NAKAYAMA ISOMORPHISMS FOR THE MAXIMAL QUOTIENT RING OF A LEFT HARADA RING

Dedicated to Professor Takeshi Sumioka on the Occasion of His Sixtieth Birthday

KAZUAKI NONOMURA

Abstract. From several results of Kado and Oshiro, we see that if the maximal quotient ring of a given left Harada ring \( R \) of type (\( \star \)) has a Nakayama automorphism, then \( R \) has a Nakayama isomorphism. This result poses a question whether if the maximal quotient ring of a given left Harada ring \( R \) has a Nakayama isomorphism, then \( R \) has a Nakayama isomorphism. In this paper, we shall show that a basic ring of the maximal quotient ring of a given Harada ring has a Nakayama isomorphism if and only if its Harada ring has a Nakayama isomorphism.

1. Introduction

Let \( R \) be a basic left Harada ring. Then we have a complete set
\[
\{e_{11}, \ldots, e_{1n(1)}, \ldots, e_{m1}, \ldots, e_{mn(m)}\}
\]
of primitive idempotents for \( R \) such that for each \( i = 1, \ldots, m \)

(a) \( e_{i1}R \) is injective as a right \( R \)-module;
(b) \( J(e_{i,k-1}R) \cong e_{ik}R \) for each \( k = 2, \ldots, n(i) \).

We call \( R \) a left Harada ring of type (\( \star \)) if there exists an unique \( g_i \) in \( \{e_{i(i)}\}_{i=1}^m \) for each \( i = 1, \ldots, m \) such that the socle of \( e_{i1}R \) is isomorphic to \( g_iR/J(g_iR) \) and the socle of \( Rg_i \) is isomorphic to \( Re_{i1}/J(Re_{i1}) \).

Oshiro [9] showed the following;

Result A ([9, Theorem 2]). Suppose that \( R \) is a left Harada ring which is not of type (\( \star \)). Then there exists a series of left Harada rings \( T_1, \ldots, T_n \) and surjective ring homomorphisms \( \phi_1, \ldots, \phi_n \):

\[
T_1 \xrightarrow{\phi_1} T_2 \xrightarrow{\phi_2} \cdots \xrightarrow{\phi_{n-1}} T_n \xrightarrow{\phi_n} R
\]
such that

(1) \( T_1 \) is of type (\( \star \)), and
(2) \( \text{Ker} \ \phi_i \) is a simple ideal of \( T_i \) for any \( i \in \{1, \ldots, n\} \).

Kado and Oshiro [7] showed the following results;

Key words and phrases. Maximal quotient rings; Harada rings; Nakayama isomorphisms.
**Result B ([7, Proposition 5.3])**. *If every basic QF rings has a Nakayama automorphism, then every basic left Harada ring of type (*)& has a Nakayama isomorphism.*

**Result C ([7, Proposition 5.4])**. *Let $S$ be a two-sided ideal of $R$ that is simple as a left ideal and as a right ideal. If $R$ has a Nakayama isomorphism, then $R/S$ has a Nakayama isomorphism.*

Moreover Kado showed the following;

**Result D ([6, Corollary])**. *The maximal quotient ring of a left Harada ring of type (*)& is a QF ring.*

Using these four results, we see that if the maximal quotient ring of a given left Harada ring $R$ of type (*)& has a Nakayama automorphism, then $R$ has a Nakayama isomorphism. So this statement poses a question whether if the maximal quotient ring of a given left Harada ring $R$ has a Nakayama isomorphism, then $R$ has a Nakayama isomorphism. In this paper, we shall show that the maximal quotient ring of a given left Harada ring $R$ has a Nakayama isomorphism iff $R$ has a Nakayama isomorphism.

Throughout this paper, we assume that all rings are associative rings with identity and all modules are unitary. We denote the set of primitive idempotents for $R$ by $\Pi(R)$, and denote a complete set of primitive idempotents for $R$ by $\pi(R)$. By $M_R$ (resp. $RM$), we mean that $M$ is a right (resp. left) $R$-module. For a module $M$, we denote the Jacobson radical of $M$ by $J(M)$, the injective hull of $M$ by $E(M)$, the socle of $M$ by $S(M)$, respectively. $L \leq M$ (resp. $L < M$) means $L$ is a submodule of $M$ (resp. $L \leq M$ and $L \neq M$).

We call a one-sided artinian ring $R$ right (resp. left) QF-3 ring if $E(\langle R \rangle_R)$ (resp. $E(\langle R \rangle_L)$) is projective, respectively.

We denote the maximal left (resp. right) quotient ring of $R$ by $Q_L(R)$ (resp. $Q_R(R)$), respectively, and denote the maximal left and maximal right quotient ring of $R$ by $Q(R)$. If a ring is QF-3, its maximal left quotient ring and its right quotient ring coincide by [12, Theorem 1.4].

**2. Maximal quotient ring**

We list some basic results, which several authors showed, for our main result in this paper. Recall that for $e, f \in \Pi(R)$, we say that the pair $(eR : Rf)$ is an $i$-pair if $S(eR) \cong fR/J(fR)$ and $S(Rf) \cong Re/J(Re)$.

**Lemma 2.1 ([5])**. *Let $R$ be a one-sided artinian ring, and let $e \in \Pi(R)$. Then the following conditions are equivalent:*

- $S(eR) \cong fR/J(fR)$ and $S(Rf) \cong Re/J(Re)$
- $L \leq M$ (resp. $L < M$)
- $L \leq M$ and $L \neq M$
(1) $eR$ is injective as a right $R$-module.
(2) There exists some $f \in \Pi(R)$ such that $(eR : Rf)$ is an $i$-pair.
In this case, $Rf$ is also injective as a left $R$-module.

