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SOME IDENTITIES RELATING MOCK THETA FUNCTIONS WHICH ARE DERIVED FROM DENOMINATOR IDENTITY

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Abstract. We show that there exists a new connection between identities satisfied by mock theta functions and special case of denominator identities for affine Lie superalgebras.

1. Introduction

In 1920, S. Ramanujan listed 17 mock theta functions of order 3, 5 and 7 together with identities satisfied by them in his last letter to G. H. Hardy. In the letter, he did not describe a formal definition of "order" nor prove these identities. However, for 3rd and 5th order, we can see that each identity consists of mock theta functions with the same order [2], [16]. Ramanujan's assertion about 7th order mock theta functions is that they are not related to each other. Some identities for order 6 , 8 and 10, which also consists of mock theta functions with the same order, were proved by G. E. Andrews, D. Hickerson, B. Gordon, R. J. McIntosh and Y.-S. Choi [3], [5], [6], [9]. In [13], mock theta functions 3rd order $\chi(q)$ and 6th order $\gamma(q)$ are related to each other. Recently, K. Bringmann and K. Ono use the theories of modular forms to understand mock theta functions[4].

In this paper, we give a new view of some typical identities satisfied by mock theta functions. It is shown that these identities are obtained by specializing the denominator identities for affine Lie superalgebras in the case $\widehat{A}(1,0)$ and $\widehat{B}(1,1)$.

We introduce the standard notation and some mock theta functions which will be used in this paper.

Definition 1. Let q be a complex number such that $|q| < 1$. We define the q-shifted factorial for all integers n by

$$(a)_\infty = (a;q)_\infty := \prod_{i=0}^{\infty}(1-aq^i) \qquad \text{and} \qquad (a)_n = (a;q)_n := \frac{(a;q)_\infty}{(aq^n;q)_\infty}.$$

For brevity, we employ the usual notation

$$(x_1,\ldots,x_r)_\infty = (x_1,\ldots,x_r;q)_\infty := (x_1)_\infty \cdots (x_r)_\infty.$$

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Moreover, we define

$$j(x,q) := (x, q/x, q)_\infty = (x)_\infty (q/x)_\infty (q)_\infty$$

for all $x \in \mathbf{C}^*$.

By Jacobi's triple product identity[1], [8], we have

$$j(x,q) = \sum_{n=-\infty}^{\infty} (-1)^n q^{\frac{n(n-1)}{2}} x^n.$$

Definition 2. (mock theta functions of order 3)

$$\begin{aligned}
f(q) &:= \sum_{n=0}^{\infty} \frac{q^{n^2}}{(-q)_n^2}, \\
\chi(q) &:= 1 + \sum_{n=1}^{\infty} \frac{q^{n^2}}{\prod_{m=1}^{n}(1 - q^m + q^{2m})}, \\
\omega(q) &:= \sum_{n=0}^{\infty} \frac{q^{2n(n+1)}}{(q;q^2)_{n+1}^2}, \\
\rho(q) &:= \sum_{n=0}^{\infty} \frac{q^{2n(n+1)}}{\prod_{m=0}^{n}(1 + q^{2m+1} + q^{4m+2})}.
\end{aligned}$$

The following mock theta functions of order 8 are found by B. Gordon and R. J. McIntosh[9].

Definition 3. (mock theta functions of order 8)

$$\begin{aligned}
U_0(q) &:= \sum_{n=0}^{\infty} \frac{q^{n^2}(-q;q^2)_n}{(-q^4;q^4)_n}, \\
U_1(q) &:= \sum_{n=0}^{\infty} \frac{q^{(n+1)^2}(-q;q^2)_n}{(-q^2;q^4)_{n+1}}, \\
V_1(q) &:= \sum_{n=0}^{\infty} \frac{q^{(n+1)^2}(-q;q^2)_n}{(q;q^2)_{n+1}} = \sum_{n=0}^{\infty} \frac{q^{2n^2+2n+1}(-q^4;q^4)_n}{(q;q^2)_{2n+2}}.
\end{aligned}$$

In Section 1, we shall prove a following new identity for 3rd order mock theta functions in Theorem 1.1 and then we shall show that the identity is equal to the special case of type $\widehat{B}(1,1)$.

