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ON THE JORDAN DECOMPOSITION OF TENSORED MATRICES OF JORDAN CANONICAL FORMS

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Abstract. Let k be an algebraically closed field of characteristic $p \geq 0$. We shall consider the problem of finding out a Jordan canonical form of $J(\alpha,\, s) \otimes_k J(\beta,\, t)$, where $J(\alpha,\, s)$ means the Jordan block with eigenvalue $\alpha \in k$ and size s.

1. Introduction

To construct graded local Frobenius algebras over an algebraically closed field k, it is important to find out a Jordan canonical form (simply, JCF) of tensor product of square matrices. In fact, it is known that any graded local Frobenius algebra is of the form of $\Lambda(\varphi,\, \gamma) = T(V)/R(\varphi,\, \gamma)$, where V is a finite dimensional k-vector space, γ an element of $GL(V)$, and $\varphi : V^{\otimes n} \to k$ a k-linear map satisfying several conditions. Further, if we decompose as $(V,\, \gamma) = \bigoplus_i (V_i,\, \gamma_i)$, then the conditions of φ can be described in terms of each $\varphi_{i_1 \dots i_r} : V_{i_1} \otimes \cdots \otimes V_{i_r} \to k$. Then, we have to consider a JCF of $\gamma_{i_1} \otimes \cdots \otimes \gamma_{i_r}$ as an element in $GL(V_{i_1} \otimes \cdots \otimes V_{i_r})$. (For detail, refer to T. Wakamatsu [9]).

Let k be an algebraically closed field of characteristic $p \geq 0$, and $J(\alpha,\, s), J(\beta,\, t)$ Jordan blocks over k. We shall consider the problem of finding out a JCF of $J(\alpha, s) \otimes J(\beta, t)$, where $\otimes$ means $\otimes_k$ $(s \leq t)$.

Over an algebraically closed base field of characteristic zero, this problem has been solved by many authors including T. Harima and J. Watanabe [4], and A. Martsinkovsky and A. Vlassov [7] etc. M. Herschend [5] solve it for extended Dynkin quivers of type $\tilde{\mathbb{A}}_n$, with arbitrary orientation and any n. In this note we solve it for any characteristic $p \geq 0$. That is, we obtain two ways to determine the Jordan decomposition of the tensored matrix $J(\alpha, s) \otimes J(\beta, t)$.

In the case of $\alpha\beta = 0$, the tensored matrix $J(\alpha, s) \otimes J(\beta, t)$ has the same direct sum decomposition as in Theorem 2.0.1 independently of characteristic of the base field k in Proposition 2.1.2. In the case of $\alpha\beta \neq 0$, our problem is reduced to the problem of finding the indecomposable decomposition of R as a $k[\theta]$-module, where R means the quotient ring $k[x, y]/(x^s, y^t)$, $\theta = x + y$ and $k[x, y]$ be a polynomial ring over k . In the section 2.1, we

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regard finding the indecomposable decomposition of R as calculating the partition $\mathbf{c} = (c_1, c_2, \ldots, c_r)$ of st in Lemma 2.1.1. Then, we are able to determine the Jordan decomposition of tensored matrix $J(\alpha, s) \otimes J(\beta, t)$. In the section 2.2, we show another algorithm. The idea is finding out elements that determine the indecomposable decomposition of R as a $k[\theta]$-module. In Theorem 2.2.2, we show that we can find out s homogeneous elements $\omega_0, \omega_1, \ldots, \omega_{s-1}$ of R such that $R = \bigoplus_{i=0}^{s-1} k[\theta]\omega_i$, where the degree of ω_i is i for each $0 \le i \le s-1$. And applying this result, we show an algorithm for computing a JCF of $J(\alpha,\, s) \otimes J(\beta,\, t)$ in Theorem 2.2.9.

2. Main results

Throughout this section, let k be an algebraically closed field. For an integer $s \ge 1$ and an element $\alpha \in k$, let

$$J(\alpha, s) = \begin{pmatrix} \alpha & 1 & & \\ & \ddots & \ddots & \\ & & \alpha & 1 \\ & & & \alpha \end{pmatrix}$$

denote the Jordan block of size $s \times s$ with an eigenvalue α.

Theorem 2.0.1. [7, Theorem 2] *Suppose that k has characteristic zero. Then the following holds for integers $s \le t$ and $\alpha, \beta \in k$:*

$$J(\alpha, s) \otimes J(\beta, t) = \begin{cases} J(0, s)^{\oplus t-s+1} \oplus \bigoplus_{i=1}^{2s-2} J(0, s - \lceil \frac{i}{2} \rceil) & \textit{if} \quad \alpha = 0 = \beta \\ J(0, s)^{\oplus t} & \textit{if} \quad \alpha = 0 \neq \beta \\ J(0, t)^{\oplus s} & \textit{if} \quad \alpha \neq 0 = \beta \\ \bigoplus_{i=1}^{s} J(\alpha\beta, s + t + 1 - 2i) & \textit{if} \quad \alpha \neq 0 \neq \beta \end{cases}.$$

Remark 1. If one of the eigenvalues α and β equals zero, then the tensored matrix $J(\alpha, s) \otimes J(\beta, t)$ has the same direct sum decomposition as in Theorem 2.0.1 independently of characteristic of the base field k (Proposition 2.1.2).

Theorem 2.0.2. *There is an algorithm to determine the Jordan decomposition of the tensored matrix $J(\alpha, s) \otimes J(\beta, t)$, which has an independent description of the characteristic of the base field k.*

Remark 2. (1)The matrix $J(\alpha, s)$ represents the action of X on $k[X]/(X - \alpha)^s$ as a $k[X]$-module.

(2)The tensored matrix $J(\alpha, s) \otimes J(\beta, t)$ is triangular. Therefore its eigenvalue is $\alpha\beta$.

(3)One has an isomorphism

$$k[X]/(X - \alpha)^s \otimes k[Y]/(Y - \beta)^t \cong k[X, Y]/((X - \alpha)^s, (Y - \beta)^t)$$

of k-algebras.

Tensored matrix $J(\alpha, s) \otimes J(\beta, t)$ represents the action of XY on $k[X, Y]/((X-\alpha)^s, (Y-\beta)^t)$ as a $k[XY]$-module.

2.1. The method for calculating numerical values of tensored matrices.

Lemma 2.1.1. *Put $R = k[X, Y]/((X-\alpha)^s, (Y-\beta)^t)$, which we regard as a $k[Z]$-module through the map $k[Z] \to R$ given by $Z \mapsto XY$. Then there is a sequence of integers such that $c_1 \geq c_2 \geq \cdots \geq c_r \geq 1$*

$$R \cong \bigoplus_{i=1}^{r} k[Z]/(Z-\alpha\beta)^{c_i}$$

of $k[Z]$-modules.

This means that $J(\alpha, s) \otimes J(\beta, t) = \bigoplus_{i=1}^{r} J(\alpha\beta, c_i)$. We can regard $\mathbf{c} = (c_1, c_2, \ldots, c_r)$ as a partition of st in obvious manner. The main problem is to determine the partition $\mathbf{c}$. For this purpose let $\mathbf{b} = (b_1, b_2, \ldots, b_{r'})$ be the partition conjugate to $\mathbf{c}$. Put $z = Z - \alpha\beta$. Note that $b_i = \#\{j | c_j \geq i\} = \dim_k(z^{i-1}R/z^iR)$. Setting $a_i = \dim_k(R/z^iR)$, we have $b_i = a_i - a_{i-1}$. Therefore, it is sufficient that we calculate the value of a_i for each case.

If one of the eigenvalues α and β equals zero, then the result is independent of the characteristic of k as we show in the next proposition.

Proposition 2.1.2. *We have the following equalities;*

$$a_i = \begin{cases} (s+t)i - i^2 \quad (1 \leq i \leq s) & \text{if} \quad \alpha = 0 = \beta \\ ti \quad (1 \leq i \leq s) & \text{if} \quad \alpha = 0 \neq \beta \\ si \quad (1 \leq i \leq t) & \text{if} \quad \alpha \neq 0 = \beta \end{cases}.$$

Therefore we get

$$J(\alpha, s) \otimes J(\beta, t) = \begin{cases} J(0, s)^{\oplus t-s+1} \oplus \bigoplus_{i=1}^{2s-2} J(0, s - \lceil \frac{i}{2} \rceil) & \text{if} \quad \alpha = 0 = \beta \\ J(0, s)^{\oplus t} & \text{if} \quad \alpha = 0 \neq \beta \\ J(0, t)^{\oplus s} & \text{if} \quad \alpha \neq 0 = \beta \end{cases}.$$

Proof. Put $x = X - \alpha$ and $y = Y - \beta$.

(1) The case $\alpha = 0 = \beta$:
Since $R/z^iR = k[x, y]/(x^s, y^t, (xy)^i)$, we have $a_i = (s+t)i - i^2$.

(2) The case $\alpha = 0 \neq \beta$:
Since $R/z^iR = k[x, y]/(x^s, y^t, x^i)$, we have $a_i = ti$.

(3) The case $\alpha \neq 0 = \beta$:
Since $R/z^iR = k[x, y]/(x^s, y^t, y^i)$, we have $a_i = si$. ∎

In the case of $\alpha \neq 0 \neq \beta$, then we have the following isomorphism of k-algebras, given by $X \mapsto x+\alpha$, $Y \mapsto \frac{-\alpha\beta}{y-\alpha}$:

$$k[X,Y]/((X-\alpha)^s,(Y-\beta)^t,(XY-\alpha\beta)^u) \cong k[x,y]/(x^s,y^t,(x+y)^u).$$

Using this isomorphism together with [3, Proposition 4.4][4, Proposition 8], we have the following proposition in the case of characteristic zero.

Proposition 2.1.3. *Suppose that $\alpha \neq 0 \neq \beta$ and that k has characteristic zero. Then we have*

$$\mathbf{b} = (\underbrace{s,s,\dots,s}_{t-s+1}, s-1, s-1, s-2, s-2, \dots, 1, 1).$$

Therefore we get $J(\alpha,s) \otimes J(\beta,t) = \bigoplus_{i=1}^{s} J(\alpha\beta, s+t+1-2i)$.

Proof. Since the linear element $x+y \in k[x,y]/(x^s,y^t)$ is a strong Lefschetz element [4]. Namely, the multiplication map $\times(x+y)^u : k[x,y]/(x^s,y^t)_i \to k[x,y]/(x^s,y^t)_{i+u}$ is either injective or surjective, for each $0 \leq i \leq s+t-2$. Then, we can easily compute $\dim_k(k[x,y]/(x^s,y^t,(x+y)^u))$ for each $1 \leq u \leq s+t-1$. The assertion follows from this. $\square$

We consider in the rest the case where $\alpha \neq 0 \neq \beta$ and that k is of positive characteristic p. Put $S = k[x,y]$, $R = k[x,y]/(x^s,y^t)$ and $A^{(u)} = R/(x+y)^u R$. To determine $a_u = \dim_k(A^{(u)})$, we may assume that $s \leq t \leq u$ without loss of generality. For each integer u satisfying $s \leq t \leq u \leq s+t-1$, we describe

$$(x+y)^u \equiv \binom{u}{s-1} x^{s-1}y^{u-s+1} + \binom{u}{s-2} x^{s-2}y^{u-s+2} + \cdots + \binom{u}{u-t+1} x^{u-t+1}y^{t-1} \pmod{(x^s,y^t)}.$$

We set $q_1 = \binom{u}{s-1}$, $q_2 = \binom{u}{s-2}, \cdots, q_r = \binom{u}{u-t+1}$ and $r = s+t-1-u$.

We obtain the representation matrix of $R \xrightarrow{(x+y)^u} R$ with respect to the natural base $\{1, x, y, x^2, xy, y^2, \dots, x^{s-1}y^{t-1}\}$ as follows;

$$\begin{pmatrix} H_0 & & & & & \\ & H_1 & & & & \\ & & H_2 & & & \\ & & & \ddots & & \\ & & & & H_{r-2} & \\ & & & & & H_{r-1} \end{pmatrix},$$

where

$$H_i = \begin{pmatrix} q_{i+1} & q_i & \cdots & q_1 \\ q_{i+2} & q_{i+1} & \cdots & q_2 \\ \vdots & \vdots & \ddots & \vdots \\ q_r & q_{r-1} & \cdots & q_{r-i} \end{pmatrix}.$$

For each $0 \le i \le r-1$ the matrix H_i is an $(r-i) \times (i+1)$ matrix whose entries are integers. We denote by $I_{i+1}(H_i)$ the ideal of $\mathbb{Z}$ generated by $(i+1)$-minors of H_i for $0 \le i \le r-1$. Obviously there exists an integer $\delta_i \ge 0$ such that $I_{i+1}(H_i) = \delta_i \mathbb{Z}$. From the argument in the case of characteristic zero in [3, Proposition 4.4], we have $I_{i+1}(H_i) \otimes_{\mathbb{Z}} \mathbb{Q} \ne 0$, particularly $\delta_i \ne 0$, for any $0 \le i \le \lfloor (r-1)/2 \rfloor$.

Proposition 2.1.4. *Under the same notation as above, for each u satisfying $1 \le s \le t \le u \le s+t-1$, and for each i satisfying $0 \le i \le \lfloor (r-1)/2 \rfloor (r = s+t-1-u)$, the following equalities hold;*

$$\delta_i = \gcd\{S_{\lambda^j}(\underbrace{1, 1, \ldots, 1}_{u}) | j = (j_1, j_2, \ldots, j_{i+1}), 1 \le j_1 < j_2 < \ldots < j_{i+1} \le r-i\},$$

where λ^j is the partition conjugate to $\mu^j = (s-j_1, s-j_2-1, \ldots, s-j_{i+1}-i)$, and S_{λ^j} is the Schur polynomial.

Proof. Computation using Jacobi-Trudi formula [2] ,[6]. □

Let

$$0 \to S(-a) \oplus S(-b) \to S(-s) \oplus S(-t) \oplus S(-u) \xrightarrow{(x^s, y^t, (x+y)^u)} S \to A^{(u)} \to 0$$

be a minimal graded S-free resolution of $A^{(u)}$, where $1 \le s \le t \le u \le a \le b$. The Hilbert-Burch theorem implies that $a + b = s + t + u$, and the Hilbert series of $A^{(u)}$ is given as

$$H_{A^{(u)}}(w) = \frac{1 - w^s - w^t - w^u + w^a + w^b}{(1-w)^2}.$$

It follows from this that $\dim_k(A^{(u)}) = st + su + tu - ab$. Letting $i_0 = \min\{i | \delta_i \equiv 0 \pmod{p}\}$, we get $a = u + i_0$ and $b = s + t - i_0$, since a is the least value of degrees of relations of $(x^s, y^t, (x+y)^u)$. Thus, we can calculate the dimension of the k-vector space $A^{(u)}$, and hence the indecomposable decomposition of $J(\alpha, s) \otimes J(\beta, t)$.

Theorem 2.1.5. *We are able to compute a Jordan canonical form of $J(\alpha,\ s) \otimes J(\beta,\ t)$ by taking the following steps:*

1. *Every $\delta_\bullet$ is determined.*
2. *For each $1 \le u \le s+t-1$, a_u is determined.*
3. *The partition $\mathbf{b}$ is determined.*

(4) *The partition* $\mathbf{c}$ *is determined.*
(5) *The Jordan decomposition of tensored matrix* $J(\alpha, s) \otimes J(\beta, t)$ *is determined.*

From the discussion in Theorem 2.1.5, one immediately obtains the following.

Theorem 2.1.6. *The tensored matrix* $J(\alpha, s) \otimes J(\beta, t)$ *has the same direct sum decomposition as in Theorem 2.0.1 if either* $\mathrm{char}(k) \geq s + t - 1$ *or* $I_{i+1}(H_i) \otimes_{\mathbb{Z}} k \neq 0$ *for any* $0 \leq i \leq \lfloor \frac{r-1}{2} \rfloor$.

2.2. **The method for finding out elements that determine the indecomposable decomposition.**

In this subsection, we show another algorithm for computing a JCF of $J(\alpha,\, s) \otimes J(\beta,\, t)$ via finding the indecomposable decomposition. We have already got the answer of our problem for case of $\alpha\beta = 0$ by Proposition 2.1.2, so we discuss only for case of $\alpha\beta \neq 0$.

We consider the indecomposable decomposition of $k[X, Y]/((X - \alpha)^s,\, (Y - \beta)^t)$ as a $k[XY]$-module. As we stated in 2.1, we have an isomorphism $k[X, Y]/((X - \alpha)^s,\, (Y - \beta)^t) \cong k[x, y]/(x^s, y^t)$. Put $R = k[x, y]/(x^s, y^t)$, and $\theta = x + y$. Thus, our problem is reduced to that of finding the indecomposable decomposition of R as a $k[\theta]$-module.

It is clear that R is a finite dimensional graded Artinian k-algebra. So we write $R = \bigoplus_{i=0}^{s+t-2} R_i$. And we immediately know that $\dim_k(R_i)$ are written as $(1, 2, \ldots, s-1, \underbrace{s, \ldots, s}_{t-s+1}, s-1, \ldots, 1)$ for $0 \leq i \leq s+t-2$. We often use a figure for R (Figure 1).

The subalgebra $k[\theta]$ of R is *uniserial*, which means that $k[\theta]$ has the only composition series as a $k[\theta]$-module. We denote by n the *nilpotency* of θ (i.e. $\theta^n \neq 0$ and $\theta^{n+1} = 0$). And then, we can choose $\langle 1,\, \theta,\, \cdots,\, \theta^n \rangle$ as a k-basis of $k[\theta]$. By easy calculation, we have the following inequality on n:

Lemma 2.2.1. *We have* $t - 1 \leq n \leq s + t - 2$*. In particular,* $n = s + t - 2$ *if* $p = 0$.

We describe the subalgebra $k[\theta]$ of R in the figure for R by drawing a polygonal line (Figure 2).

Since the algebra $k[\theta]$ is uniserial, any indecomposable summand M of R as a $k[\theta]$-module can be written as $k[\theta]\omega$ for some element ω in R. Hence we can write the indecomposable decomposition of R as a $k[\theta]$-module such as:

$$R = \bigoplus_{i=1}^{r} k[\theta]\omega_i \quad (\omega_i \in R). \tag{$*$}$$

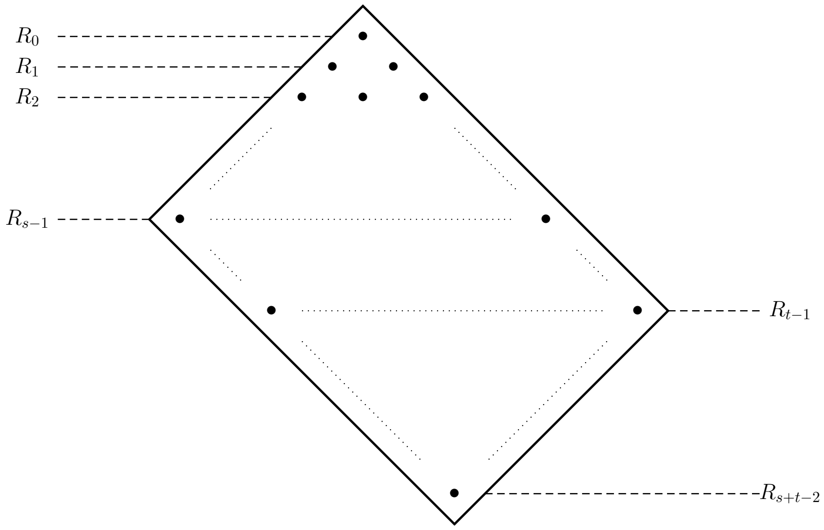

FIGURE 1. An illustration of a basis of R. A bullet $\bullet$ stands for a base of R.

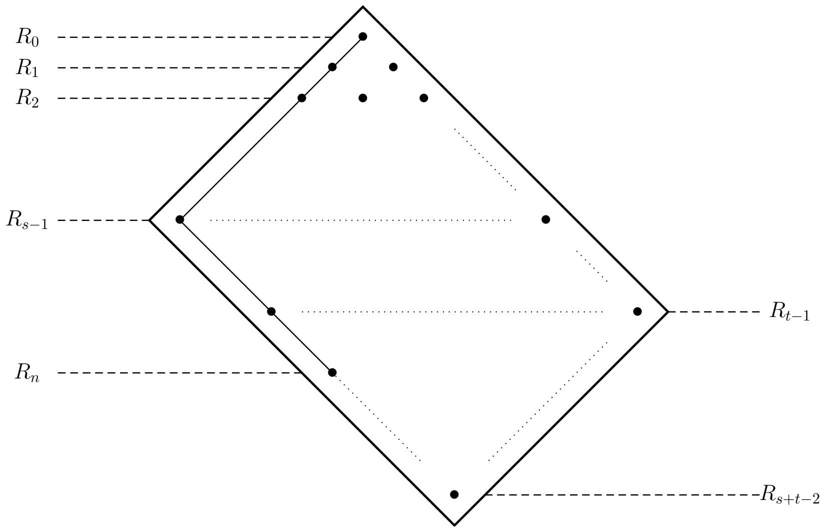

FIGURE 2. An illustration of the subalgebra $k[\theta]$ of R. We consider the bullets on the polygonal line as $\langle 1, \theta, \cdots, \theta^n \rangle$.

We shall call each element ω_i a *generator* (for an indecomposable summand of R), and the set $\{\omega_1, \ldots, \omega_r\}$, which consists of the generators in $(*)$, a *generating set* (for the indecomposable decomposition of R). A generating set is not unique. However, we can prove the number of generators and that there exists the generating set which consists of homogeneous elements.

Theorem 2.2.2. *There exists a generating set $\{\omega_0, \omega_1, \ldots, \omega_{s-1}\}$ whose generator ω_i is an i-th degree homogeneous element. Hence,*

$$R = \bigoplus_{i=0}^{s-1} k[\theta]\omega_i \quad (\omega_i \in R_i).$$

In order to prove this theorem, we have to prepare some lemmas and notations.

For a uniserial $k[\theta]$-submodule M of R generated by some homogeneous elements of R, we denote by $\sigma(M)$ the *socle degree* of M as a $k[\theta]$-module. In short, $\sigma(M) = d$ if $\mathrm{soc}_{k[\theta]}(M) \subseteq R_d$. For example, $\sigma(k[\theta]) = n$. And if $\theta^n x \neq 0$, then $\sigma(k[\theta]x) = n + 1$. The following lemma is checked easily:

Lemma 2.2.3. *Let ζ, η be homogeneous elements of R. If $\sigma(k[\theta]\zeta) \neq \sigma(k[\theta]\eta)$, then $k[\theta]\zeta \cap k[\theta]\eta = \{0\}$ holds. Hence $k[\theta]\zeta + k[\theta]\eta = k[\theta]\zeta \oplus k[\theta]\eta$.*

Lemma 2.2.4. *Let κ be a homogeneous element of R, and put $d = \sigma(k[\theta]\kappa)$. If $t - 1 \leq d < s + t - 2$, then,*

$$\sum_{i=0}^{s+t-2-d} k[\theta]\kappa x^i = \bigoplus_{i=0}^{s+t-2-d} k[\theta]\kappa x^i.$$

Proof. Put $d' = s + t - 2 - d$. And let the degree of κ be m. We now check $\theta^{d-m}\kappa x^{d'} \neq 0$. Since $\theta^{d-m}\kappa$ is an element of R_d, whose dimension as a k-vector space is $d' + 1$, we can write

$$\theta^{d-m}\kappa = \sum_{i=0}^{d'} c_i x^{s-1-i} y^{t-1-d'+i} \quad (c_i \in k).$$

Then we have $c_i + c_{i+1} = 0$ for each i, because $\theta \times \theta^{d-m}\kappa = 0$ holds. Hence we find that all c_i are non-zero. Therefore $\theta^{d-m}\kappa \times x^{d'} = c_{d'} x^{s-1} y^{t-1} \neq 0$. Applying Lemma 2.2.3, we finish the proof of this lemma. □

The multiplication map $\times\theta^j : R_i \to R_{i+j}$ is a k-linear map. We denote by $K(i, i+j)$ the kernel of this map, and by $M(i, i+j)$ the matrix representation with respect to the canonical bases.

Lemma 2.2.5. *For each $0 \leq i \leq s - 1$, we have the following:*

(1) *The map $\times\theta^{t-1-i} : R_i \to R_{t-1}$ is injective.*
(2) *The map $\times\theta^{s+t-1-2i} : R_i \to R_{s+t-1-i}$ is not injective.*

Hence, any non-zero element $\kappa_i \in K(i, s+t-1-i) \subseteq R_i$ satisfies both $\theta^{s+t-1-2i}\kappa_i = 0$ and $\theta^{t-1-i}\kappa_i \neq 0$.

Proof. (1): The map $\times\theta^{t-1-i} : R_i \to R_{t-1}$ is represented by the $s \times (i+1)$ matrix:

$$M(i,\, t-1) = \begin{pmatrix} \binom{t-1-i}{t-s} & \binom{t-1-i}{t-s-1} & \cdots & \cdots & \binom{t-1-i}{t-s-i} \\ \binom{t-1-i}{t-s+1} & \binom{t-1-i}{t-s} & \cdots & \cdots & \binom{t-1-i}{t-s+1-i} \\ \vdots & \vdots & & & \vdots \\ \binom{t-1-i}{t-1-i} & \binom{t-1-i}{t-2-i} & \cdots & \cdots & \binom{t-1-i}{t-1-2i} \\ 0 & \binom{t-1-i}{t-1-i} & \cdots & \cdots & \binom{t-1-i}{t-2i} \\ \vdots & \ddots & \ddots & & \vdots \\ \vdots & & \ddots & \ddots & \vdots \\ 0 & \cdots & \cdots & 0 & \binom{t-1-i}{t-1-i} \end{pmatrix}.$$

Hence the map is injective since the rank of $M(i,\, t-1)$ is $i+1$.

(2): It is clear; because $i+1 = \dim_k R_i > \dim_k R_{s+t-1-i} = i$. $\square$

We now prove Theorem 2.2.2.

Proof of Theorem 2.2.2. We put $n_0 = n$ and $m_0 = s+t-2-n_0$. If $m_0 > 0$, then we have

$$\sum_{i_0=0}^{m_0} k[\theta]x^{i_0} = \bigoplus_{i_0=0}^{m_0} k[\theta]x^{i_0} \subseteq R$$

by Lemma 2.2.4. If this direct sum coincides with R, then we finish the proof. Suppose not. By Lemma 2.2.5, we can take an element $\kappa_{(1)} \in K(m_0+1,\, n_0)$, and then we have $t-1 \leq \sigma(k[\theta]\kappa_{(1)}) \leq n_0 - 1$. We put $n_1 = \sigma(k[\theta]\kappa_{(1)})$ and $m_1 = (n_0-1) - n_1$. If $m_1 > 0$, then we have

$$(\bigoplus_{i_0=0}^{m_0} k[\theta]x^{i_0}) + (\sum_{i_1=0}^{m_1} k[\theta]\kappa_{(1)}x^{i_1}) = \bigoplus_{i_0=0}^{m_0} k[\theta]x^{i_0} \oplus \bigoplus_{i_1=0}^{m_1} k[\theta]\kappa_{(1)}x^{i_1} \subseteq R$$

from Lemma 2.2.4. Thus, we can construct the direct sum of $k[\theta]$-submodules of R. However, since R is finite dimensional, this construction will be over in finite steps. And it is clear that this construction finishes just when s-th direct summand is constructed. By *the Krull-Remak-Schmidt theorem*, this decomposition is the indecomposable decomposition of R as a $k[\theta]$-module. And this argument does work if some m_i is zero. $\square$

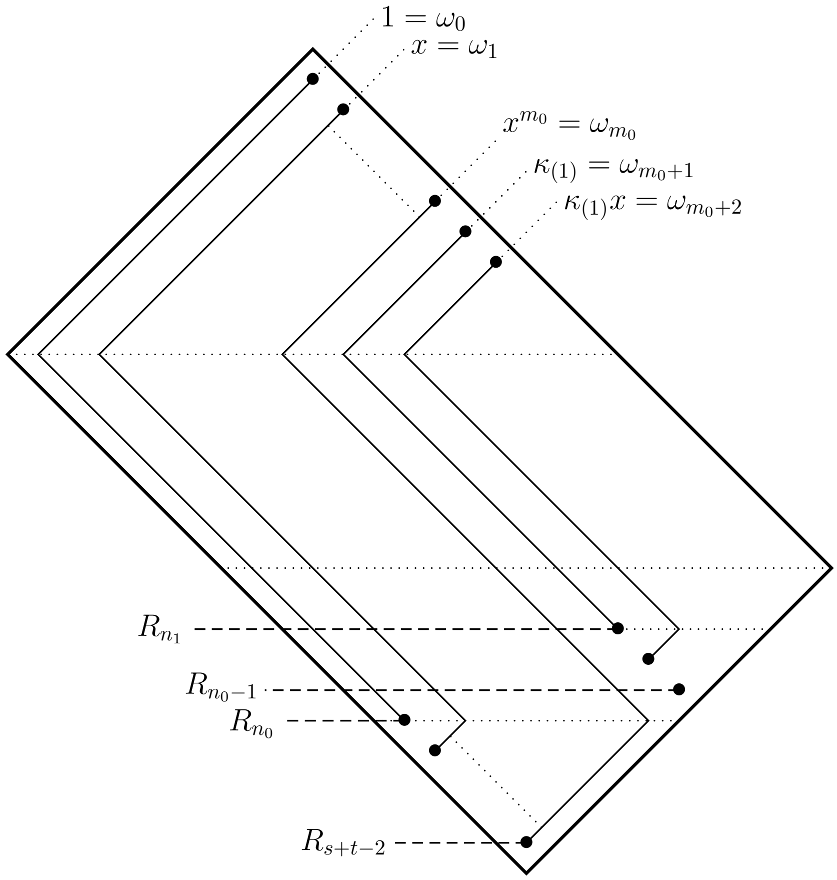

FIGURE 3. Construction of $k[\theta]\omega_i$. Each polygonal line stands for $k[\theta]\omega_i$.

Remark 3. (1) This proof gives concretely the indecomposable summands of R such as:

$$\begin{aligned} &k[\theta],\, k[\theta]x,\, \ldots,\, k[\theta]x^{m_0}, \\ &\qquad\qquad k[\theta]\kappa_{(1)},\, k[\theta]\kappa_{(1)}x,\, \ldots,\, k[\theta]\kappa_{(1)}x^{m_1}, \\ &\qquad\qquad\qquad\qquad \cdots\cdots \\ &\qquad\qquad\qquad\qquad k[\theta]\kappa_{(r-1)},\, k[\theta]\kappa_{(r-1)}x,\, \ldots,\, k[\theta]\kappa_{(r-1)}x^{m_{r-1}}, \end{aligned}$$

where $\kappa_{(i)}$ means some element in $K(\sum_{j=0}^{i-1}(m_j+1),\, n_{i-1})$ and $m_i = (n_{i-1} - 1) - n_i$, $n_i = \sigma(k[\theta]\kappa_{(i)})$. Thus, these $\kappa_{(i)}$, m_i, n_i are determined by the following order:

$$(n =)\, n_0 \to m_0 \to \kappa_{(1)} \to n_1 \to m_1 \to \kappa_{(2)} \to \cdots \to n_{i-1} \to m_{i-1} \to \kappa_{(i)} \to \cdots .$$

(Then we define $n_{-1} = s+t-1$, $m_{-1} = 0$, and $\kappa_{(0)} = 1_R$ for convenience).

(2) We have to discuss on whether the value of $n_i = \sigma(k[\theta]\kappa_{(i)})$ depends on the choice of an element $\kappa_{(i)} \in K(\sum_{j=0}^{i-1}(m_j+1),\, n_{i-1})$. However, we immediately find that the numbers $(n_0, n_1, \dots, n_{r-1})$ have to be unique by the uniqueness of the indecomposable decomposition. Therefore we can choose $\kappa_{(i)}$ free.

(3) Theorem 2.2.2 declares the number of Jordan blocks of $J(\alpha, s) \otimes J(\beta, t)$ is s.

Definition 1. Thus, the particular indecomposable summands

$$(k[\theta] =)\, k[\theta]\kappa_{(0)},\, k[\theta]\kappa_{(1)},\, \dots,\, k[\theta]\kappa_{(r-1)}$$

of R characterize the indecomposable decomposition of R. So, we shall call each $k[\theta]\kappa_{(i)}$ a *leading module* (of R). And we call the number of the indecomposable summands of R whose lengths are equal to that of $k[\theta]\kappa_{(i)}$ the *leading degree* of $k[\theta]\kappa_{(i)}$.

By this result, if there are r leading modules $k[\theta]\kappa_{(0)}, k[\theta]\kappa_{(1)}, \dots, k[\theta]\kappa_{(r-1)}$, then we have

$$J(\alpha, s) \otimes J(\beta, t) = \bigoplus_{i=0}^{r-1} J(\alpha\beta,\, \ell_i)^{\oplus d_i},$$

where ℓ_i and d_i mean the length and leading degree of $k[\theta]\kappa_{(i)}$ respectively.

Next, we show a good way to compute a JCF of $J(\alpha, s) \otimes J(\beta, t)$. To compute it, we find the lengths and the leading degrees of the leading modules.

For each $0 \le i \le s-1$, we define a function such as

$$D_p(i) = \begin{cases} 0 & \text{(if the map } \times\theta^{s+t-2-2i} : R_i \to R_{s+t-2-i} \text{ is bijective)} \\ 1 & \text{(if the map } \times\theta^{s+t-2-2i} : R_i \to R_{s+t-2-i} \text{ is not bijective)} \end{cases}.$$

And we put

$$\varDelta_p = (D_p(0),\, D_p(1),\, \dots,\, D_p(s-1)).$$

Remark 4. By Lemma 2.2.5 (1), we have known the map $\times\theta^{t-s} : R_{s-1} \to R_{t-1}$ is always injective (hence, bijective) independently of the value of characteristic p. So $D_p(s-1) = 0$ holds.

By Theorem 2.2.2, we can assume that R is of the form of $\bigoplus_{i=0}^{s-1} k[\theta]\omega_i$, i.e. any base of R is that of $\theta^j\omega_i$. Then the following lemmas hold:

Lemma 2.2.6. *If an indecomposable summand $k[\theta]\omega_i$ is a leading module and $D_p(i) = 0$. Then we have the following:*

(1) *$\sigma(k[\theta]\omega_i) = s + t - 2 - i$. Hence the length and the leading degree of $k[\theta]\omega_i$ are $s + t - 1 - 2i$ and one respectively.*

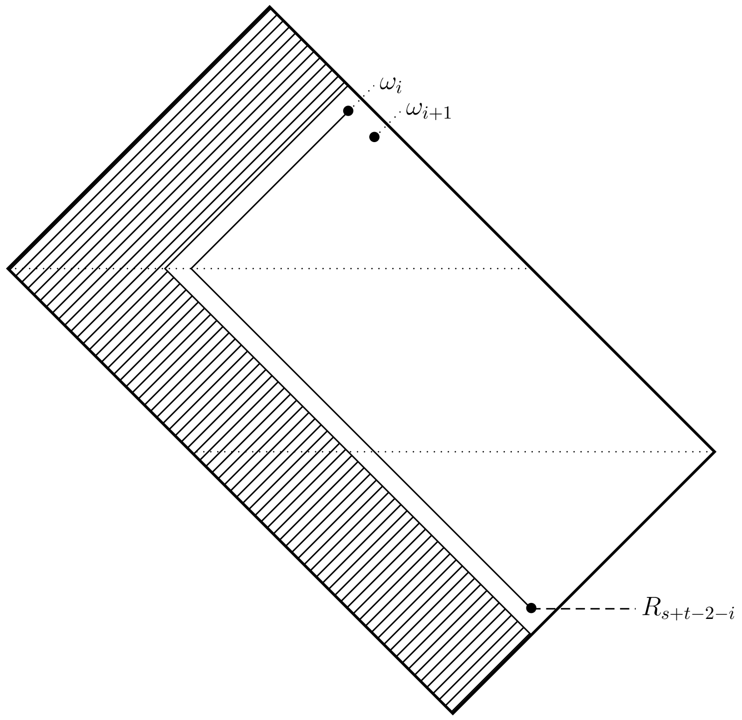

FIGURE 4. The result of Lemma 2.2.6.

(2) *The next indecomposable summand* $k[\theta]\omega_{i+1}$ *is a leading module.*

Proof. (1): The map $\times\theta^{s+t-2-2i} : R_i \to R_{s+t-2-i}$ is bijective by assumption. This procedures $\theta^{s+t-2-2i}\omega_i \neq 0$ for the generator ω_i. Hence $\sigma(k[\theta]\omega_i)$ is $s+t-2-i$, and the other statements hold clearly.

(2): It is trivial since the leading degree of $k[\theta]\omega_i$ is one.

□

Lemma 2.2.7. *If an indecomposable summand* $k[\theta]\omega_i$ *is a leading module,* $D_p(i) = D_p(i+1) = \cdots = D_p(i+f-1) = 1\,(f > 0)$, *and* $D_p(i+f) = 0$. *Then we have the following:*

(1) $\sigma(k[\theta]\omega_i) = s+t-2-i-f$. *Hence the length and the leading degree of* $k[\theta]\omega_i$ *are* $s+t-1-2i-f$ *and* $f+1$ *respectively.*
(2) *The indecomposable summand* $k[\theta]\omega_{i+f+1}$ *is a leading module.*

Proof. (1): Put $\nu = \sigma(k[\theta]\omega_i)$. Since $D_p(i+f) = 0$, we have $\theta^{s+t-2-2(i+f)} \times \theta^f\omega_i = \theta^{s+t-2-2i-f}\omega_i \neq 0$. So $s+t-2-i-f \leq \nu \leq s+t-2-i$ holds. Put $\mu = \nu - (s+t-2-i-f)$ and suppose $\mu > 0$. Then

$$\langle \theta^{s+t-2-(i+f-\mu)}\omega_0, \ldots, \theta^{s+t-2-2(i+f-\mu)}\omega_{i+f-\mu}\rangle$$

is a basis of R_ν because the socle of the leading module $K[\theta]\omega_i$ is in R_ν. Now $\langle \theta^{i+f-\mu}\omega_0, \ldots, \omega_{i+f-\mu}\rangle$ is a basis of $R_{i+f-\mu}$. Hence it is shown

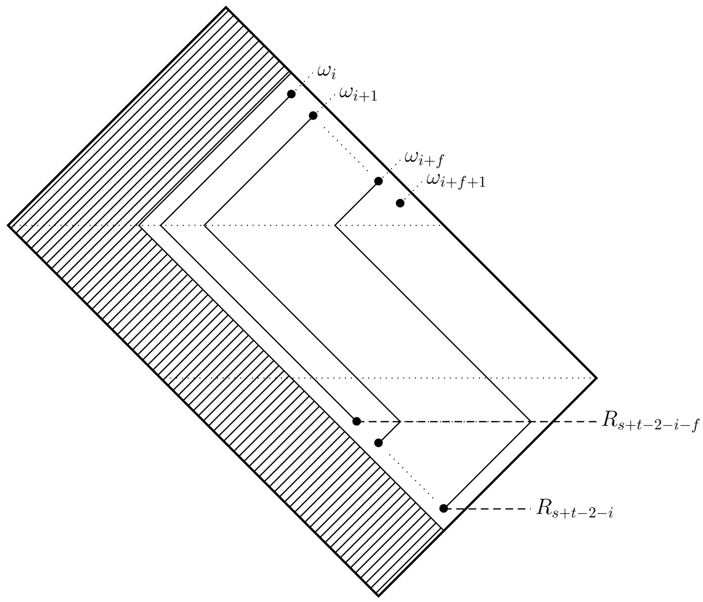

FIGURE 5. The result of Lemma 2.2.7.

that $\times\theta^{s+t-2-2(i+f-\mu)} : R_{i+f-\mu} \to R_{s+t-2-(i+f-\mu)}$ is bijective. However, this contradicts $D_p(i+f-\mu)=1$. Therefore $\nu = s+t-2-i-f$. And the other statements hold clearly.

(2): It is trivial from (1). $\square$

Since the indecomposable summand $k[\theta]\omega_0$ is a leading module, we can apply Lemma 2.2.6 and 2.2.7 to the components of Δ_p inductively. Thus, via the sequence Δ_p, we can compute the lengths and the leading degrees of the leading modules concretely:

Theorem 2.2.8. *We can compute a JCF of $J(\alpha, s) \otimes J(\beta, t)$ by using the sequence Δ_p.*

And we can easily compute the determinant $D(i)$ of the linear map $\times\theta^{s+t-2-2i} : R_i \to R_{s+t-2-i}$. In fact, the matrix representation $M(i, s+t-2-i)$ is of the form of

$$\begin{pmatrix} \binom{s+t-2-2i}{t-1-i} & \binom{s+t-2-2i}{t-2-i} & \cdots & \cdots & \binom{s+t-2-2i}{t-1-2i} \\ \binom{s+t-2-2i}{t-i} & \binom{s+t-2-2i}{t-1-i} & \cdots & \cdots & \binom{s+t-2-2i}{t-2i} \\ \vdots & \vdots & \ddots & & \vdots \\ \vdots & \vdots & & \ddots & \vdots \\ \binom{s+t-2-2i}{t-1} & \binom{s+t-2-2i}{t-2} & \cdots & \cdots & \binom{s+t-2-2i}{t-1-i} \end{pmatrix}.$$

If $p = 0$, it is shown by P. C. Roberts [8] that the determinant of the matrix of this form is computed as follows:

$$D(i) = \prod_{j=0}^{i} \frac{\binom{s+t-2-2i+j}{t-1-i}}{\binom{t-1-i+j}{t-1-i}}.$$

And this is true if $p > 0$, because $D(i)$ is an integer.

Thus, we get an algorithm for computing a JCF of $J(\alpha, s) \otimes J(\beta, t)$:

Theorem 2.2.9. *We are able to compute a JCF of $J(\alpha, s) \otimes J(\beta, t)$ by taking the following steps:*

(1) *Computing $D(i)$ for each $0 \le i \le s-1$.*
(2) *Computing the sequence $\varDelta_p$. $D_p(i) = 0$ iff $D(i) \not\equiv 0\,(\mathit{mod}\ p)$.*
(3) *Applying Theorem 2.2.8.*

Example 2.2.10. Let us compute a JCF of $J(\alpha, 4) \otimes J(\beta, 5)\,(\alpha\beta \neq 0)$. The determinants $D(i)$ are

$$D(0) = \frac{\binom{7}{4}}{\binom{4}{4}} = 5{\cdot}7,\, D(1) = \frac{\binom{5}{3}\binom{6}{3}}{\binom{3}{3}\binom{4}{3}} = 2{\cdot}5^2,\, D(2) = \frac{\binom{3}{2}\binom{4}{2}\binom{5}{2}}{\binom{2}{2}\binom{3}{2}\binom{4}{2}} = 2{\cdot}5,\, D(3) = 1.$$

So the sequence $\varDelta_p$ is

$$\begin{aligned} \varDelta_p &= (0,\, 0,\, 0,\, 0) \ \ (p \neq 2,\, 5,\, 7), \\ \varDelta_2 &= (0,\, 1,\, 1,\, 0), \\ \varDelta_5 &= (1,\, 1,\, 1,\, 0), \\ \varDelta_7 &= (1,\, 0,\, 0,\, 0). \end{aligned}$$

Therefore

$$J(\alpha, 4) \otimes J(\beta, 5) = \begin{cases} J(\alpha\beta, 8) \oplus J(\alpha\beta, 6) \oplus J(\alpha\beta, 4) \oplus J(\alpha\beta, 2) & (p \neq 2, 5, 7) \\ J(\alpha\beta, 8) \oplus J(\alpha\beta, 4)^{\oplus 3} & (p = 2) \\ J(\alpha\beta, 5)^{\oplus 4} & (p = 5) \\ J(\alpha\beta, 7)^{\oplus 2} \oplus J(\alpha\beta, 4) \oplus J(\alpha\beta, 2) & (p = 7) \end{cases}.$$

If $p = 0$ or $p \ge s + t - 1$, then the determinants $D(i)$ are clearly all non-zero. Hence:

Corollary 2.2.11. *If $p = 0$ or $p \ge s + t - 1$, then*

$$J(\alpha, s) \otimes J(\beta, t) = \bigoplus_{i=0}^{s-1} J(\alpha\beta,\, s + t - 1 - 2i).$$

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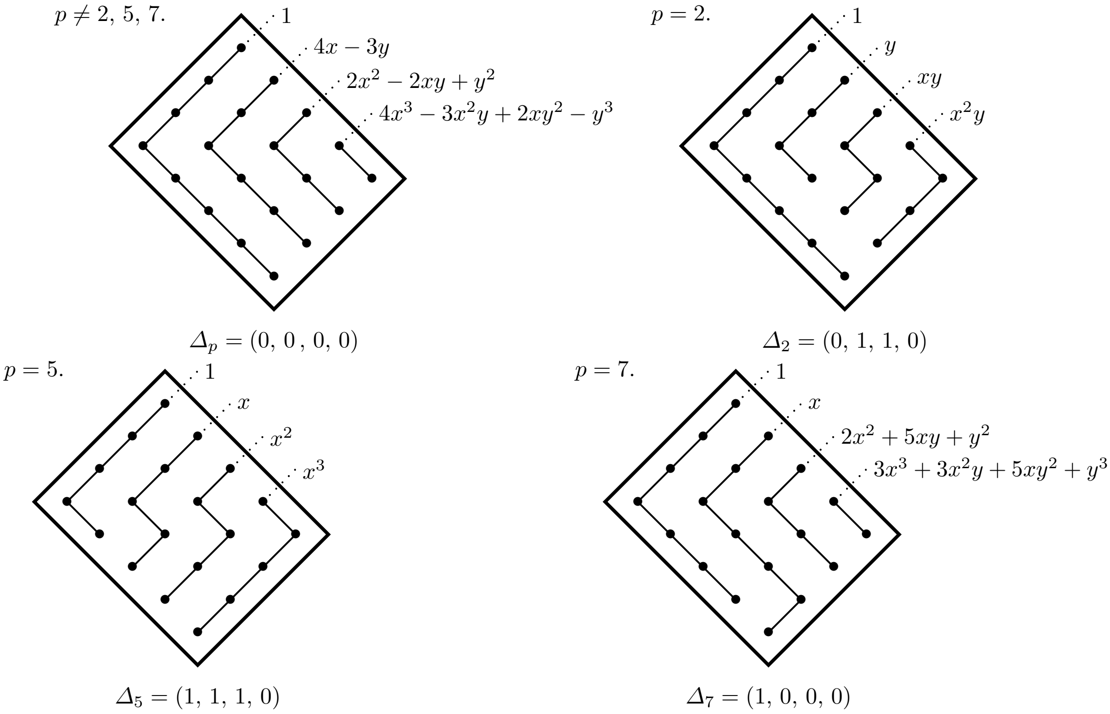

Figure 6. The result of Example 2.2.10.

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