Let $R$ be a left perfect ring. Then $R$ has a primitive idempotent $e$ with $S(R_R)e \neq 0$. If $R$ is QF-3, then the primitive idempotent $e$ with $S(R_R)e \neq 0$ are characterized as follows;

**Lemma 2.2** ([4, Theorem 2.1]). Let $R$ be a one-sided artinian QF-3 ring, and let $e \in \Pi(R)$. Then $RRe$ is injective if and only if $S(R_R)e \neq 0$.

We call $e \in \Pi(R)$ right (resp. left) $S$-primitive if $S(R_R)e \neq 0$ (resp. $eS(R_R) \neq 0$), respectively.

The following statement, which Storrer [11, Proposition 4.8] showed, is helpful in this paper.

**Lemma 2.3** ([11, Proposition 4.8]). Let $R$ and $Q = Q(R)$ be left perfect. Then

(1) If $e$ is a right $S$-primitive idempotent for $R$, then so is it for $Q$.
(2) If $e_1, e_2$ are right $S$-primitive idempotents for $R$, then $e_1 R \cong e_2 R$ if and only if $e_1 Q \cong e_2 Q$.
(3) If $e$ is a right $S$-primitive idempotent for $Q$, then there exists a right $S$-primitive idempotent $e' \in R$ such that $eQ \cong e'R$.

A ring $R$ is called a left Harada ring if it is left artinian and its complete set $\Pi(R)$ of orthogonal primitive idempotents is arranged as follows:

$$\Pi(R) = \bigcup_{i=1}^{m} \{e_{ij}\}_{j=1}^{n(i)},$$

where

(a) each $e_{ij}R_R$ is an injective module for each $i = 1, 2, \ldots, m$.
(b) $e_{i,k-1}R_R \cong e_{ik}R$, or $J(e_{i,k-1}R_R) \cong e_{ik}R$ for each $i$ and each $k = 2, 3, \ldots, n(i)$.
(c) $e_{ik}R \not\cong e_{jl}R$ for $i \neq j$.

**Remark.** Let $R$ be a left Harada ring. Then $Q(R)$ is also a left Harada ring (See [6, Theorem 4]) and a complete set $\Pi(R)$ of orthogonal primitive idempotents for $R$ is also the one of $Q$ (See [6, p.248]).

Using Remark 2, Kado showed the following;

**Proposition 2.4** ([6, Proposition 2]). Let $R$ be a left Harada ring, and let $(eR : Rf)$ be an $i$-pair for $e, f \in \Pi(R)$. Then $(eQ(R) : Q(R)f)$ is an $i$-pair.
Remark. Let $R$ be a basic and left Harada ring. Then we have a complete set of orthogonal primitive idempotents $\pi(R) = \bigcup_{i=1}^{m} \{e_{ij}\}_{j=1}^{n(i)}$ for $R$ satisfying the following conditions:

(a) $e_{i1}R_R$ is injective for each $i = 1, \ldots, m$,
(b) $e_{i,j+1}R_R \cong J(e_{i}R_R)$ for each $j = 1, \ldots, n(i) - 1$.

We have a complete set $\{Rg_1, \ldots, Rg_m\}$ of pairwise non-isomorphic indecomposable injective projective left $R$-modules, such that the $(e_{i1}R : Rg_i)$ are $i$-pair for each $i = 1, \ldots, m$ since $R$ is basic and artinian QF-3. So the number of right $S$-primitive is $m$ by Lemma 2.2.

Recall the following notation [6, p.249]. Let $\theta : fR \to eR$ be an $R$-monomorphism such that $\text{Im} \theta = J(eR)$, where $e, f \in \Pi(R)$. Then by [11, Proposition 4.3], $\theta$ can be uniquely extended to a $Q_r(R)$-homomorphism $\theta^* : fQ_r(R) \to eQ_r(R)$.

We shall need the following results.

**Lemma 2.5** ([6, Proposition 3]). Let $R$ be a basic and left Harada ring, and $Q = Q(R)$ and $\theta$ as above. Then the following hold.

1. If $e$ is not right $S$-primitive, then the extension $\theta^* : fQ \to eQ$ is an isomorphism.
2. If $e$ is right $S$-primitive, then the extension $\theta^* : fQ \to eQ$ is a monomorphism such that $\text{Im} \theta^* = J(eQ)$.

Remark (cf. [11, Lemma 4.2]). Let $\{g_i\} \cup \{f_j\}$ be a complete set of orthogonal primitive idempotents for $R$, where the $g_i$ are right $S$-primitive and the $f_j$ are not right $S$-primitive. We denote $g_0$ by $g_0 = \sum g_i$. Then $Q(R)g = Rg$ and $Q(R)g_0 = Rg_0$ for every right $S$-primitive idempotent $g$ of $R$.

Let $R$ be a basic left artinian ring, and let $\{e_1, e_2, \ldots, e_n\}$ be a complete set of orthogonal primitive idempotents for $R$ and let

$$S = \text{End}_R(\oplus_{i=1}^{n} E(Re_i/J(Re_i)))$$

be the endomorphism ring of a minimal injective cogenerator for the category of left $R$-modules. Let $f_i$ be the primitive idempotent for $S$ corresponding to the projection

$$\oplus_{i=1}^{n} E(Re_i/J(Re_i)) \to E(Re_i/J(Re_i)).$$

Then we call a ring isomorphism $\tau : R \to S$ a Nakayama isomorphism if $\tau(e_i) = f_i$ for each $i = 1, 2, \ldots, n$. By [3, p.42], the existence of a Nakayama isomorphism does not depend on the choice of the complete set $\{e_1, e_2, \ldots, e_n\}$ of orthogonal primitive idempotents. (See [7, Remark on p.387].)
It is important whether the maximal quotient ring of a basic artinian ring is basic since a Nakayama isomorphism is defined on a basic ring. Here we shall study the case that the maximal quotient ring of a given left Harada ring is basic.