Theorem 1.1. *For 3rd order mock theta functions $\rho(q)$ and $\omega(q)$, the following relation holds.*

$$(q^2;q^2)_\infty\left((\rho(q)+\rho(-q))+\frac{1}{2}(\omega(q)+\omega(-q))\right)$$
$$=\frac{3\ (q^{12};q^{12})_\infty(-q^{12};q^{24})_\infty(-q^{12};q^{24})_\infty(q^{24};q^{24})_\infty}{(q^6;q^{12})_\infty}.$$

In Section 2, we shall see that the identity

$$(q)_\infty\,(4\chi(q)-f(q))=3\frac{(q^3;q^3)^2_\infty}{(-q^3;q^3)^2_\infty} \tag{1.1}$$

is equal to the case of type $\widehat{A}(1,0)$. This identity (1.1) is found in Ramanujan's last letter. We also see that some identities for mock theta functions are derived from a specialization of the denominator identity.

We give generalized Lambert series for mock theta functions in order to prove these identities satisfied by mock theta functions. By expressing mock theta functions with the generalized Lambert series, we can handle the mock theta functions more easily. The following generalized Lambert series for 3rd order mock theta functions are obtained by G. N. Watson[16].

$$(q)_\infty\ f(q) = 1+4\sum_{n=1}^{\infty}\frac{(-1)^n q^{n(3n+1)/2}}{1+q^n}, \tag{1.2}$$

$$(q)_\infty\ \chi(q) = 1+\sum_{n=1}^{\infty}\frac{(-1)^n(1+q^n) q^{n(3n+1)/2}}{1-q^n+q^{2n}}, \tag{1.3}$$

$$(q^2;q^2)_\infty\ \omega(q) = \sum_{n=0}^{\infty}\frac{(-1)^n(1+q^{2n+1}) q^{3n(n+1)}}{1-q^{2n+1}}, \tag{1.4}$$

$$(q^2;q^2)_\infty\ \rho(q) = \sum_{n=0}^{\infty}\frac{(-1)^n(1-q^{4n+2}) q^{3n(n+1)}}{1+q^{2n+1}+q^{4n+2}}. \tag{1.5}$$

The following generalized Lambert series for 8th order mock theta functions are obtained by B. Gordon and R. J. McIntosh[9].

$$U_0(q) = 2\frac{(-q;q^2)_\infty}{(q^2;q^2)_\infty}\sum_{n=-\infty}^{\infty}\frac{(-1)^n q^{n(2n+1)}}{1+q^{4n}}, \tag{1.6}$$

$$U_1(q) = \frac{(-q;q^2)_\infty}{(q^2;q^2)_\infty}\sum_{n=-\infty}^{\infty}\frac{(-1)^n q^{(n+1)(2n+1)}}{1+q^{4n+2}}, \tag{1.7}$$

$$V_1(q) = \frac{(-q^4;q^4)_\infty}{(q^4;q^4)_\infty}\sum_{n=-\infty}^{\infty}\frac{(-1)^n q^{(2n+1)^2}}{1-q^{4n+1}}. \tag{1.8}$$

We close this section by preparation of two identities as a special case of denominator identity for affine Lie superalgebras. We quote the denominator identity for affine Lie superalgebras which was discovered by V. G. Kac and M. Wakimoto [10], [15]. The identity is written in the following form

$$\frac{\prod_{\alpha\in\Delta^+_{\text{even}}}(1-e^{-\alpha})^{\text{mult}(\alpha)}}{\prod_{\alpha\in\Delta^+_{\text{odd}}}(1+e^{-\alpha})^{\text{mult}(\alpha)}} = e^{-\rho}\sum_{w\in W}\varepsilon(w)\ w\left(\frac{e^{\rho}}{\prod_{i=1}^{k}(1+e^{-\beta_i})}\right).$$

