-13)^7}.
\end{aligned}$$

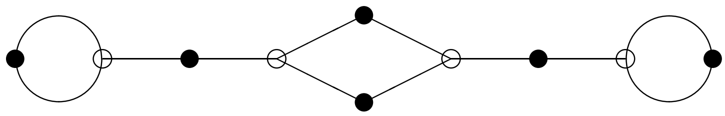
The ramification chart is the following:



The case D : $\Delta(4,8,8) \subset \Delta(2,3,8)$, *index* $= 12$.
The monodromy permutation is given by

$$\begin{cases} \sigma_0 & = & (1\ 2)(3\ 4)(5\ 6)(7\ 8)(9\ 10)(11\ 12), \\ \sigma_1 & = & (1\ 2\ 3)(4\ 5\ 12)(6\ 7\ 11)(8\ 9\ 10), \\ \sigma_\infty & = & (1)(2\ 3\ 12\ 7\ 10\ 8\ 6\ 4)(5\ 11)(9). \end{cases}$$

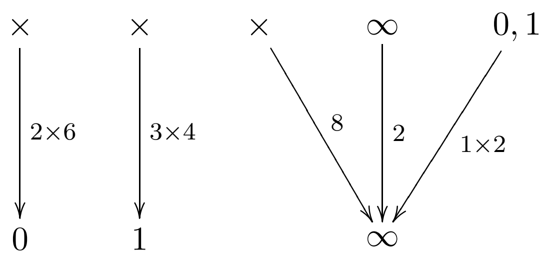
The corresponding dessin is illustrated as follows:



This corresponds to the cover $\beta : X_0(8) \to X(1)$. By [F] and normalization, we get

$$\begin{aligned} \beta_D(t) &= \frac{(4t^2-4t-1)^2(16t^4-32t^3+152t^2-136t+1)^2}{2^4\cdot 3^3 t(t-1)(2t-1)^8}, \\ 1-\beta_D(t) &= -\frac{(16t^4-32t^3-40t^2+56t+1)^3}{2^4\cdot 3^3 t(t-1)(2t-1)^8}. \end{aligned}$$

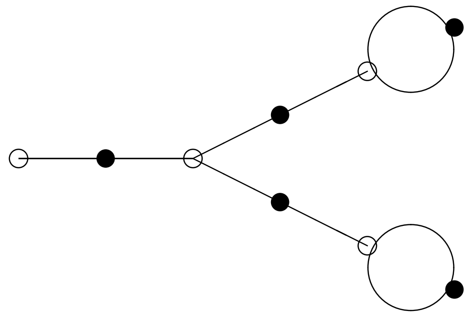
The ramification chart is the following:



The case E : $\Delta(3,8,8) \subset \Delta(2,3,8)$, *index* $= 10$.
The monodromy permutation is given by

$$\begin{cases} \sigma_0 & = & (1\ 2)(3\ 4)(5\ 6)(7\ 8)(9\ 10), \\ \sigma_1 & = & (1\ 2\ 3)(4\ 8\ 10)(5\ 6\ 7)(9), \\ \sigma_\infty & = & (1)(2\ 3\ 10\ 9\ 8\ 6\ 7\ 4)(5). \end{cases}$$

The corresponding dessin is illustrated as follows:



To compute the Belyi function β_E, we may put

$$\frac{1}{1-\beta_E(t)} = \frac{t^8 f_2(t)}{(f_3(t))^3},$$

where $f_n(t)$ denotes a polynomial with $\deg(f_n(t)) = n$. Note that the constant term of $f_3(t)$ can be supposed to equal 1. From the condition of ramification over 1, we derive that

$$f_3(t)^3 - t^8 f_2(t) = f_5(t)^2 \tag{2}$$

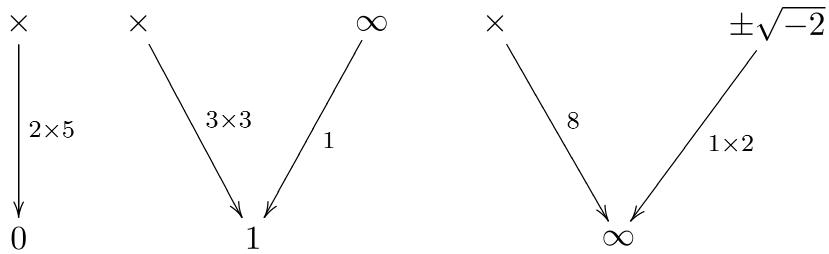
Suppose that $f_3(t) = a_3t^3 + a_2t^2 + a_1t + 1$ and $f_5(t) = b_5t^5 + b_4t^4 + b_3t^3 + b_2t^2 + b_1t + b_0$. Then we may assume $a_1 = 0$ or $a_1 = 1$. By comparing the coefficients of (2), we obtain