**Theorem 2.6 (cf. [2, Corollary 22]).** Let \( R \) be a basic and left Harada ring and \( Q = Q(R) \). Then \( Q \) is a basic ring if and only if \( R \) either is QF or satisfies the following; \( n(i) = 1 \) or \( 2 \) and \( RRe_{i1} \) is injective for any \( i \). In this case \( R = Q \).

**Proof.** Note that both \( R \) and \( Q \) are artinian QF-3. Assume that \( Q \) is basic. Let \( e_{i,k+1}, e_{ik} \in \{e_{ij}\}_{j=2}^{n(i)} \). Then we have an \( R \)-monomorphism \( \theta_{ik} : e_{i,k+1}R \to e_{ik}R \) such that \( \text{Im} \theta_{ik} = J(e_{ik}R) \). If \( e_{ik} \) is not right \( S \)-primitive, then \( e_{ik}Q \cong e_{ik}Q \) by Lemma 2.5. This contradicts that \( Q \) is basic. Hence \( e_{ik} \) is right \( S \)-primitive for \( k = 1, 2, \ldots, n(i) - 1 \). Since the \( RRe_{ik} \) are injective for each \( k = 1, 2, \ldots, n(i) - 1 \) by Lemma 2.2, there exists some \( Rg \) in \( \{Rg_1, \ldots, Rg_m\} \) such that \( RRe_{ik} \cong Rg \). However \( R \) is basic, so we see that \( n(i) = 1 \) or \( 2 \) and \( e_{i1} \) is right \( S \)-primitive.

In case \( n(i) = 1 \) for every \( i = 1, \ldots, m \), then \( R \) is QF.

In case \( n(i) = 2 \) for some \( i \in \{1, \ldots, m\} \). If \( e_{in(i)} \) is right \( S \)-primitive, then \( RRe_{in(i)} \) is injective by Lemma 2.2. Hence \( e_{in(i)} \) is not right \( S \)-primitive since \( RRe_{i1} \) is injective and so \( \{Rg_1, \ldots, Rg_m\} = \{Re_{11}, \ldots, Re_{m1}\} \).

Conversely, first, assume that \( R \) is QF. Since \( RRe \) is injective for any \( e \in \pi(R) \), \( e \) is right \( S \)-primitive by Lemma 2.2. Thus, \( eQ \not\cong fQ \) for any \( e, f \in \pi(R) = \pi(Q) \) by Lemma 2.3. Therefore \( Q \) is basic. Next, assume that \( R \) satisfies \( n(i) = 1 \) or \( 2 \) and \( RRe_{i1} \) is injective for any \( i \). Then \( e_{i1} \) is left \( S \)-primitive and so \( eQ = eR \) by Remark 2. Hence \( J(eQ) = J(eR) \). Therefore it is also clear to see that \( R = Q \). \( \square \)

**Example.** We shall give a basic left Harada ring \( R \) with \( J(R)^5 = 0 \), which is not QF. Let \( R \) be an algebra over a field \( K \) defined by the following quiver;

\[
\begin{array}{ccc}
\gamma & \overrightarrow{1} & \gamma' \\
\beta & \alpha & 4 \\
\beta' & \beta & \gamma
\end{array}
\]

with the relations \( \gamma\beta = \gamma'\beta' \), \( \alpha\gamma\beta = 0 \), and \( \beta'\alpha\gamma = 0 \).
The composition diagrams of the Loewy factors of the indecomposable projective modules of \( R \) is the following.

\[
\begin{array}{c}
e_{R}/eJ & 1 & 2 & 3 & 4 \\
e_{J}/eJ^2 & 2 & 3 & 4 & \text{ } & 1 & 1 \\
e_{J^2}/eJ^3 & 3 & 4 & 1 & 2 & 2 & \text{ } \\
e_{J^4} & 1 & \text{ } & 4 & 3 & \text{ } & \text{ }
\end{array}
\]

Then \( R \) is a left Harada ring which is not QF since \( e_{1}R, e_{3}R \) and \( e_{4}R \) are injective and \( e_{2}R \cong J(e_{1}R) \). Moreover \( e_{1}, e_{3}, e_{4} \) are right \( S \)-primitive. Hence \( e_{1}Q(R) = e_{1}R, e_{3}Q(R) = e_{3}R \) and \( e_{4}Q(R) = e_{4}R \) are injective and \( e_{2}Q(R) \cong J(e_{1}Q(R)) \). Therefore \( R = Q(R) \).

Example. We shall give a basic Harada ring \( R \) with \( J(R)^{6} = 0 \), but \( Q(R) \) is not basic. Let \( R \) be an algebra over a field \( K \) defined by the following quiver:

\[
\begin{array}{c}
\gamma \vee 1 \gamma' \\
\alpha \\
\beta \vee 2 \beta'
\end{array}
\]

with the relations \( 0 = \beta\alpha\gamma\beta = \beta'\alpha\gamma'\beta' = \beta\alpha\gamma = \beta'\alpha\gamma' \), and \( \gamma\beta = \gamma'\beta' \). Then the composition diagrams of the Loewy factors of the indecomposable projective modules of \( R \) is the following.

\[
\begin{array}{c}
e_{i}R/e_{i}J & 1 & 2 & 3 & 4 \\
e_{i}J/e_{i}J^2 & 2 & 3 & 4 & \text{ } & 1 & 1 \\
e_{i}J^2/e_{i}J^3 & 3 & 4 & 1 & 2 & 2 & \text{ } \\
e_{i}J^4/e_{i}J^5 & 1 & \text{ } & 4 & 3 & \text{ } & \text{ }
\end{array}
\]

Then since \( e_{1}R, e_{3}R \) and \( e_{4}R \) are injective and \( e_{2}R \cong J(e_{1}R) \), \( R \) is a left Harada ring which is not QF. Hence \( e_{2}Q(R) \cong e_{1}Q(R) \) since \( e_{1} \) is not right \( S \)-primitive. Therefore \( Q(R) \) is not basic.

3. Nakayama isomorphism

In this section, we study the Nakayama isomorphisms for the representative matrix ring of a basic left Harada ring and its maximal quotient ring. Let \( R \) be a basic left Harada ring, and let \( \pi_{i}(R) = \bigcup_{j=1}^{n} \{ e_{ij} \} \) be a complete set of orthogonal primitive idempotents as in Remark 2. Furthermore,
let $R^*$ be the representative matrix ring of $R$. $R^*$ is represented as block matrices as follows:

$$R^* = \begin{pmatrix} R_{11}^* & \cdots & R_{1m}^* \\ \vdots & \ddots & \vdots \\ R_{m1}^* & \cdots & R_{mm}^* \end{pmatrix},$$

where $R_{ij}^* = P_{ij}$ for $j \neq \sigma(i)$ and $R_{i\sigma(i)}^* = P_{i\sigma(i)}^*$ (See [7, Section 4]).

Here, adding one row and one column to $R^*$, we make an extended matrix ring $W_i(R)$ of $R$ as follows:

$$W_i(R) = \begin{pmatrix} R_{11}^* & \cdots & R_{1i}^* & Y_1 & R_{1,i+1}^* & \cdots & R_{1m}^* \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots \\ R_{i1}^* & \cdots & R_{ii}^* & Y_i & R_{i,i+1}^* & \cdots & R_{im}^* \\ X_1 & \cdots & X_{i-1} & X_i & Q & X_{i+1} & \cdots & X_m \\ R_{i+1,1}^* & \cdots & R_{i+1,i}^* & Y_{i+1} & R_{i+1,i+1}^* & \cdots & R_{i+1,m}^* \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots \\ R_{m1}^* & \cdots & R_{mi}^* & Y_m & R_{m,i+1}^* & \cdots & R_{mm}^* \end{pmatrix},$$

where $X_k$ is the last row of $R_{ik}^*$ $(k = 1, \ldots, m, \ k \neq i)$, $Y_k$ is the last column of $R_{ki}^*$ $(k = 1, \ldots, m)$, $X_i = (P_{in(i),i1}^* \cdots P_{in(i),in(i)-1}^* J(P_{in(i),in(i)}^*))$, and $Q = P_{in(i),in(i)}^*$.

Then $W_i(R)$ naturally becomes a ring by operations of $R^*$. We call this the $i$-th extended ring of $R$.

**Proposition 3.1** ([7, Proposition 5.11]). If $W_i(R)$ has a Nakayama isomorphism, then $R$ also has a Nakayama isomorphism.

Let $R$ be a basic and left Harada ring, and let

$$\pi(R) = \bigcup_{i=1}^{m} \{ e_{ij} \}_{j=1}^{n(i)}$$

be a complete set of orthogonal primitive idempotents for $R$ as given in Remark 2. Then (See [7, p.388]), for any $e_{ij}$ in $\pi(R)$, there exists some $g_i$ in $\pi(R)$ with $Rg_i$ injective such that $E(Re_{ij}/J(Re_{ij})) \cong Rg_i/S_{j-1}(Rg_i)$, where $S_j(Rg_i)$ is the $j$-th socle of $Rg_i$. We denote the generator $g_i + S_{j-1}(Rg_i)$ of $Rg_i/S_{j-1}(Rg_i)$ by $g_{ij}$ for each $i = 1, \ldots, m, j = 1, \ldots, n(i)$. By [7, Proposition 3.2], a minimal injective cogenerator $G = \bigoplus_{i,j} Rg_{ij}$ is finitely generated. Therefore we note that $R$ is left Morita dual to $\text{End}_R(G)$ by [1, Theorem 30.4]. We call this $\text{End}_R(G)$ the dual ring of $R$. We denote the dual ring of $R$ by $T(R)$. 


For the proof of proposition 3.2 below, we denote
\[
\begin{pmatrix}
0 & \cdots & 0 \\
0 & \cdots & R_{ij}^* & 0 & \cdots & 0 \\
0 & \cdots & 0
\end{pmatrix} \subseteq R^*
\]
by \([R_{ij}^*]\) and

\[
\begin{pmatrix}
0 & \cdots & 0 \\
0 & \cdots & R_{ij}^* & 0 & \cdots & 0 \\
0 & \cdots & 0
\end{pmatrix} \subseteq W_i(R)
\]
by \([R_{ij}^*]_w\),

\[
\begin{pmatrix}
0 & \cdots & 0 \\
0 & \cdots & X_k & 0 & \cdots & 0 \\
0 & \cdots & 0
\end{pmatrix} \subseteq W_i(R)
\]
by \([X_k]_w\),

\[
\begin{pmatrix}
0 & \cdots & 0 \\
0 & \cdots & Y_l & 0 & \cdots & 0 \\
0 & \cdots & 0
\end{pmatrix} \subseteq W_i(R)
\]
by \([Y_l]_w\),

\[
\begin{pmatrix}
0 & \cdots & 0 \\
0 & \cdots & Q & 0 & \cdots & 0 \\
0 & \cdots & 0
\end{pmatrix} \subseteq W_i(R)
\]
by \([Q]_w\).

By using the result that Kado and Oshiro [7, Proposition 5.11] showed, we shall show the following proposition. The proposition is essential in this paper.

**Proposition 3.2.** \(W_i(R)\) has a Nakayama isomorphism if and only if so does \(R\).

**Proof.** \((\Rightarrow)\). By Proposition 3.1 ([7, Proposition 5.11]). \((\Leftarrow)\). As [7, Proposition 5.11], let \(e_{ij}\) be the matrix of \(R^*\) such that the \((ij, ij)\)-component is the unity and other components are zero, and let \(w_{ij}\) be the matrix of \(W_i(R)\) such that the \((ij, ij)\)-component is the unity and other components are zero. Note that the size of the columns in \(W_i(R)\) is \(n(i) + 1\). Let \(\Psi\) be the natural
embedding homomorphism;

\[ \begin{pmatrix}
R_{11}^* & \cdots & R_{im}^* \\
\vdots & & \vdots \\
R_{m1}^* & \cdots & R_{mm}^*
\end{pmatrix}
\]

where \( R_{ij}^* \rightarrow R_{ij}^* \) are identity maps for all \( i, j \). Moreover let \( h_{ij} \) be the matrix of \( T(R) \) such that the \((i, j)\)-component is the unity and other components are zero, and let \( v_{ij} \) be the matrix of \( W_i(T(R)) \) such that the \((i, j)\)-component is the unity and other components are zero. Note that the size of the columns in \( W_i(T(R)) \) is \( n(i) + 1 \). Let

\[ T(R)_{1i} \quad \cdots \quad T(R)_{1m} \\
T(R)_{m1} \quad \cdots \quad T(R)_{mm} \]

be the representative matrix ring \( T(R)^* \) of \( T(R) \), and let \( T(W_i(R)) \) be the dual ring of \( W_i(R) \) as follows;

\[
\begin{pmatrix}
T(R)_{11} & \cdots & T(R)_{1i} & tY_1 & T(R)_{1,i+1} & \cdots & T(R)_{1m} \\
\vdots & & \vdots & \vdots & \vdots & & \vdots \\
T(R)_{i1} & \cdots & T(R)_{ii} & tY_i & T(R)_{i,i+1} & \cdots & T(R)_{im} \\
tX_1 & \cdots & tX_i & tQ & tX_{i+1} & \cdots & tX_m \\
T(R)_{i+1,1} & \cdots & T(R)_{i+1,i} & tY_{i+1} & T(R)_{i+1,i+1} & \cdots & T(R)_{i+1,m} \\
\vdots & & \vdots & \vdots & \vdots & & \vdots \\
T(R)_{m1} & \cdots & T(R)_{mi} & tY_m & T(R)_{m,i+1} & \cdots & T(R)_{mm}
\end{pmatrix}
\]

Let \( \Psi_{T(R)} \) be the natural embedding homomorphism;

\[
\begin{pmatrix}
T(R)_{11} & \cdots & T(R)_{1m} \\
\vdots & & \vdots \\
T(R)_{m1} & \cdots & T(R)_{mm}
\end{pmatrix}
\]

\[ \downarrow \Psi_{T(R)} \]
Here we define a map \( \overline{\varphi} \) following diagram;

\[
\begin{pmatrix}
T(R)_{11} & \cdots & \cdots & T(R)_{1i} & 0 & T(R)_{1,i+1} & \cdots & T(R)_{1m} \\
\vdots & \ddots & \ddots & \vdots & \vdots & \vdots & & \vdots \\
T(R)_{i1} & \cdots & \cdots & T(R)_{ii} & 0 & T(R)_{i,i+1} & \cdots & T(R)_{im} \\
0 & \cdots & 0 & 0 & 0 & 0 & \cdots & 0 \\
T(R)_{i+1,1} & \cdots & \cdots & T(R)_{i+1,i} & 0 & T(R)_{i+1,i+1} & \cdots & T(R)_{i+1,m} \\
\vdots & \ddots & \ddots & \vdots & \vdots & \vdots & & \vdots \\
T(R)_{m1} & \cdots & \cdots & T(R)_{mi} & 0 & T(R)_{m,i+1} & \cdots & T(R)_{mm}
\end{pmatrix}_{i+1},
\]

where \( T(R)_{ij} \rightarrow T(R)_{ij} \) are identity maps for all \( i, j \). We note that \( T(W_i(R)) = W_i(T(R)) \) (See [7, Proposition 5.11]).

Assume that \( \varphi : R^* \rightarrow T(R)^* \) is a Nakayama isomorphism with \( \varphi(e_{ij}) = h_{ij} \). (i.e., \( \varphi([r_{kl}]) \in [T(R)_{kl}] \) for any \([r_{kl}] \in [R^*_{ij}] \), where \((k,l)\)-componentwise of \( R^*_{ij} \) corresponds to \((k,l)\)-componentwise of \( T(R)_{ij} \).) We consider the following diagram:

\[
\begin{array}{ccc}
W_i(R) & \xrightarrow{\Psi} & W_i(T(R)) \\
\uparrow & & \uparrow \\
R^* & \xrightarrow{\varphi} & T(R)^* \\
\end{array}
\]

Here we define a map \( \widetilde{\varphi} : W_i(R) \rightarrow W_i(T(R)) \) as follows:

(a) \( \widetilde{\varphi}([r_{kl}]^w) = [\varphi([r_{kl}])]^w \in [T(R)_{kl}]^w \) for any \([r_{kl}]^w \in [R^*_{kl}]^w; 1 \leq k \leq m, 1 \leq l \leq m; \)
(b) \( \widetilde{\varphi}([x]^w) \in [tX_k]^w \) for any \([x]^w \in [X_k]; k = 1, \ldots, m; \)
(c) \( \widetilde{\varphi}([y]^w) \in [tY_l]^w \) for any \([y]^w \in [Y_l]^w; l = 1, \ldots, m; \)
(d) \( \widetilde{\varphi}([q]^w) \in [tQ]^w \) for any \([q]^w \in [Q]^w. \)

Since \( \varphi(e_{ij}) = h_{ij} \), \( \widetilde{\varphi} \) is well-defined. Moreover it is satisfied \( \widetilde{\varphi}([r_{kl}]^w) = \psi_{i,n(i)+1} \cdot [r_{kl}]^w \in [R^*_{kl}]^w \) implies \([r_{kl}] \in [R^*_{kl}] \). So we can easily check that \( \widetilde{\varphi} \) is a ring homomorphism. Then since \( \varphi \) is a Nakayama isomorphism, we see that \( \widetilde{\varphi} \) is also injective and surjective. Therefore \( \widetilde{\varphi} \) is a Nakayama isomorphism. \( \square \)

**Remark.** We shall define a special case of an extended ring for a given ring \( R \). Let \( \{e_1, e_2, \ldots, e_n\} \) be a complete set of orthogonal primitive idempotents.
for $R$. Then for a primitive idempotent $e_i$ in $R$, we define $R_{e_i}$ as follows:

$$
\begin{pmatrix}
  e_1Re_1 & \cdots & e_1Re_i & Y_1 & e_1Re_{i+1} & \cdots & e_1Re_n \\
  \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\
  e_iRe_1 & \cdots & e_iRe_i & Y_i & e_iRe_{i+1} & \cdots & e_iRe_n \\
  X_1 & \cdots & X_i & U & X_{i+1} & \cdots & X_n \\
  e_{i+1}Re_1 & \cdots & e_{i+1}Re_i & Y_{i+1} & e_{i+1}Re_{i+1} & \cdots & e_{i+1}Re_n \\
  \vdots & \cdots & \vdots & \vdots & \vdots & \ddots & \vdots \\
  e_nRe_1 & \cdots & e_nRe_i & Y_n & e_nRe_{i+1} & \cdots & e_nRe_n 
\end{pmatrix},
$$

where the $X_j$ are $e_iRe_j$ for $j = 1, \ldots, i - 1, i + 1, \ldots, n$, $X_i$ is $J(e_iRe_i)$, the $Y_k$ are $e_kRe_i$ for $k = 1, \ldots, n$ and $U$ is $e_iRe_i$. Then $R_{e_i}$ is a ring by usual matrix operations.

**Remark.** Proposition 3.2 says that a basic left Harada ring $R$ has a Nakayama isomorphism if and only if so does $R_e$ for $e \in \pi(R) = \bigcup_{i=1}^m \{ e_{ij} \}^{n(i)}$.

**Remark.** If $R$ is a one-sided artinian QF-3 ring, the number of right $S$-primitive idempotents for $R$ coincides with that of left $S$-primitive idempotents for $R$.

We denote a basic ring of $Q(R)$ by $Q^b(R)$.

Let $R$ be a basic and left Harada ring, let $Q = Q(R)$, and let $\pi(R) = \bigcup_{i=1}^m \{ e_{ij} \}^{n(i)}$ be a complete set of primitive idempotents for $R$ as given in Remark 2.

(a). First, we consider the following three cases.

(i). We take $\{ e_{ij} \}^{n(i)}$ without right $S$-primitive idempotents. Then $e_{i1}Q \cong e_{ij}Q$ for $j = 2, \ldots, n(i)$ by Lemma 2.5. So $Q^b$ has $e_{i1}$ as a primitive idempotent. Note that if we have $\{ e_{ij} \}^{n(i)}$ without right $S$-primitive idempotents, there exists some $k \neq i \in \{ 1, \ldots, m \}$ such that $\{ e_{kj} \}^{n(k)}$ has two or more right $S$-primitive idempotents by Remark 3.

(ii). We take $\{ e_{ij} \}^{n(i)}$ with a right $S$-primitive idempotent. Let $e_{ik}$ be a right $S$-primitive idempotent. Then by Lemma 2.5 it is satisfied the following:

$$
\begin{cases}
  e_{i1}Q \cong e_{ij}Q & \text{for } j = 2, \ldots, k; \\
  e_{i,k+1}Q \cong J(e_{ik}Q) & \text{and} \\
  e_{i,k+1}Q \cong e_{ij}Q & \text{for } j = k + 2, \ldots, n(i).
\end{cases}
$$

So $Q^b$ has $e_{i1}, e_{ik}$ as primitive idempotents. Note that if $e_{in(i)}$ is a right $S$-primitive idempotent, then $e_{i1}Q \cong e_{ij}Q$ for $j = 2, \ldots, n(i)$ by Lemma 2.5.

(iii). We take $\{ e_{ij} \}^{n(i)}$ with two or more right $S$-primitive idempotents. Let $e_{ikt}$ $(2 \leq t < n(i))$ be right $S$-primitive idempotents. Then by Lemma
2.5 it is satisfied the following sequence:

\[
e_{i1}Q > e_{i1} J(Q) \\
| \uparrow \ |
\]
\[
e_{i,k1+1}Q > J(e_{i,k1+1}Q) \\
| \uparrow \ |
\]
\[
e_{i,k2+1}Q > J(e_{i,k2+1}Q) \\
| \uparrow \ |
\]
\[
e_{i,k3+1}Q \quad \ldots
\]

So \( Q^b \) has \( e_{i1}, e_{ikt+1} \) as primitive idempotents.

Note that if every \( \{ e_{ij} \}_{j=1}^{n(i)} \) for any \( i = 1, \ldots, m \) has only one right \( S \)-primitive idempotent, say \( e_{ik(i)} \), then by (ii), \( \bigcup_{i=1}^{m} \{ e_{i1}, e_{ik(i)}+1 \} \) is a complete set of the primitive idempotents \( \pi(Q^b) \) for \( Q^b \) with \( e_{i1} Q^b \) is injective. Since \( e_{i1} \) is left \( S \)-primitive, \( e_{i1} R = e_{i1} Q \) by Remark 2 and so \( e_{i1} Re_{i1} = e_{i1} Q e_{i1} \). Moreover if we have some \( i \in \{ 1, \ldots, m \} \) such that \( \{ e_{ij} \}_{j=1}^{n(i)} \) has no right \( S \)-primitive idempotents, then there exist some \( k \neq i \in \{ 1, \ldots, m \} \) such that \( \{ e_{kj} \}_{k=1}^{n(k)} \) has two or more right \( S \)-primitive idempotents by Remark 2. Let \( e = \sum_{i=1}^{m} e_{i1} + \sum e_{ikt+1} \), where the \( e_{ik} \) are right \( S \)-primitive. Therefore if we cooperate (i), (ii) or (iii), we can make the basic ring \( Q^b \) isomorphic to \( eRe \). Furthermore we see that \( Q^b \) is isomorphic to \( eRe \) for some idempotent \( e \) of \( R \) if \( Q \) is not basic.

(b). Next, we consider the following three conditions:

(iv). If some \( \{ e_{h1} \}_{h=1}^{n(h)} \subset \pi(R) \) has the right \( S \)-primitive \( e_{hn(h)} \), then putting \( e_h = e_{h1} + \cdots + e_{hn(h)} \), by Lemma 2.3 and Lemma 2.5, we see \( e_h R = e_h Q \).

(v). If \( \{ e_{h1} \}_{h=1}^{n(h)} \) has no right \( S \)-primitive, then by Remark 3, \( Q^b_{e_{h1}} \) is isomorphism to a ring with the complete set \( \pi(Q^b) \cup \{ e_{h2} \} \) of primitive idempotents.

Let

\[
Q^b = \begin{pmatrix}
* & e_{11} Re_{h1} & * \\
\vdots & \ldots & \ldots & \ldots \\
e_{h1} Re_{11} & \ldots & e_{h1} Re_{h1} & \ldots & e_{h1} Re_{m1} & \ldots \\
* & \ldots & \ldots & \ldots & e_{m1} Re_{h1} & * \\
\vdots & \ldots & \ldots & \ldots & \ldots & \ldots
\end{pmatrix}
\]
Then by Remark 3,

\[
Q_{e_{h_1}}^b = \begin{pmatrix}
  * & e_{11}Re_{h_1} & e_{11}Re_{h_1} & * \\
  e_{h_1}Re_{11} & \cdots & e_{h_1}Re_{h_1} & e_{h_1}Re_{h_1} & \cdots & e_{h_1}Re_{m_1} & \cdots \\
  e_{h_1}Re_{11} & \cdots & J(e_{h_1}Re_{h_1}) & e_{h_1}Re_{h_1} & \cdots & e_{h_1}Re_{m_1} & \cdots \\
  * & e_{m_1}Re_{h_1} & e_{m_1}Re_{h_1} & * \\
  \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
\end{pmatrix}
\]

For two ideals \(A, B\) of \(Q_{e_{h_1}}^b\) as follows:

\[
A = \begin{pmatrix}
  0 & \cdots & \cdots & \cdots & 0 \\
  e_{h_1}Re_{11} & e_{h_1}Re_{h_1} & e_{h_1}Re_{h_1} & \cdots & e_{h_1}Re_{m_1} \\
  0 & \cdots & \cdots & \cdots & 0 \\
\end{pmatrix},
\]

\[
B = \begin{pmatrix}
  0 & \cdots & \cdots & 0 \\
  0 & \cdots & \cdots & \cdots & 0 \\
  e_{h_1}Re_{11} & J(e_{h_1}Re_{h_1}) & e_{h_1}Re_{h_1} & e_{h_1}Re_{m_1} & \cdots \\
  0 & \cdots & \cdots & \cdots & 0 \\
\end{pmatrix}
\]

we have \(J(A) \cong B\) by [10, Theorem 1].

Hence we have, as a ring isomorphism,

\[
\begin{pmatrix}
  * & e_{11}Re_{h_1} & e_{11}Re_{h_2} & * \\
  e_{h_1}Re_{11} & \cdots & e_{h_1}Re_{h_1} & e_{h_1}Re_{h_2} & \cdots & e_{h_1}Re_{m_1} & \cdots \\
  e_{h_2}Re_{11} & \cdots & e_{h_2}Re_{h_1} & e_{h_2}Re_{h_2} & \cdots & e_{h_2}Re_{m_1} & \cdots \\
  * & e_{m_1}Re_{h_1} & e_{m_1}Re_{h_2} & * \\
  \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
\end{pmatrix}
\]
We denote a right extended ring with the complete set $\pi(Q)$. Then the complete set $\pi(Q)$ consists of primitive idempotents. We denote a right $S$-primitive idempotent of $Q$ by $e_{hk}$. We reset

$$
\{e_{h1}\}_{h=1}^{n(h)} = \{e_{h1}, \ldots, e_{hk}, \ldots, e_{hk}, \ldots\}.
$$

Then the complete set $\pi(Q)$ of $Q$ is $\bigcup_{t \geq 1} \{e_{i1}, e_{i,k1+1}\}$. First by the same argument above for $e_{i1}, e_{i,k1+1}$, we have a ring isomorphic to a ring with the complete set $\{e_{i1}, \ldots, e_{i,k1+1}\} \subset \pi(R)$. Next, by [10, Theorem 1], repeating the same argument like as (iv), for $e_{i,k1+1}, e_{i,k2+1}$, we have a ring isomorphism to a ring with the complete set $\{e_{i1}, \ldots, e_{ik1}, e_{ik1+1}, \ldots, e_{ik2}, e_{i,k2+1}\}$. Hence the suitable extended ring of $Q$ is isomorphic to $R$.

Therefore by (a)-(i),(ii),(iii) and (b)-(iv),(v),(vi) above together with Proposition 3.2 (Remark 3), we get the following main theorem:

**Theorem 3.3.** Let $R$ be a basic and left Harada ring and let $Q = Q(R)$. Then $Q$ has a Nakayama isomorphism if and only if so does $R$.

**Example.** Let

\[
V = \begin{pmatrix}
Q_1 & Q_1 & Q_1 & A & A \\
J_1 & Q_1 & Q_1 & A & A \\
J_1 & J_1 & Q_1 & A & A \\
B & B & B & B & Q_2
\end{pmatrix}
\]

and

\[
K = \begin{pmatrix}
0 & 0 & 0 & 0 & S(AQ_2) \\
0 & 0 & 0 & 0 & S(AQ_2) \\
0 & 0 & 0 & 0 & S(AQ_2) \\
0 & 0 & S(BQ_1) & S(BQ_1) & 0 \\
0 & 0 & S(BQ_1) & S(BQ_1) & 0
\end{pmatrix},
\]

by [10, Theorem 1] again. Similarly repeating $n(h) - 2$ times, we can make an extended ring with the complete set $\pi(Q) \cup \{e_{hj}\}_{j=2}^{n(h)}$ of primitive idempotents.

(vi). Assume that $\{e_{h1}\}_{h=1}^{n(h)} \subset \pi(R)$ has one or more right $S$-primitive idempotents. We denote a right $S$-primitive idempotent of $\{e_{h1}\}_{h=1}^{n(h)}$ by $e_{hk}$. We reset

\[
\{e_{h1}\}_{h=1}^{n(h)} = \{e_{h1}, \ldots, e_{hk}, \ldots, e_{hk}, \ldots\}.
\]
where $Q_i$ is local, and $J_i = J(Q_i)$ for $i = 1, 2$. We put $R = V/K$. We abbreviate this as

$$R = \left( \begin{array}{cccccc}
Q_1 & Q_1 & Q_1 & A & \bar{A} \\
J_1 & Q_1 & Q_1 & A & \bar{A} \\
J_1 & J_1 & Q_1 & A & \bar{A} \\
J_1 & J_1 & J_1 & Q_1 & A & \bar{A} \\
B & B & \bar{B} & \bar{B} & Q_2 & Q_2 \\
B & B & \bar{B} & \bar{B} & J_2 & Q_2
\end{array} \right).$$

Then $R$ is a basic left Harada ring, and we have a complete set

$$\{e_{11}, e_{12}, e_{13}, e_{14}, e_{21}, e_{22}\}$$

of orthogonal primitive idempotents for $R$, where $(e_{11}R; Re_{21})$ and $(e_{21}R; Re_{12})$ are $i$-pairs. First, let

$$e_{11}R \vee e_{12}R \to e_{11}J(R)$$

$$e_{13}R \vee e_{14}R \to e_{12}J(R)$$

$$e_{14}R \to e_{13}J(R)$$

$$e_{21}R \vee e_{22}R \to e_{21}J(R)$$

be projective covers. Then since $e_{12}, e_{21}$ are right $S$-primitive, we have, by Lemma 2.5, the following:

$$e_{12}Q(R) \cong e_{11}Q(R)$$

$$e_{14}Q(R) \cong e_{13}Q(R) \cong e_{12}J(Q(R))$$

$$e_{21}Q(R) \vee e_{22}Q(R) \to e_{21}J(Q(R)).$$

Hence we see

$$Q(R) \cong \left( \begin{array}{cccccc}
Q_1 & Q_1 & Q_1 & A & \bar{A} \\
Q_1 & Q_1 & Q_1 & A & \bar{A} \\
J_1 & J_1 & Q_1 & A & \bar{A} \\
J_1 & J_1 & Q_1 & A & \bar{A} \\
B & B & \bar{B} & \bar{B} & Q_2 & Q_2 \\
B & B & \bar{B} & \bar{B} & J_2 & Q_2
\end{array} \right).$$
So a basic ring of $Q(R)$ is the following:

$$Q^b(R) \cong \begin{pmatrix} Q_1 & Q_1 & A & \bar{A} \\ J_1 & Q_1 & A & \bar{A} \\ B & \bar{B} & Q_2 & Q_2 \\ B & \bar{B} & J_2 & Q_2 \end{pmatrix}.$$

Therefore we see that, as a ring isomorphism,

$$\begin{pmatrix} Q_1 & Q_1 & A & \bar{A} \\ J_1 & Q_1 & A & \bar{A} \\ B & \bar{B} & Q_2 & Q_2 \\ B & \bar{B} & J_2 & Q_2 \end{pmatrix} \cong (e_{11} + e_{13} + e_{21} + e_{22})R(e_{11} + e_{13} + e_{21} + e_{22}).$$

Next, adding $e_{11}$ to $Q^b$ yields

$$\begin{pmatrix} Q_1 & Q_1 & A & \bar{A} \\ J_1 & Q_1 & A & \bar{A} \\ B & \bar{B} & Q_2 & Q_2 \\ B & \bar{B} & J_2 & Q_2 \end{pmatrix} \cong (e_{11} + e_{12} + e_{13} + e_{21} + e_{22})R(e_{11} + e_{12} + e_{13} + e_{21} + e_{22}).$$

Then we get a ring isomorphism
Moreover adding $e_{14}$ to $Q^b_{e_{11}}$ \( \cong \begin{pmatrix} Q_1 & Q_1 & Q_1 & A & \overline{A} \\ J_1 & Q_1 & Q_1 & A & \overline{A} \\ J_1 & J_1 & Q_1 & A & \overline{A} \\ B & B & B & Q_2 & Q_2 \\ B & B & B & J_2 & Q_2 \end{pmatrix} \), according to Remark 3, \((Q^b_{e_{11}})_{e_{14}}\) is isomorphic to
\[
\begin{pmatrix} Q_1 & Q_1 & Q_1 & A & \overline{A} \\ J_1 & Q_1 & Q_1 & A & \overline{A} \\ J_1 & J_1 & Q_1 & A & \overline{A} \\ B & B & B & \overline{B} & Q_2 \end{pmatrix}
\]
\( \cong R. \)

4. Another question

Oshiro’s result (Result A) in the introduction also poses another question whether there exist surjective ring homomorphisms $\bar{\phi}_1, \ldots, \bar{\phi}_n$ with the following commutative diagrams:

\[
\begin{array}{c}
\begin{array}{ccc}
Q(T_1) & \bar{\phi}_1 & Q(T_2) \\
\vee & \vee & \vee \\
T_1 & \phi_1 & T_2 \\
\end{array}
\end{array}
\quad \begin{array}{ccc}
\ldots & \phi_{n-1} & Q(T_n) \\
\vee & \vee & \vee \\
T_1 & \phi_{n-1} & T_n \\
\end{array}
\quad \begin{array}{ccc}
Q(R) & \bar{\phi}_n & