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ON LIE IDEALS AND LEFT JORDAN σ-CENTRALIZERS OF 2-TORSION-FREE RINGS

Dedicated to Miguel Ferrero on his $70^{th.}$ birthday

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Abstract. B. Zalar proved that any left (resp. right) Jordan centralizer on a 2-torsion-free semiprime ring is a left (resp. right) centralizer. We prove this question changing the semiprimality condition on R.

The main result of this paper is the following. Let R be a 2-torsion-free ring which has a commutator right (resp. left) nonzero divisor and let G: $R \to R$ be a left (resp. right) Jordan σ-centralizer mapping of R, where σ is an automorphism of R. Then G is a left (resp. right) σ-centralizer mapping of R.

1. Introduction

This research has been motivated by the works of J. Vukman [8], J. Vukman and I.K.-Ulbl [9] and B. Zalar [10].

Throughout the paper R will denote an associative ring, with center $Z(R)$, not necessarily with an identity element, unless otherwise stated, and U denotes a Lie ideal of R. As usual, $[x, y]$ denotes the commutator $xy - yx$. Recall that if R is a ring, R has a Lie structure by the bracket product $[x, y]$, for $x, y \in R$. A *Lie ideal* of R is any additive subgroup U of R with $[u, r] = ur - ru \in U$ for every $u \in U$ and $r \in R$ [7].

An additive mapping G: $R \to R$ is called a *left (resp. right) centralizer*, if $G(xy) = G(x)y$ (resp. $G(xy) = xG(y)$) holds for all $x, y \in R$. If $a \in R$, then $L_a(x) = ax$ is a left centralizer and $R_a(x) = xa$ is a right centralizer. Following B. Zalar [10], G is a *centralizer* mapping if G is both left and right centralizer.

If R is a ring with involution $\star$, then every additive mapping E: $R \to R$ which satisfies $E(x^2) = E(x)x^\star + xE(x)$ for all $x \in R$ is called a *Jordan $\star$-derivation.* Following [10], these mappings are closely connected with a question of representability of quadratic forms by bilinear forms. In [3,

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proof of Theorem 2.1], M. Brešar and B. Zalar obtained a representation of Jordan $\star$-derivations in terms of left and right centralizers on the algebra of compact operators on a Hilbert space.

If we introduce a new product in an associative ring R, given by $x \circ y = xy + yx$, then a *Jordan derivation* is an additive mapping d which satisfies $d(x \circ y) = d(x) \circ y + x \circ d(y)$, for every $x, y \in R$, and a *Jordan homomorphism* is an additive mapping h which satisfies $h(x \circ y) = h(x) \circ h(y)$, for all $x, y \in R$. Therefore, we can define a Jordan centralizer to be an additive mapping T which satisfies $T(x \circ y) = T(x) \circ y = x \circ T(y)$, for every $x, y \in R$. Since the product $\circ$ is commutative, there is no difference between the left and right Jordan centralizers.

An easy computation shows that every centralizer is also a Jordan centralizer. In [10], B. Zalar proves that every Jordan centralizer of a semiprime ring is a centralizer.

Let $G\colon R \to R$ be an additive mapping and σ an endomorphism of R. We call G a *left (resp. right) Jordan σ-centralizer* if $G(x^2) = G(x)\sigma(x)$ (resp. $G(x^2) = \sigma(x)G(x)$) holds for all $x \in R$. Similarly, if U is a Lie ideal of R, then an additive mapping $G\colon R \to R$ is said to be a left (resp. right) Jordan σ-centralizer *of U into R* in case that the above corresponding conditions are satisfied for all $x \in U$. When $\sigma = id_R$ we have the usual well-known definitions of left and right Jordan centralizer mappings.

There exists a type of Lie ideals that occurs in some works involving usual derivations [2], namely *square closed*. A Lie ideal U of R is said to be square closed if it verifies $u^2 \in U$, for every $u \in U$. It follows that for a square closed Lie ideal U of R, $uv + vu \in U$ and $2uv \in U$, for any $u, v \in U$ [5, page 251]. These remarks will be freely used in the paper.

In the whole paper, we will consider σ as an automorphism of a ring R.