In this formula, Δ^+_{even} (resp. Δ^+_{odd}) is the set of all even (resp. odd) and positive roots of the affine Lie superalgebra $\widehat{\mathfrak{g}}$ and $\text{mult}(\alpha)$ is the dimension of the root space $\widehat{\mathfrak{g}}_\alpha$ and W is the Weyl group of $\widehat{\mathfrak{g}}$ and $\varepsilon(w)$ is the signature of $w\in W$ and ρ is the Weyl vector of $\widehat{\mathfrak{g}}$ and $\{\beta_1,\cdots,\beta_k\}$ is a maximal set of simple odd roots satisfying the inner product $(\beta_i|\beta_j)=0$ for all $i,j=1,\cdots,k$ (see [10], [11], [14], [15] for complete explanation and details).
In [15] , we can find that the denominator identity for $\widehat{A}(1,0)$ is

$$e^{\rho}R = \sum_{w\in W}\varepsilon(w)w\left(\frac{e^{\rho}}{1+e^{-\alpha_2}}\right),$$

where

$$R=\prod_{n=1}^{\infty}\frac{(1-e^{-n\delta})^2(1-e^{-(n-1)\delta-\alpha_1})(1-e^{-n\delta+\alpha_1})}{(1+e^{-(n-1)\delta-\alpha_2})(1+e^{-n\delta+\alpha_2})(1+e^{-(n-1)\delta-\alpha_1-\alpha_2})(1+e^{-n\delta+\alpha_1+\alpha_2})},$$

α_1 is an even simple root, α_2 is an odd simple root and δ is a primitive imaginary root. The identity is rewritten as follows:

$$\begin{aligned}(1.9)\quad & e^{\rho}\prod_{n=1}^{\infty}\frac{(1-e^{-n\delta})^2(1-e^{-(n-1)\delta-\alpha_1})}{(1+e^{-(n-1)\delta-\alpha_2})(1+e^{-n\delta+\alpha_2})}\\ &\qquad\times\frac{(1-e^{-n\delta+\alpha_1})}{(1+e^{-(n-1)\delta-\alpha_1-\alpha_2})(1+e^{-n\delta+\alpha_1+\alpha_2})}\\ &= e^{\rho}\left(\sum_{n=-\infty}^{\infty}\frac{e^{-\delta n(n+1)}e^{n\alpha_1}}{1+e^{-\alpha_2}e^{-n\delta}}-\sum_{n=-\infty}^{\infty}\frac{e^{-\delta n(n+1)}e^{-\alpha_1(-n+1)}}{1+e^{-\alpha_1}e^{-\alpha_2}e^{-n\delta}}\right).\end{aligned}$$

By putting $q:=e^{-\delta}$, $x:=e^{-\alpha_1}$ and $y:=e^{-\alpha_2}$, (1.9) is calculated as follows:

$$\begin{aligned}(1.10)\qquad & \sum_{n=-\infty}^{\infty}\frac{x^{-n}q^{n(n+1)}}{1+yq^n}-\sum_{n=-\infty}^{\infty}\frac{x^{n+1}q^{n(n+1)}}{1+xyq^n}\\ &=\frac{(q,q,x,q/x)_\infty}{(-y,-q/y,-xy,-q/xy)_\infty}.\end{aligned}$$

By replacing q by q^2 and substituting $x = q$ and $y = z$ in (1.10), we have

$$A(z;q) := \sum_{n=-\infty}^{\infty} \frac{(-1)^n q^{n(n+1)/2}}{1+zq^n} = \frac{(q,q)_\infty}{(-z,-z^{-1}q)_\infty}, \tag{1.11}$$

where $z \neq -q^n \quad (n \in \mathbf{Z})$.

In [15], we can also find that the denominator identity for $\widehat{B}(1,1)$ is

$$e^\rho R = \sum_{w\in W} \varepsilon(w) w \left(\frac{e^\rho}{1+e^{-\alpha_1}} \right),$$

where

$$\begin{aligned} R \;&=\; \prod_{n=1}^{\infty} \frac{(1-e^{-n\delta})^2(1-e^{-(n-1)\delta-\alpha_2})(1-e^{-n\delta+\alpha_2})}{(1+e^{-(n-1)\delta-\alpha_1})(1+e^{-n\delta+\alpha_1})(1+e^{-(n-1)\delta-\alpha_1-\alpha_2})} \\ &\quad \times \frac{(1-e^{-(n-1)\delta-2\alpha_1-2\alpha_2})(1-e^{-n\delta+2\alpha_1+2\alpha_2})}{(1+e^{-n\delta+(\alpha_1+\alpha_2)})(1+e^{-(n-1)\delta-\alpha_1-2\alpha_2})(1+e^{-n\delta+\alpha_1+2\alpha_2})}, \end{aligned}$$

α_1 is an odd simple root and α_2 is an even simple root. The identity is rewritten as follows:

$$\begin{aligned} &e^\rho \prod_{n=1}^{\infty} \frac{(1-e^{-n\delta})^2(1-e^{-(n-1)\delta-\alpha_2})(1-e^{-n\delta+\alpha_2})}{(1+e^{-(n-1)\delta-\alpha_1})(1+e^{-n\delta+\alpha_1})(1+e^{-(n-1)\delta-\alpha_1-\alpha_2})} \\ &\quad \times \frac{(1-e^{-(n-1)\delta-2\alpha_1-2\alpha_2})(1-e^{-n\delta+2\alpha_1+2\alpha_2})}{(1+e^{-(n-1)\delta-\alpha_1-2\alpha_2})(1+e^{-n\delta+\alpha_1+2\alpha_2})(1+e^{-n\delta+(\alpha_1+\alpha_2)})} \\ &= e^\rho \left(\sum_{n=-\infty}^{\infty} \frac{e^{-\delta(\frac{1}{2}n^2+\frac{1}{2}n)} e^{-n(\alpha_1+\alpha_2)}}{1+e^{-\alpha_1}e^{-n\delta}} - \sum_{n=-\infty}^{\infty} \frac{e^{-\delta(\frac{1}{2}n^2+\frac{1}{2}n)} e^{-n(\alpha_1+\alpha_2)} e^{-\alpha_2}}{1+e^{-\alpha_1}e^{-2\alpha_2}e^{-n\delta}} \right). \end{aligned} \tag{1.12}$$

By putting $q := e^{-\delta}$, $x := e^{-\alpha_1}$ and $y := e^{-\alpha_2}$, (1.12) is calculated as follows:

$$\begin{aligned} &\sum_{n=-\infty}^{\infty} \frac{q^{\frac{1}{2}n^2+\frac{1}{2}n}(xy)^n}{1+xq^n} - \sum_{n=-\infty}^{\infty} \frac{q^{\frac{1}{2}n^2+\frac{1}{2}n}(xy)^{-n-1}}{1+x^{-1}y^{-2}q^n} \\ &\qquad = \frac{(q,q,y,q/y,x^2y^2,q/x^2y^2)_\infty}{(-x,-q/x,-xy,-q/xy,-xy^2,-q/xy^2)_\infty}. \end{aligned} \tag{1.13}$$

By replacing q by q^2 and substituting $x = zq^{1/2}$ and $y = z^{-1}q^{-1}$ in (1.13), we have

$$\begin{aligned} B(z;q) &:= \sum_{n=-\infty}^{\infty} \frac{(-1)^n q^{n(n+1)/4}}{1+zq^{n+\frac{1}{2}}} \\ &= \frac{(q^{\frac{1}{2}},q)_\infty (q^2,zq,z^{-1}q;q^2)_\infty}{(-zq^{\frac{1}{2}},-z^{-1}q^{\frac{1}{2}})_\infty}, \end{aligned} \tag{1.14}$$

where $z \neq -q^{n+\frac{1}{2}} \quad (n \in \mathbf{Z})$.

2. The proof of Theorem 1.1

Proof. We need the following two identities.

$$(q^2;q^2)_\infty \left(\omega(q)+\omega(-q)\right) = 2\sum_{n=-\infty}^{\infty} \frac{(-1)^n q^{3n(n+1)}}{1-q^{4n+2}}, \tag{2.1}$$

$$(q^2;q^2)_\infty \left(\rho(q)+\rho(-q)\right) = 2\sum_{n=-\infty}^{\infty} \frac{(-1)^n q^{3n(n+1)}}{1+q^{4n+2}+q^{8n+4}}. \tag{2.2}$$

The identity (2.1) follows from (1.4) :

$$\begin{aligned}
(q^2;q^2)_\infty \left(\omega(q)+\omega(-q)\right) &= \sum_{n=0}^{\infty}(-1)^n q^{3n(n+1)} \left(\frac{1+q^{2n+1}}{1-q^{2n+1}} + \frac{1-q^{2n+1}}{1+q^{2n+1}}\right)\\
&= \sum_{n=0}^{\infty}(-1)^n q^{3n(n+1)} \left(\frac{2(1+q^{4n+2})}{1-q^{4n+2}}\right)\\
&= 2\sum_{n=-\infty}^{\infty} \frac{(-1)^n q^{3n(n+1)}}{1-q^{4n+2}}.
\end{aligned}$$

Similarly, the identity (2.2) follows from (1.5). Two identities (2.1) and (2.2) are rewritten as follows:

$$\begin{aligned}
&(q^2;q^2)_\infty \left(\frac{\omega(q)+\omega(-q)}{2}\right)\\
&= \sum_{n=-\infty}^{\infty} \frac{(-1)^n q^{3n(n+1)}}{1-q^{4n+2}} \times \frac{1+q^{4n+2}+q^{8n+4}}{1+q^{4n+2}+q^{8n+4}}\\
&= \sum_{n=-\infty}^{\infty} \frac{(-1)^n q^{3n(n+1)}}{1-q^{12n+6}} + \sum_{n=-\infty}^{\infty} \frac{(-1)^n q^{3n(n+1)+4n+2}}{1-q^{12n+6}}\\
&\qquad + \sum_{n=-\infty}^{\infty} \frac{(-1)^n q^{3n(n+1)+8n+4}}{1-q^{12n+6}}\\
&= \sum_{n=-\infty}^{\infty} \frac{(-1)^n q^{3n(n+1)}}{1-q^{12n+6}} + \sum_{n=-\infty}^{\infty} \frac{(-1)^n q^{3n(n+1)+4n+2}}{1-q^{12n+6}}\\
&\qquad + \sum_{n=-\infty}^{\infty} \frac{(-1)^{-n} q^{3(-n)(-n+1)+8(-n)+4}}{1-q^{12(-n)+6}}
\end{aligned}$$

$$
\begin{aligned}
&= \sum_{n=-\infty}^{\infty} \frac{(-1)^n q^{3n(n+1)}}{1-q^{12n+6}} + \sum_{n=-\infty}^{\infty} \frac{(-1)^n q^{3n(n+1)+4n+2}}{1-q^{12n+6}} \\
&\quad + \sum_{n=-\infty}^{\infty} \frac{(-1)^{-n} q^{3(-n)(-n+1)+8(-n)+4}}{-q^{-12n+6}(1-q^{12n-6})} \\
&= \sum_{n=-\infty}^{\infty} \frac{(-1)^n q^{3n(n+1)}}{1-q^{12n+6}} + \sum_{n=-\infty}^{\infty} \frac{(-1)^n q^{3n(n+1)+4n+2}}{1-q^{12n+6}} \\
&\quad - \sum_{n=-\infty}^{\infty} \frac{(-1)^n q^{3n(n-1)+4n-2}}{1-q^{12n-6}} \\
&= \sum_{n=-\infty}^{\infty} \frac{(-1)^n q^{3n(n+1)}}{1-q^{12n+6}} + \sum_{n=-\infty}^{\infty} \frac{(-1)^n q^{3n(n+1)+4n+2}}{1-q^{12n+6}} \\
&\quad - \sum_{n=-\infty}^{\infty} \frac{(-1)^{n+1} q^{3(n+1)n+4(n+1)-2}}{1-q^{12(n+1)-6}} \\
&= \sum_{n=-\infty}^{\infty} \frac{(-1)^n q^{3n(n+1)}}{1-q^{12n+6}} + 2\sum_{n=-\infty}^{\infty} \frac{(-1)^n q^{3n(n+1)+4n+2}}{1-q^{12n+6}}
\end{aligned}
$$