$$\begin{aligned}
b_0 &= \pm 1,\\
b_1 &= \pm\frac{3}{2}a_1,\\
b_2 &= \pm\frac{3}{8}(4a_2 + a_1^2),\\
b_3 &= \pm\frac{1}{16}(12a_1a_2 + 24a_3 - a_1^3),\\
b_4 &= \pm\frac{3}{128}(32a_1a_3 - 8a_1^2a_2 + 16a_2^2 + a_1^4),\\
b_5 &= \pm\frac{3}{256}(64a_2a_3 - 16a_1a_2^2 - 16a_1^2a_3 + 8a_1^3a_2 - a_1^5).
\end{aligned}$$

Here we substitute these equations for (2). If $a_1 = 0$, then $a_2 = a_3 = 0$, so we obtain $a_1 = 1, a_2 = 5/12, a_3 = 1/18$ and by $t \mapsto (8t-14)/9$, we get

$$\begin{aligned}
\beta_E(t) &= \frac{(2^8t^5 + 2^8\cdot 5^2t^4 + 2^5\cdot 13\cdot 23t^3 + 2^4\cdot 5^2\cdot 71t^2 + 17783t + 2^3\cdot 5\cdot 11\cdot 73)^2}{(t^2+2)(4t-7)^8},\\
1-\beta_E(t) &= -\frac{2(2^7t^3 + 2^3\cdot 3\cdot 17t^2 + 2^3\cdot 3\cdot 13t + 17\cdot 47)^3}{(t^2+2)(4t-7)^8}.
\end{aligned}$$

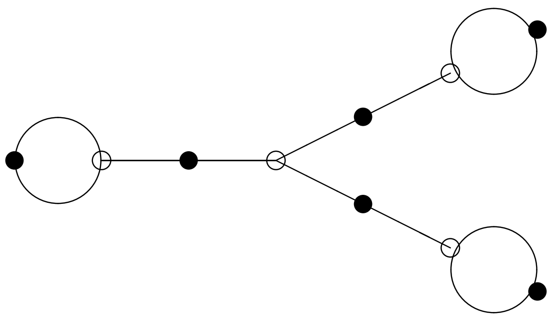
The ramification chart is the following:



The case F : $\Delta(9,9,9) \subset \Delta(2,3,9)$, *index* $= 12$.
The monodromy permutation is given by

$$\begin{cases} \sigma_0 & = & (1\ 2)(3\ 4)(5\ 6)(7\ 8)(9\ 10)(11\ 12), \\ \sigma_1 & = & (1\ 2\ 3)(4\ 8\ 12)(5\ 6\ 7)(9\ 10\ 11), \\ \sigma_\infty & = & (1)(2\ 3\ 12\ 10\ 11\ 8\ 6\ 7\ 4)(5)(9). \end{cases}$$

The corresponding dessin is illustrated as follows:



Observing that this dessin is a triple cover of the dessin in the case J (see §6 below), we see that it is essentially $\beta_J(t^3)$. Taking our normalization into accounts, it turns out that

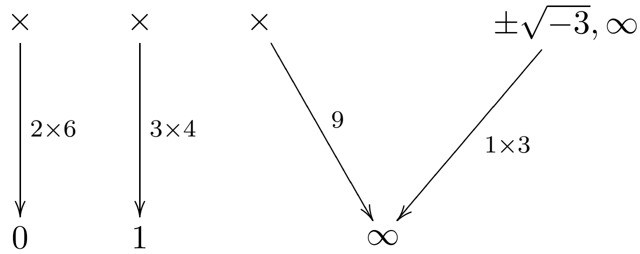
$$\beta_F(t) = 1 - \beta_J\left(\left(\frac{t+3}{t-3}\right)^3\right).$$

We calculate :

$$\beta_F(t) = -\frac{(t^6 - 2\cdot 3^2 \cdot 19t^5 - 3^3 \cdot 211t^4 - 2^2 3^3 5 \cdot 19t^3 - 3^5 139t^2 - 2\cdot 3^6 19t - 3^6 71)^2}{2^7 3^2 (t-3)^9 (t^2+3)},$$

$$1 - \beta_F(t) = \frac{(t+3)^3 (t^3 + 3^2 \cdot 17t^2 + 3^3 t + 3^3 \cdot 17)^3}{2^7 3^2 (t-3)^9 (t^2+3)}.$$

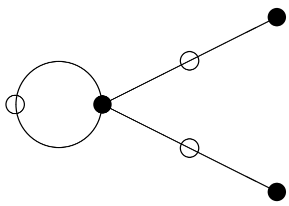
The ramification chart is the following:



The case G : $\Delta(4,4,5) \subset \Delta(2,4,5)$, *index* $= 6$.
The monodromy permutation is given by

$$\begin{cases} \sigma_0 & = & (1)(2\ 3\ 4\ 5)(6), \\ \sigma_1 & = & (1\ 2)(3\ 4)(5\ 6), \\ \sigma_\infty & = & (1\ 2\ 6\ 5\ 3)(4). \end{cases}$$

The corresponding dessin is illustrated as follows:



To compute the Belyi function β_G, we may put

$$\beta_G(t) = \frac{t^4 f_2(t)}{b_0(f_1(t))},$$

where b_0 is constant. Suppose that $f_2(t) = t^2 - 2t + a_0, f_1(t) = t - c_0$. We consider ramification points. By taking derivatives, we get

$$\beta'_G(t) = \frac{t^3}{b_0 f_1(t)^2} \cdot g_3(t),$$

with

$$g_3(t) = 5t^3 - (6c_0 + 8)t^2 + (3a_0 + 10c_0)t - 4a_0c_0.$$

By considering ramification over 1, we obtain

$$1 - \beta_G = -\frac{1}{25b_0 f_1(t)} \cdot g_6(t),$$

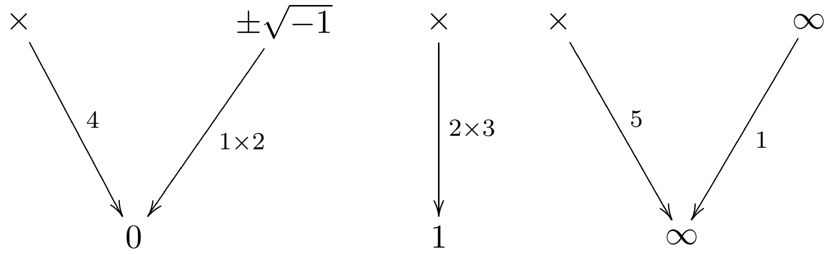
where

$$g_6(t) = 25t^6 - 50t^5 + 25a_0t^4 - 25b_0t + 25b_0c_0.$$

Hence $g_3(t)^2 = g_6(t)$, we get the Belyi function. By normalization $t \mapsto \frac{-2t+11}{4t+3}$, we obtain

$$\begin{aligned}\beta_G(t) &= -\frac{(t^2+1)(2t-11)^4}{(4t+3)^5}\\ 1-\beta_G(t) &= \frac{(4t^3+84t^2-37t+122)^2}{(4t+3)^5}.\end{aligned}$$

The ramification chart is the following:



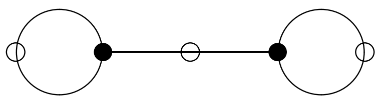
6. Non-compact cases

Non-compact cases $H \sim K$ are calculated by H.Nakamura [3]. Here, we recollect his results for the sake of completeness of presentation of Main Theorem.

The case H : $\Delta(n,4n,4n) \subset \Delta(2,3,4n)$, *index* $= 6$.
The monodromy permutation is given by

$$\begin{cases}\sigma_0 &= (1\,2\,3)(4\,5\,6),\\ \sigma_1 &= (1\,2)(3\,4)(5\,6),\\ \sigma_\infty &= (1\,4\,5\,3)(2)(6).\end{cases}$$

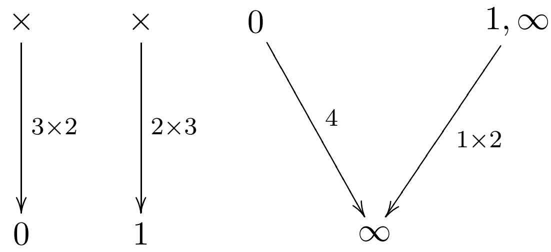
The corresponding dessin is illustrated as follows:



This corresponds to the cover $\beta_H : X_0(4) \to X(1)$. It is given by

$$\begin{aligned}\beta_H(t) &= -\frac{(t^2-16t+16)^3}{108t^4(t-1)},\\ 1-\beta_H(t) &= \frac{(t^2+32t-32)^2(t-2)^2}{108t^4(t-1)}.\end{aligned}$$

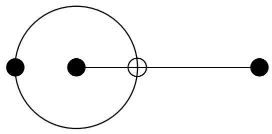
The ramification chart is the following:



The case I : $\Delta(n, 2n, 2n) \subset \Delta(2, 4, 2n),$ *index* $= 4.$
The monodromy permutation is given by

$$\begin{cases} \sigma_0 &= (1\ 3)(2)(4), \\ \sigma_1 &= (1\ 2\ 3\ 4), \\ \sigma_\infty &= (1\ 2)(3\ 4). \end{cases}$$

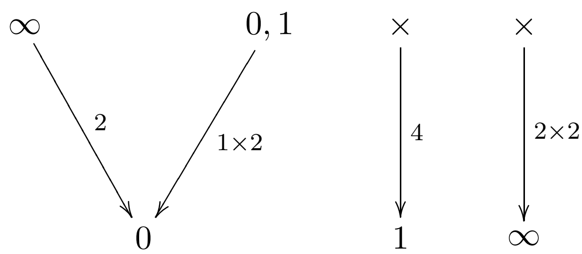
The corresponding dessin is illustrated as follows:



This is the cover $\beta_I : X_0(4) \to {X_0}^*(2)$, where ${X_0}^*(2)$ is the quotient of $X_0(2)$ by the Fricke involution. So β_I is given by

$$\begin{aligned} \beta_I(t) &= -\frac{16t(t-1)}{(4t^2-4t-1)^2}, \\ 1-\beta_I(t) &= \frac{(2t-1)^4}{(4t^2-4t-1)^2}. \end{aligned}$$

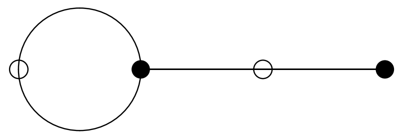
The ramification chart is the following:



The case J : $\Delta(3, n, 3n) \subset \Delta(2, 3, 3n),$ *index* $= 4.$
The monodromy permutation is given by

$$\begin{cases} \sigma_0 &= (1)(2\ 3\ 4), \\ \sigma_1 &= (1\ 2)(3\ 4), \\ \sigma_\infty &= (1\ 2\ 3\ )(4). \end{cases}$$

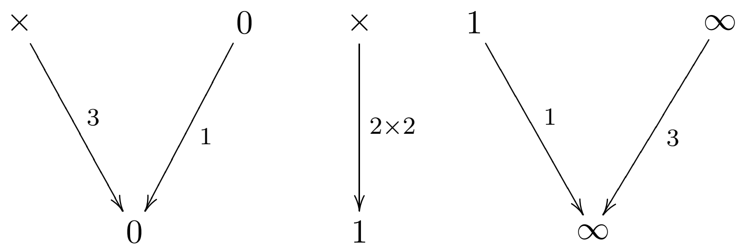
The corresponding dessin is illustrated as follows:



This is the cover $\beta_J : X_0(3) \to X(1)$. It is given by

$$\begin{aligned} \beta_J(t) &= \frac{t(9t-8)^3}{64(t-1)}, \\ 1-\beta_J(t) &= -\frac{(27t^2-36t+8)^2}{64(t-1)}. \end{aligned}$$

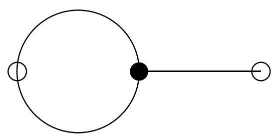
The ramification chart is the following:



The case K : $\Delta(2,n,2n) \subset \Delta(2,3,2n)$, *index* $= 3$.
The monodromy permutation is given by

$$\begin{cases} \sigma_0 &= (1\,2\,3), \\ \sigma_1 &= (1)(2\,3), \\ \sigma_\infty &= (1\,2)(3). \end{cases}$$

The corresponding dessin is illustrated as follows:



This corresponds to the cover $X_0(2) \to X(1)$. The Belyi function is given by

$$\begin{aligned} \beta_K(t) &= \frac{(4t-3)^3}{27(t-1)}, \\ 1-\beta_K(t) &= -\frac{t(8t-9)^2}{27(t-1)}. \end{aligned}$$

The ramification chart is the following:

$$\begin{tikzcd}
\times \arrow[d, "3"] & \times \arrow[dr, "2"] & & 0 \arrow[dl, "1"'] & \infty \arrow[dr, "2"] & & 1 \arrow[dl, "1"'] \\
0 & & 1 & & & \infty &
\end{tikzcd}$$

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