2. Results

B. Zalar [10, Proposition 1.4] proved that any left (resp. right) Jordan centralizer of a 2-torsion-free semiprime ring is a left (resp. right) centralizer.

It is our aim in this paper to prove the result above changing the semiprimality condition on R by the existence of a commutator right (resp. left) nonzero divisor. This last condition seems to be artificial. Then, for the sake of completeness, we include at the end of the paper some examples of rings that are not semiprime and having commutators nonzero divisors.

Furthermore, we state this result for Lie ideals, proving the following

Theorem 2.1. *Let R be a 2-torsion-free ring, U a square closed Lie ideal of R which has a commutator right (resp. left) nonzero divisor, and $G\colon R \to R$ a left (resp. right) Jordan σ-centralizer mapping of U into R. Then G is a left (resp. right) σ-centralizer mapping of U into R.*

Since $U = R$ is obviously a square closed Lie ideal of R, Theorem 2.1 is also true for left (resp. right) Jordan σ-centralizer mappings of R. We include an example by M. Brešar showing that the semiprimality and the right (resp. left) nonzero divisor commutator assumptions are independent each other [4, Example].

Furthermore, if R is a 2-torsion-free ring, U is a square closed Lie ideal of R, and G: $R \to R$ is a left (resp. right) σ-centralizer of U into R, then an easy computation gives that $G(xyz) = G(x)\sigma(y)\sigma(z)$ (resp. $G(xyz) = \sigma(x)\sigma(y)G(z)$), for every $x, y, z \in U$. A natural question is to ask whether the converse is also true. We prove the following

Theorem 2.2. *Let R be a 2-torsion-free ring, U a square closed Lie ideal of R which has a commutator right (resp. left) nonzero divisor, and G: $R \to R$ an additive mapping that satisfies $G(xyz) = G(x)\sigma(y)\sigma(z)$ (resp. $G(xyz) = \sigma(x)\sigma(y)G(z)$), for every $x, y, z \in U$. Then G is a left (resp. right) σ-centralizer of U into R.*

Finally, if G: $R \to R$ is a left and right σ-centralizer map of a ring R, we easily have $2G(xyz) = G(x)\sigma(y)\sigma(z) + \sigma(x)\sigma(y)G(z)$, for each $x, y, z \in R$. We obtained a result that provides a converse of this fact.

Theorem 2.3. *Let R be a 2-torsion-free ring, U a square closed Lie ideal of R which has an element that is nonzero divisor, and G: $R \to R$ an additive mapping that satisfies $2G(xyz) = G(x)\sigma(y)\sigma(z) + \sigma(x)\sigma(y)G(z)$, for every $x, y, \in U$. Then G is a σ-centralizer of U into R.*

If there is no possibility of misunderstanding, we will consider the mappings G: $R \to R$ always as additive mappings.

3. Proofs

Before proving Theorem 2.1, let us point out that in case R has an identity element, Theorem 2.1 can be easily proved for $U = R$. Namely, in this case one puts $x + 1$ for x in $G(x^2) = G(x)\sigma(x)$ (resp. $G(x^2) = \sigma(x)G(x)$), where 1 denotes the identity element, which gives $G(x) = G(1)\sigma(x)$ (resp. $G(x) = \sigma(x)G(1)$). Thus G is left (resp. right) σ-centralizer.

Now let U be a Lie ideal of R. To facilitate our discussion we define a mapping δ_l: $R^2 \to R$ (resp. δ_r: $R^2 \to R$) such that $\delta_l(x, y) := G(xy) - G(x)\sigma(y)$, for G: $R \to R$ a left Jordan σ-centralizer mapping of U into R, (resp. $\delta_r(x, y) := G(xy) - \sigma(x)G(y)$, for G: $R \to R$ a right Jordan σ-centralizer of U into R). It is easy to see that δ_l and δ_r are additive with respect to both arguments. Moreover, if δ_l (resp. δ_r) is zero, then G is a left (resp. right) σ-centralizer mapping of U into R.

We need the following Lemma:

Lemma 3.1. *Let R be a 2-torsion-free ring, U a square closed Lie ideal of R and $G\colon R \to R$ a left (resp. right) Jordan σ-centralizer mapping of U into R. Then, for every $a, b, c \in U$, the following statements hold:*

(i). $G(ab + ba) = G(a)\sigma(b) + G(b)\sigma(a)$ *(resp.* $G(ab + ba) = \sigma(a)G(b) + \sigma(b)G(a)$*);*

(ii). $G(aba) = G(a)\sigma(b)\sigma(a)$ *(resp.* $G(aba) = \sigma(a)\sigma(b)G(a)$*);*

(iii). $G(abc + cba) = G(a)\sigma(b)\sigma(c) + G(c)\sigma(b)\sigma(a)$ *(resp.* $G(abc + cba) = \sigma(a)\sigma(b)G(c) + \sigma(c)\sigma(b)G(a)$*);*

(iv). $\delta_l(a, b)\sigma([a, b]) = 0$ *(resp.* $\sigma([a, b])\delta_r(a, b) = 0$*).*

Proof. We only show the left Jordan σ-centralizer case, because the right case can be proved in an analogous way.

(i) and (ii) are easily obtained in the way similar to that in the proof of [5, Theorem 1.3]; while (iii) follows easily as in the proof of [4, Lemma 2.1 (iii)].

Take now $a, b \in U$. Replacing c by $2ab$ in (iii), we get for $\beta = 8ab(ab) + 8(ab)ba$,

$$G(\beta) = 8G(ab(ab) + (ab)ba) = 8(G((ab)^2) + G(ab^2a)).$$

Since R is 2-torsion-free, this shows (iv). □

Note that in Lemma 3.1, item (i), we do not need the 2-torsion-free condition on R. Besides that, Lemma 3.1 holds in case σ is just an endomorphism of the ring R.

Proof of Theorem 2.1. We only show the left Jordan σ-centralizer case, because the right case is similar. By assumption, there exist elements a and b of U such that $c[a, b] = 0$ implies $c = 0$ for every $c \in R$. Since σ is an automorphism of R, this implies that $c \cdot \sigma([a, b]) = 0 \Rightarrow \ c = 0$. Then by Lemma 3.1 (iv), we have

$$\delta_l(a, b) = 0. \tag{1}$$

Our aim is to show that $\delta_l(x, y) = 0$, for all $x, y \in U$. From Lemma 3.1 (iv), we have

$$\delta_l(x, y)\sigma([x, y]) = 0, \text{ for all } x, y \in U. \tag{2}$$

Replacing x by $x + a$ in (2), we get

$$\delta_l(x, y)\sigma([a, y]) + \delta_l(a, y)\sigma([x, y]) = 0, \text{ for all } x, y \in U. \tag{3}$$

Now substitute y by $y + b$ in (3). We get

$$\delta_l(x, y)\sigma([a, b]) + \delta_l(x, b)\sigma([a, y]) + \delta_l(x, b)\sigma([a, b]) + \delta_l(a, y)\sigma([x, b]) = 0, \tag{4}$$

for all $x, y \in U$.

Changing x by a in (4), using (1) and since R is a 2-torsion-free ring, we obtain $\delta_l(a,y)\sigma([a,b]) = 0$, for every $y \in U$. Hence we have

$$\delta_l(a,y) = 0, \text{ for all } y \in U. \tag{5}$$

Again replacing y by b in (3) and using (1), we get $\delta_l(x,b)\sigma([a,b]) = 0$, for every $x \in U$. Since $\sigma([a,b])$ is also a right nonzero divisor, we find that

$$\delta_l(x,b) = 0, \text{ for all } x \in U. \tag{6}$$

Combining (4), (5) and (6) we find that $\delta_l(x,y)\sigma([a,b]) = 0$ and hence $\delta_l(x,y) = 0$, for every $x, y \in U$, that is, G is a left σ-centralizer of U into R. □

Proof of Theorem 2.2. The proof will be done only for the left side case, because the right one is analogous.

By assumption we easily obtain

$$G(xyz + zyx) = G(x)\sigma(y)\sigma(z) + G(z)\sigma(y)\sigma(x), \ \forall x, y, z \in U, \tag{7}$$

since R is 2-torsion-free and U is square closed.

Putting $z = 2xy \in U$ in (7), we get $G(xy \cdot (2xy) + (2xy) \cdot yx) = 2G(x)\sigma(y)\sigma(x)\sigma(y) + 2G(xy)\sigma(y)\sigma(x)$, for every $x, y \in U$. On the other hand, by assumption, we have

$$G(x(2yx)y) + 2G(xy^2x) = 2G(x)\sigma(y)\sigma(x)\sigma(y) + 2G(x)\sigma(y^2)\sigma(x),$$

for every $x, y \in U$.

As before, we will denote $\delta_l(x,y) = G(xy) - G(x)\sigma(y)$, for every $x, y \in U$. Thus, the expression above gives $\delta_l(x,y)\sigma(y)\sigma(x) = 0$ for each $x, y \in U$.

Furthermore, by assumption, we have

$$G((2xy)xy) + G((2xy)yx) = 2G(xy)\sigma(x)\sigma(y) + 2G(xy)\sigma(y)\sigma(x), \ \forall\, x, y \in U.$$

On the other hand,

$$G(xy \cdot (2xy) + (2xy) \cdot yx) = 2G(x)\sigma(y)\sigma(x)\sigma(y) + 2G(xy)\sigma(y)\sigma(x), \ \forall x, y \in U.$$

Hence $\delta_l(x,y)xy = 0$, for each $x, y \in U$, since R is 2-torsion-free.

Hence, we get $\delta_l(x,y) \cdot \sigma([x,y]) = 0$, for every $x, y \in U$.

Now we are ready following the same steps as in the proof of Theorem 2.1. □

Proof of Theorem 2.3. By our assumptions on R, U and G: $R \to R$, we easily obtain

$$2G(xyx) = G(x)\sigma(y)\sigma(x) + \sigma(x)\sigma(y)G(x), \tag{8}$$

for every $x, y \in U$, and

$$2G(xyz+zyx) = G(x)\sigma(y)\sigma(z)+\sigma(x)\sigma(y)G(z)+ +G(z)\sigma(y)\sigma(x)+\sigma(z)\sigma(y)G(x), \tag{9}$$

for each $x, y, z \in U$.

In particular, for $z = 2xy \in U$, we have

$$2G(xy\cdot(2xy))+4G(xy^2x) = 2G(x)\sigma(y)\sigma(x)\sigma(y)+\sigma(x)\sigma(y)G(2xy)+ +2G(x)\sigma(y^2)\sigma(x)+2\sigma(x)\sigma(y^2)G(x),$$

for every $x, y \in U$. On the other hand,

$$2G(xy\cdot(2xy)+(2xy)\cdot yx) = 2G(x)\sigma(y)\sigma(x)\sigma(y)+G(2xy)\sigma(y)\sigma(x)+ +2\sigma(x)\sigma(y^2)G(x)+\sigma(x)\sigma(y)G(2xy),$$

for each $x, y \in U$.

Denoting again $\delta_l(x,y) = G(xy) - G(x)\sigma(y)$ for every $x, y \in U$ as above, we have that

$$\delta_l(x,y)\cdot\sigma(y)\sigma(x) = 0,\ \forall x, y \in U, \tag{10}$$

since R is 2-torsion-free.

Analogously, for $z = 2yx \in U$, we get

$$4G(xy^2x)+2G((2yx)yx) = 2G(x)\sigma(y^2)\sigma(x)+2\sigma(x)\sigma(y^2)G(x)+ +G(2yx)\sigma(y)\sigma(x)+2\sigma(y)\sigma(x)\sigma(y)G(x),$$

for every $x, y \in U$. On the other hand,

$$2G(xy\cdot(2yx)+(2yx)\cdot yx) = 2G(x)\sigma(y^2)\sigma(x)+G(2yx)\sigma(y)\sigma(x)+ +2\sigma(y)\sigma(x)\sigma(y)G(x)+\sigma(x)\sigma(y)G(2yx),$$

for each $x, y \in U$.

As above, denoting $\delta_r(x,y) = G(xy) - \sigma(x)G(y)$ for every $x, y \in U$, we have that

$$\sigma(x)\sigma(y)\cdot\delta_r(y,x) = 0,\ \forall x, y \in U, \tag{11}$$

since R is 2-torsion-free.

Let $b \in U$ be a nonzero divisor. Swap now x by b in (10) to obtain

$$\delta_l(b,y)\sigma(y) = 0,\ \forall y \in U, \tag{12}$$

since $\sigma(b)$ is a nonzero divisor, too.

Replacing x by $x+b$ in (10), and using (10) and (12), we get $\delta_l(x,y)\sigma(y)\sigma(b) = 0$, for every $x, y \in U$. Since $\sigma(b)$ is not a zero divisor, we have that

$$\delta_l(x,y)\sigma(y) = 0,\ \forall x, y \in U. \tag{13}$$

Substitute y by b in (13) to obtain

$$\delta_l(x,b) = 0,\ \forall x \in U. \tag{14}$$

Changing y by $y + b$ in (13), and using (13) and (14), we obtain $\delta_l(x, y)\sigma(b) = 0$, for every $x, y \in U$. Therefore $\delta_l(x, y) = 0$, for all $x, y \in U$, and G is a left σ-centralizer of U into R.

Replace x by b in (11) to obtain

$$\sigma(y)\delta_r(y, b) = 0, \ \forall y \in U. \tag{15}$$

Substituting x by $x + b$ in (11), using (11) and (15), we obtain $\sigma(b)\sigma(y)\delta_r(y, x) = 0$, for each $x, y \in U$, and since $\sigma(b)$ is a nonzero divisor, it follows that

$$\sigma(y)\delta_r(y, x) = 0, \ \forall x, y \in U. \tag{16}$$

Change now y by b in (16) to obtain

$$\delta_r(b, x) = 0, \ \forall x \in U. \tag{17}$$

Replacing y by $y+b$ in (16), and using (16) and (17), we get $\sigma(b)\delta_r(y, x) = 0$, for every $x, y \in U$. Hence, $\delta_r(y, x) = 0$, for all $x, y \in U$, because $\sigma(b)$ is a nonzero divisor, and G is a right σ-centralizer of U into R. The proof is complete. □

The following example shows, for the sake of completeness, that the assumptions of our main Theorem 2.1 and [10, Proposition 1.4] are independent each other. This example is due to M. Brešar who kindly allowed us to include it here.

Example [4, Example]. A semiprime ring may not contain a commutator nonzero divisor (after all, take commutative semiprime rings, or more generally, semiprime rings R containing a nonzero central idempotent element $e \in R$ such that eR is commutative). Conversely, a ring may contain a commutator nonzero divisor, but is not semiprime. For example, let $R = T_2(A_1)$ be the ring of the 2×2 upper triangular matrices whose entries are elements from the Weyl algebra A_1 (polynomials in x, y such that $xy - yx = 1$). Then R is not semiprime, but the commutator of scalar matrices generated by x and y is the identity matrix.

Finally, we give some well-known examples of not semiprime rings that have commutators nonzero divisors.

Example. Consider the 2×2 matrix algebra $\mathcal{M}_2(R)$ over any ring R with 1. Let E_{ij} be the usual matrix units. Then the commutator $[E_{12}, E_{21}] = E_{11} - E_{22}$ is invertible.

When $char(R) \neq 2$, in $\mathcal{M}_3(R)$ we have that $[E_{12} + E_{23}, E_{21} - E_{32}] = E_{11} - 2E_{22} + E_{33}$ is a nonzero divisor.

Further examples can be seen in [4, Example].

Remark. In the year 1999, J. Vukman [8, Theorem 1] proved that if R is a 2-torsion-free semiprime ring and G: $R \to R$ is an additive mapping such that $2G(x^2) = G(x)x + xG(x)$ holds for every $x \in R$, then G is a left and right centralizer.

If an additive mapping G: $R \to R$, where R is an arbitrary ring, is both left and right Jordan σ-centralizer, then obviously G satisfies the relation $2G(x^2) = G(x)\sigma(x) + \sigma(x)G(x)$, for every $x \in R$. It seems natural to ask whether the result of ([8], Theorem 1) is also true for the relation above on Lie ideals. We are unable to answer this question.

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