and

$$
\begin{aligned}
&(q^2;q^2)_\infty \left(\rho(q)+\rho(-q)\right) \\
&= 2\left(\sum_{n=-\infty}^{\infty} \frac{(-1)^n q^{3n(n+1)}}{1+q^{4n+2}+q^{8n+4}} \times \frac{1-q^{4n+2}}{1-q^{4n+2}}\right) \\
&= 2\left(\sum_{n=-\infty}^{\infty} \frac{(-1)^n q^{3n(n+1)}}{1-q^{12n+6}} - \sum_{n=-\infty}^{\infty} \frac{(-1)^n q^{3n(n+1)+4n+2}}{1-q^{12n+6}}\right),
\end{aligned}
$$

respectively. From these identities, the left hand side of Theorem 1.1 simplifies to

$$
\begin{aligned}
(2.3) \qquad & (q^2;q^2)_\infty \left(\left(\rho(q)+\rho(-q)\right) + \left(\frac{\omega(q)+\omega(-q)}{2}\right)\right) \\
&= 3\sum_{n=-\infty}^{\infty} \frac{(-1)^n q^{3n(n+1)}}{1-q^{12n+6}} \\
&= 3B(-1;q^{12}).
\end{aligned}
$$

Replacing q by q^{12} and substituting $z=-1$ in (1.14), we have

$$
(2.4) \qquad B(-1;q^{12}) = \frac{(q^{12};q^{12})_\infty(-q^{12},\ -q^{12},\ q^{24};q^{24})_\infty}{(q^6;q^{12})_\infty}.
$$

By (2.3) and (2.4), the proof completes. □

This proof implies that the relation in Theorem 1.1 equals to the special case of the denominator identity for affine Lie superalgebra in the case $\widehat{B}(1,1)$.

3. Some identities

We will prove some identities relating mock theta functions by using the denominator identity. First we see that the identity (1.1):

$$(q)_\infty\,(4\chi(q)-f(q)) = 3\frac{(q^3;q^3)_\infty^2}{(-q^3;q^3)_\infty^2}$$

is derived from the denominator identity for $\widehat{A}(1,0)$. From (1.2) and (1.3), we have

$$(q)_\infty\,(4\chi(q)-f(q)) = 6\sum_{n=-\infty}^{\infty}\frac{(-1)^n q^{3n(n+1)/2}}{1+q^{3n}} = 6A(1;q^3). \tag{3.1}$$

Replacing q by q^3 and substituting $z=1$ in (1.11), we have

$$A(1;q^3) = \frac{(q^3;q^3)_\infty^2}{2\,(-q^3;q^3)_\infty^2}. \tag{3.2}$$

Hence, from (3.1) and (3.2), we can see that the identity (1.1) is a special case of the denominator identity for $\widehat{A}(1,0)$.

Next, we prove

$$(q^2;q^2)_\infty\left(\rho(q)+\frac{1}{2}\,\omega(q)\right) = \frac{3}{2}\,\frac{(q^6;q^6)_\infty^2}{(q^3;q^6)_\infty^2}, \tag{3.3}$$

which is obtained by G. N. Watson in [16]. From (1.4) and (1.5), we have

$$(q^2;q^2)_\infty\left(\rho(q)+\frac{1}{2}\omega(q)\right) = \frac{3}{2}\sum_{n=-\infty}^{\infty}\frac{(-1)^n q^{3n(n+1)}}{1-q^{6n+3}} = \frac{3}{2}A(-q^3;q^6).$$

Replacing q by q^6 and substituting $z=-q^3$ in (1.11) which is type $\widehat{A}(1,0)$, we have

$$A(-q^3;q^6) = \frac{(q^6;q^6)_\infty^2}{(q^3;q^6)_\infty^2}.$$

Hence we can prove (3.3).

Finally, we prove two identities relating mock theta functions of order 8 in [9]:

$$U_0(q)+2U_1(q) \;=\; (-q;q^2)_\infty^3(q^2;q^2)_\infty(q^2;q^4)_\infty, \tag{3.4}$$

$$V_1(q) - V_1(-q) = 2q(-q^2;q^2)_\infty(-q^4;q^4)_\infty^2(q^8;q^8)_\infty. \tag{3.5}$$

From (1.6) and (1.7), we have

$$\begin{aligned} U_0(-q) + 2U_1(-q) &= 2\frac{(q;q^2)_\infty}{(q^2;q^2)_\infty}\sum_{n=-\infty}^{\infty}\frac{(-1)^n q^{\frac{n(n+1)}{2}}}{1+q^{2n}} \\ &= 2\frac{(q;q^2)_\infty}{(q^2;q^2)_\infty}B(q^{-1};q^2). \end{aligned} \tag{3.6}$$

Replacing q by q^2 and substituting $z = q^{-1}$ in (1.14), we have

$$B(q^{-1};q^2) = \frac{1}{2}\frac{(q)_\infty^2}{(-q^2;q^2)_\infty}. \tag{3.7}$$

From (3.6) and (3.7), we prove (3.4) by replacing q by $-q$. Similarly, from (1.8), we have

$$\begin{aligned} V_1(q) - V_1(-q) &= 2q\frac{(-q^4;q^4)_\infty}{(q^4;q^4)_\infty}\sum_{n=-\infty}^{\infty}\frac{(-1)^n q^{4n(n+1)}}{1-q^{8n+2}} \\ &= 2q\frac{(-q^4;q^4)_\infty}{(q^4;q^4)_\infty}A(-q^2;q^8). \end{aligned}$$

Replacing q by q^8 and substituting $z = -q^2$ in (1.11), we have

$$A(-q^2;q^8) = \frac{(q^8;q^8)_\infty^2}{(q^2;q^8)_\infty(q^6;q^8)_\infty} = \frac{(q^8;q^8)_\infty^2}{(q^2;q^4)_\infty}.$$

Hence, we prove (3.5).

Remark. The denominator identities for affine Lie superalgebras in the case $\widehat{A}(1,0)$ and $\widehat{B}(1,1)$ are related to Ramanujan's summation formula ${}_1\psi_1$ series and Bailey's summation formula of a very-well-poised-balanced ${}_6\psi_6$ series, respectively.

4. Appendix

Here is an additional remark. We shall show that the identity in Theorem 1.1 can be seen as a relation between theta constants. From (3.3), the identity in Theorem 1.1 becomes

$$j(-q^3,q^{12})^2 + j(q^3,q^{12})^2 = 2\,j(-q^6,q^{24})\,j(-q^{12},q^{24}). \tag{4.1}$$

Now, let v be a complex number and τ be a complex number whose imaginary part is positive. Theta functions are defined by

$$\vartheta_3(v|\tau) = \vartheta_{00}(v,\tau) := \sum_{n=-\infty}^{\infty}\mathbf{e}\left(\frac{1}{2}n^2\tau + nv\right),$$

$$\vartheta_4(v|\tau) = \vartheta_{01}(v,\tau) \quad := \quad \sum_{n=-\infty}^{\infty} \mathbf{e}\left(\frac{1}{2}n^2\tau + n\left(v+\frac{1}{2}\right)\right),$$

where $\mathbf{e}(x) := e^{2\pi ix}$ [12]. Putting $z = e^{v\pi i}$ and $q = e^{\tau\pi i}$, theta functions can be rewritten as product formulas:

$$\vartheta_{00}(v,\tau) = j(-z^2q, q^2), \quad \vartheta_{01}(v,\tau) = j(z^2q, q^2).$$

In the following relation[7, §13.23.(Transformations of the second order)]

$$\vartheta_{00}(v,\tau)^2 + \vartheta_{01}(v,\tau)^2 = 2\ \vartheta_{00}(0,2\tau)\ \vartheta_{00}(2v,2\tau), \tag{4.2}$$

replacing τ by 6τ and v by $\frac{3}{2}\tau$ and using $j(x,q) = j(q/x,q)$ yields (4.1). Hence, we can see that the identity in Theorem 1.1 is also the special case of (4.2).

5. Conclusion

In [15, p195], M. Wakimoto states that "The denominator identities for the simplest affine Lie superalgebras are Ramanujan's mock theta functions. In this sense, denominator identities of affine Lie superalgebras provide a general class of mock theta functions." However, the specific instance for 3rd or 8th order mock theta functions is not given there. We have found that some identities satisfied by mock theta functions are special cases of the denominator identity. It is plausible that these connections will assist in giving the true meaning of mock theta functions.

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