THE SPACE $L_q$ OF DOUBLE SEQUENCES

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Abstract. The spaces $BS$, $BS(t)$, $CS_p$, $CS_{bp}$, $CS_r$ and $BV$ of double sequences have recently been studied by Altay and Başar [J. Math. Anal. Appl. 309(1)(2005), 70–90]. In this work, following Altay and Başar [1], we introduce the Banach space $L_q$ of double sequences corresponding to the well-known space $\ell_q$ of single sequences and examine some properties of the space $L_q$. Furthermore, we determine the $\beta(\upsilon)$-dual of the space and establish that the $\alpha$- and $\gamma$-duals of the space $L_q$ coincide with the $\beta(\upsilon)$-dual; where $1 \leq q < \infty$ and $\upsilon \in \{p, bp, r\}$.

1. Introduction

By $w$ and $\Omega$, we denote the set of all real valued single and double sequences which are the vector spaces with coordinatewise addition and scalar multiplication. Any vector subspaces of $w$ and $\Omega$ are called as the single and double sequence spaces, respectively. The space $M_u$ of all bounded double sequences is defined by

$$M_u := \left\{ x = (x_{mn}) \in \Omega : \|x\|_\infty = \sup_{m,n \in \mathbb{N}} |x_{mn}| < \infty \right\},$$

which is a Banach space with the norm $\| \cdot \|_\infty$; where $\mathbb{N}$ denotes the set of all positive integers. Consider a sequence $x = (x_{mn}) \in \Omega$. If for every $\varepsilon > 0$ there exists $n_0 = n_0(\varepsilon) \in \mathbb{N}$ and $l \in \mathbb{R}$ such that

$$|x_{mn} - l| < \varepsilon$$

for all $m, n > n_0$ then we call that the double sequence $x$ is convergent in the Pringsheim’s sense to the limit $l$ and write $p - \lim x_{mn} = l$; where $\mathbb{R}$ denotes the real field. By $C_p$ we denote the space of all convergent double sequences in the Pringsheim’s sense. It is well-known that there are such sequences in the space $C_p$ but not in the space $M_u$. So, we can consider the space $C_{bp}$ of the double sequences which are both convergent in the Pringsheim’s sense and bounded, i.e., $C_{bp} = C_p \cap M_u$. A sequence in the space $C_p$ is said to be regularly convergent if it is a single convergent sequence with respect to each index and the set of all such sequences denoted by $C_r$. Also by $C_{bp0}$ and $C_{r0}$, we denote the spaces of all double sequences converging to 0 contained

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in the sequence spaces $C_{bp}$ and $C_r$, respectively. Móricz [7] proved that $C_{bp}$,
$C_{bp0}$, $C_r$ and $C_{r0}$ are Banach spaces with the norm $\| \cdot \|_{\infty}$.

Let us consider the isomorphism $T$ which plays an essential role for the
present study, defined by Zeltser [11, p. 36] as

$$T : \Omega \rightarrow w \quad x \mapsto z = (z_i) := (x_{\psi^{-1}(i)}),$$

(1.1)

where $\psi : \mathbb{N} \times \mathbb{N} \rightarrow \mathbb{N}$ is a bijection defined by

$$
\begin{align*}
\psi[(1, 1)] &= 1, \\
\psi[(1, 2)] &= 2, \quad \psi[(2, 2)] = 3, \quad \psi[(2, 1)] = 4, \\
\vdots \quad \psi[(1, n)] &= (n - 1)^2 + 1, \quad \psi[(2, n)] = (n - 1)^2 + 2, \quad \ldots, \\
\psi[(n, n)] &= (n - 1)^2 + n, \quad \psi[(n, n - 1)] = n^2 - n + 2, \quad \ldots, \quad \psi[(n, 1)] = n^2, \\
& \quad \vdots
\end{align*}
$$

Let us consider a double sequence $x = (x_{mn})$ and define the sequence
$s = (s_{mn})$ via $x$ by

(1.2) 

$$s_{mn} := \sum_{i,j=1}^{m,n} x_{ij} ; \quad (m, n \in \mathbb{N}),$$

which will be used throughout. For the sake of brevity, here and in what
follows, we abbreviate the summations $\sum_{i=1}^{\infty} \sum_{j=1}^{\infty}$, $\sum_{i=1}^{m} \sum_{j=1}^{n}$ and
$\sum_{i=1}^{n} \sum_{j=1}^{n}$ by $\sum_{i,j}$, $\sum_{i,j}^{m,n}$ and $\sum_{i,j}^{n}$, respectively. Then the pair $(x, s)$
and the sequence $s = (s_{mn})$ are called as a double series and the sequence
of partial sums of the double series, respectively. Let $\lambda$ be the space of
double sequences, converging with respect to some linear convergence rule
$v - \lim : \lambda \rightarrow \mathbb{R}$. The sum of a double series $\sum_{i,j} x_{ij}$ with respect to this
rule is defined by $v - \sum_{i,j} x_{ij} := v - \lim s_{mn}$.

Let us define the following sets of double sequences:

$$
\begin{align*}
M_u(t) := \left\{ (x_{mn}) \in \Omega : \sup_{m,n \in \mathbb{N}} |x_{mn}|^{t_{mn}} < \infty \right\}, \\
C_p(t) := \left\{ (x_{mn}) \in \Omega : p - \lim_{m,n \rightarrow \infty} |x_{mn} - l|^{t_{mn}} = 0 \quad \text{for some} \ l \in \mathbb{C} \right\}, \\
C_{0p}(t) := \left\{ (x_{mn}) \in \Omega : p - \lim_{m,n \rightarrow \infty} |x_{mn}|^{t_{mn}} = 0 \right\}, \\
L_u(t) := \left\{ (x_{mn}) \in \Omega : \sum_{m,n} |x_{mn}|^{t_{mn}} < \infty \right\},
\end{align*}
$$
\[ C_{bp}(t) := C_p(t) \cap M_u(t) \quad \text{and} \quad C_{0bp}(t) := C_{0p}(t) \cap M_u(t); \]

where \( t = (t_{mn}) \) is the sequence of strictly positive reals \( t_{mn} \) for all \( m, n \in \mathbb{N} \). In the case \( t_{mn} = 1 \) for all \( m, n \in \mathbb{N} \), \( M_u(t) \), \( C_p(t) \), \( C_{0p}(t) \), \( L_u(t) \), \( C_{bp}(t) \) and \( C_{0bp}(t) \) reduce to the sets \( M_u \), \( C_p \), \( C_{0p} \), \( L_u \), \( C_{bp} \) and \( C_{0bp} \), respectively. Now, we may summarize the knowledge given in some document related to the double sequence spaces. Gökhan and Çolak [4, 5] have proved that \( M_u(t) \) and \( C_p(t) \), \( C_{bp}(t) \) are complete paranormed spaces of double sequences and gave the \( \alpha \)-, \( \beta \)-, \( \gamma \)-duals of the spaces \( M_u(t) \) and \( C_{bp}(t) \). Quite recently, in her PhD thesis, Zeltser [11] has essentially studied both the theory of topological double sequence spaces and the theory of summability of double sequences. Mursaleen and Edely [8] have recently introduced the statistical convergence and Cauchy for double sequences and given the relation between statistical convergent and strongly Cesàro summable double sequences. Nextly, Mursaleen [9] and Mursaleen and Edely [10] have defined the almost strong regularity of matrices for double sequences and applied these matrices to establish a core theorem and introduced the \( M \)-core for double sequences and determined those four dimensional matrices transforming every bounded double sequence \( x = (x_{jk}) \) into one whose core is a subset of the \( M \)-core of \( x \). More recently, Altay and Başar [1] have defined the spaces \( BS, BS(t), CS_p, CS_{bp}, CS_r \) and \( BV \) of double sequences consisting of all double series whose sequence of partial sums are in the spaces \( M_u, M_u(t), C_p, C_{bp}, C_r \) and \( L_u \), respectively, and also examined some properties of those sequence spaces and determined the \( \alpha \)-duals of the spaces \( BS, BV, CS_{bp} \) and the \( \beta(\vartheta) \)-duals of the spaces \( CS_{bp} \) and \( CS_r \) of double series.

In the present paper, we introduce the space \( \mathcal{L}_q \)

\[
\mathcal{L}_q := \left\{ (x_{ij}) \in \Omega : \sum_{i,j} |x_{ij}|^q < \infty \right\}, \quad (1 \leq q < \infty)
\]

of double sequences corresponding to the space \( \ell_q \) of single sequences and examine some properties of the space.

2. The Double Sequence Space \( \mathcal{L}_q \)

In this section, we give the theorem which states that \( \mathcal{L}_q \) is a sequence space and is a Banach space with the norm \( \| \cdot \|_q \), firstly. Subsequent to giving some inclusion relations concerning the space \( \mathcal{L}_q \), we establish that the \( \alpha \)- and \( \gamma \)-duals of a space of double sequences are identical whenever it is solid, and \( \mathcal{L}_q \) is said if \( q > 1 \) and determine the \( \beta(\vartheta) \)-dual of the space \( \mathcal{L}_q \) for \( \vartheta \in \{ p, bp, r \} \) which coincides with the \( \alpha \)- and \( \gamma \)-duals of the space \( \mathcal{L}_q \).
Theorem 2.1. The set \( L_q \) becomes a linear space with the coordinatewise addition and scalar multiplication and \( L_q \) is a Banach space with the norm

\[
\|x\|_q = \left( \sum_{i,j} |x_{ij}|^q \right)^{1/q},
\]

where \( 1 \leq q < \infty \).

Proof. The proof of the first part of the theorem is a routine verification and so we omit the detail.

Furthermore, the statement "a sequence space \( \nu \) is a Banach space with the norm \( \| \cdot \|_\nu \) if and only if the sequence space \( T^{-1}(\nu) = \lambda \) is a Banach space with the norm \( \| \cdot \|_\lambda \)" holds by Boos [3, Corollary 6.3.41]. Therefore, the restriction of the transformation defined by (1.1) to the space \( L_q \) which is norm preserving isomorphism yields the fact that \( L_q = T^{-1}(\ell_q) \) is also a Banach space with the norm \( \| \cdot \|_q \) defined by (2.1) because of \( \ell_q \) is a Banach space.

This step concludes the proof. \( \square \)

Theorem 2.2. Let \( 1 \leq q < s < \infty \). Then, the inclusions \( L_q \subset L_s \subset C_{r0} \subset \mathcal{M}_u \) hold.

Proof. Let us take any \( x = (x_{ij}) \in L_q \). Then, \( \sum_{\max\{i,j\} > n_0} |x_{ij}|^q < \varepsilon < 1 \) for sufficiently large \( n_0 \in \mathbb{N} \). Since \( q < s \), it is obvious that \( |x_{ij}|^q \geq |x_{ij}|^s \) for all \( i, j \in \mathbb{N} \) such that \( \max\{i, j\} > n_0 \). Thus,

\[
\sum_{i,j} |x_{ij}|^s = \sum_{i,j=1}^{n_0} |x_{ij}|^s + \sum_{\max\{i,j\} > n_0} |x_{ij}|^s \\
\leq A + \sum_{\max\{i,j\} > n_0} |x_{ij}|^q \\
\leq A + \varepsilon,
\]

which leads us to the fact that \( x \in L_s \), where \( A = \sum_{i,j=1}^{n_0} |x_{ij}|^s \). Hence, \( L_q \subset L_s \).

Besides one can easily deduce, by means of the suitable restrictions of the isomorphism \( T \) defined by (1.1) and taking into account the fact that the space \( C_{r0} \) consists of all sequences \( x = (x_{mn}) \) such that \( \lim_{\max\{m,n\} \to \infty} x_{mn} = 0 \), from the known inclusions \( \ell_s \subset c_0 \subset \ell_\infty \) for \( 1 \leq s < \infty \) that

\[
T^{-1}(\ell_s) = L_s \subset C_{r0} = T^{-1}(c_0) \subset T^{-1}(\ell_\infty) = \mathcal{M}_u.
\]

This step completes the proof. \( \square \)
The $\alpha$-dual $\lambda^\alpha$, $\beta(v)$-dual $\lambda^{\beta(v)}$ with respect to the $v$-convergence for $v \in \{p, bp, r\}$ and the $\gamma$-dual $\lambda^\gamma$ of a double sequence space $\lambda$ are respectively defined by

$$
\lambda^\alpha := \left\{ (a_{ij}) \in \Omega : \sum_{i,j} |a_{ij}x_{ij}| < \infty \text{ for all } (x_{ij}) \in \lambda \right\},
$$

$$
\lambda^{\beta(v)} := \left\{ (a_{ij}) \in \Omega : v - \sum_{i,j} a_{ij}x_{ij} \text{ exists for all } (x_{ij}) \in \lambda \right\}
$$

and

$$
\lambda^\gamma := \left\{ (a_{ij}) \in \Omega : \sup_{k,l \in \mathbb{N}} \left| \sum_{i,j=1}^{k,l} a_{ij}x_{ij} \right| < \infty \text{ for all } (x_{ij}) \in \lambda \right\}.
$$

It is easy to see for any two spaces $\lambda$, $\mu$ of double sequences that $\mu^\alpha \subset \lambda^\alpha$ whenever $\lambda \subset \mu$ and $\lambda^\alpha \subset \lambda^\gamma$. Additionally, it is known that the inclusion $\lambda^\alpha \subset \lambda^{\beta(v)}$ holds while the inclusion $\lambda^{\beta(v)} \subset \lambda^\gamma$ does not hold, since the $v$-convergence of the sequence of partial sums of a double series does not imply its boundedness.

The space $\lambda$ of double sequences is said to be solid if and only if

$$
\tilde{\lambda} = \{(u_{kl}) \in \Omega : \exists (x_{kl}) \in \lambda \text{ such that } |u_{kl}| \leq |x_{kl}| \text{ for all } k, l \in \mathbb{N} \}
$$

is a bounded set of real numbers.

The space $\lambda$ of double sequences is also said to be monotone if and only if $m_0\lambda \subset \lambda$, where $m_0$ is the span of the set of all sequences of zeros and ones and $m_0\lambda = \{ax = (a_{ij}x_{ij}) : a \in m_0, x \in \lambda\}$. If $\lambda$ is monotone, then $\lambda^\alpha = \lambda^{\beta(v)}$ (cf. Zeltser [11, p. 36]) and $\lambda$ is monotone whenever $\lambda$ is solid.

Prior to giving the theorem which asserts that the $\alpha$- and $\gamma$-duals of a solid space of double sequences are identical, we quote two lemmas which are needed in proving the theorem.

**Lemma 2.3.** [6, Theorem 2, p. 279] A positive term double series converges to its l.u.b. (that is the l.u.b. of its partial sums) if it is bounded above. Otherwise it diverges to $+\infty$.

**Lemma 2.4.** [2, p. 382] A double series is absolutely convergent if and only if the set

$$
\left\{ \sum_{i,j=1}^{m,n} |x_{ij}| : m, n \in \mathbb{N} \right\}
$$

is a bounded set of real numbers.

Now, we may give the theorem
Theorem 2.5. If a given double sequence space $\lambda$ is solid, then the equality $\lambda^\alpha = \lambda^\gamma$ holds.

Proof. To prove the theorem, it is enough to show that the inclusion $\lambda^\gamma \subset \lambda^\alpha$ holds. Suppose that the sequence space $\lambda$ is solid and take any $y = (y_{kl}) \in \lambda^\gamma$. Then,

$$\sup_{m,n \in \mathbb{N}} \left| \sum_{k,l=1}^{m,n} x_{kl} y_{kl} \right| < \infty$$

for any $x = (x_{kl}) \in \lambda$. Now, define the sequence $z = (z_{kl})$ via the sequence $x = (x_{kl}) \in \lambda$ by $z_{kl} := x_{kl} \text{sgn}(x_{kl} y_{kl})$ for all $k, l \in \mathbb{N}$. Then, $z = (z_{kl}) \in \lambda$ since $\lambda$ is solid and $|z_{kl}| \leq |x_{kl}|$ for all $k, l \in \mathbb{N}$. Therefore,

$$\sup_{m,n \in \mathbb{N}} \left| \sum_{k,l=1}^{m,n} x_{kl} y_{kl} \text{sgn}(x_{kl} y_{kl}) \right| = \sup_{m,n \in \mathbb{N}} \left| \sum_{k,l=1}^{m,n} y_{kl} z_{kl} \right| < \infty.$$ 

This shows that the positive term double series $\sum_{k,l} |x_{kl} y_{kl}|$ is bounded which is convergent by Lemma 2.3. Therefore, one can see by Lemma 2.4 that $(x_{kl} y_{kl})_{k,l \in \mathbb{N}} \in L_1$. Since $x \in \lambda$ is arbitrary, $y$ must be in $\lambda^\alpha$, i.e., the inclusion $\lambda^\gamma \subset \lambda^\alpha$ holds.

This step terminates the proof. ∎

As an easy consequence of Theorem 2.5, we have

Corollary 2.6. If $\lambda$ is solid then $\lambda^\alpha = \lambda^{\beta(\nu)} = \lambda^\gamma$.

One can easily observe that the double sequence space $L_q$ is solid, if $q > 1$. This yields to us that the double sequence space $L_q$ is monotone which implies the fact that the $\alpha$- and the $\beta(\nu)$-duals of the space $L_q$ are identical.

Now, we may give the theorem on the $\beta(\nu)$-dual of the space $L_q$.

Theorem 2.7. The $\beta(\nu)$-dual of the space $L_q$ is the space $L_{q'}$, where $q > 1$ and $q^{-1} + q'^{-1} = 1$.

Proof. Let $q > 1$ and $q^{-1} + q'^{-1} = 1$. Let us take any $x \in L_{q'}$ and $y \in L_q$. Consider the inequalities

$$|x_{mn} y_{mn}| \leq \frac{|x_{mn}| q'}{q} + \frac{|y_{mn}| q}{q} \leq |x_{mn}| q' + |y_{mn}| q.$$
satisfied for all \( m, n \in \mathbb{N} \). Therefore, we derive that

\[
\sum_{m,n} |x_{mn}y_{mn}| \leq \sum_{m,n} |x_{mn}|^{q'} + \sum_{m,n} |y_{mn}|q < \infty,
\]

which leads us to the fact that \( x \in L_q^\alpha \), i.e., the inclusions

(2.2) \( L_q' \subset L_q^\alpha \subset L_q^{\beta(v)} \)

hold.

Conversely, take any \( y = (y_{mn}) \in L_q^{\beta(v)} \). For establishing the inclusion \( L_q^{\beta(v)} \subset L_q' \), we use the analogous idea employing by Boos [3, p. 344, Theorem 7.1.11.c] for single sequences. Let us consider the linear functional \( f_n \) and the double sequence \( y^{[n]} \) defined by

\[
f_n : L_q \longrightarrow \mathbb{R} \quad x = (x_{kl}) \longmapsto f_n(x) := \sum_{k,l=1}^n x_{kl}y_{kl}
\]

and

\[
y^{[n]} = \begin{bmatrix}
y_{11} & y_{12} & y_{13} & \cdots & y_{1n} & 0 & \cdots \\
y_{21} & y_{22} & y_{23} & \cdots & y_{2n} & 0 & \cdots \\
y_{31} & y_{32} & y_{33} & \cdots & y_{3n} & 0 & \cdots \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots \\
y_{n1} & y_{n2} & y_{n3} & \cdots & y_{nn} & 0 & \cdots \\
0 & 0 & 0 & \cdots & 0 & 0 & \cdots \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots
\end{bmatrix}
\]

for every \( n \in \mathbb{N} \). Then, since \( y^{[n]} \in L_q' \), we obtain by Hölder’s inequality that

\[
|f_n(x)| \leq \sum_{k,l=1}^n |x_{kl}y_{kl}| = \sum_{k,l} |x_{kl}y_{kl}^{[n]}| \leq \|x\|_q \left\|y^{[n]}\right\|_{q'}
\]

for each \( x = (x_{kl}) \in L_q \) which yields the continuity of the linear functionals \( f_n \). Therefore, we have

(2.3) \( \|f_n\| \leq \left\|y^{[n]}\right\|_{q'} \) for each \( n \in \mathbb{N} \).

Let us consider the sequence \( x^{(n)} = \{x_{kl}^{(n)}\}_{k,l \in \mathbb{N}} \) to prove the reverse inequality, defined by

\[
x_{kl}^{(n)} := \begin{cases} 
\frac{|y_{kl}|^{q'}}{y_{kl}}, & \text{if } y_{kl} \neq 0, \text{ and } k, l \leq n, \\
0, & \text{otherwise}.
\end{cases}
\]
Then, it is clear that \( x^{(n)} \in L_q \) and one can see that

\[
\left\| x^{(n)} \right\|_q = \left( \sum_{k,l=1}^{n} |y_{kl}|^{(q'-1)q} \right)^{1/q} = \left( \sum_{k,l=1}^{n} |y_{kl}|^{q'} \right)^{1/q} = \left( \left\| y^{[n]} \right\|_{q'}^{q'/q} \right).
\]

This leads us to the consequence for all \( n \in \mathbb{N} \) that

\[
\frac{|f_n(x^{(n)})|}{\| x^{(n)} \|_q} = \frac{\sum_{k,l=1}^{n} |y_{kl}|^{q'}}{\| x^{(n)} \|_q} = \left\| y^{[n]} \right\|_{q'}.
\]

Hence,

\[
\left\| y^{[n]} \right\|_{q'} \leq \| f_n \| \quad \text{for all} \quad n \in \mathbb{N}.
\]

Therefore, we have by (2.3) and (2.4) that

\[
\| f_n \| = \left\| y^{[n]} \right\|_{q'} \quad \text{for all} \quad n \in \mathbb{N}.
\]

By applying the Banach-Steinhauss Theorem, one can observe by our hypothesis that the sequence \((f_n)\) of linear functionals converges pointwise. Since \((L_q, \| \cdot \|_q)\) and \((C, | \cdot |)\) are the Banach spaces, the linear functional defined by

\[
f_y : L_q \to \mathbb{R}, \quad x = (x_{kl}) \mapsto f_y(x) := \lim_{n \to \infty} f_n(x) = \sum_{k,l} x_{kl} y_{kl}
\]

is continuous, and

\[
\| f_y \| \leq \sup_{n \in \mathbb{N}} \| f_n \| = \sup_{n \in \mathbb{N}} \left\| y^{[n]} \right\|_{q'} < \infty
\]

holds. Thus, we have \( y \in L_{q'} \), because of

\[
\| f_y \| \leq \left( \sum_{k,l=1}^{n} |y_{kl}|^{q'} \right)^{1/q'} = \left( \sum_{k,l} |y_{kl}|^{q'} \right)^{1/q'} < \infty.
\]

That is to say that the inclusion

\[
L_{q}^{\beta(v)} \subset L_{q'}
\]

holds.

By combining the inclusions (2.2) and (2.5), the desired result immediately follows.

This completes the proof. \( \square \)

As a direct consequence of Theorem 2.7, we have

**Corollary 2.8.** The \( \alpha-, \beta(v)- \) and \( \gamma-\)duals of the space \( L_q \) are the space \( L_{q'} \), where \( q > 1 \) and \( q^{-1} + q'^{-1} = 1 \).
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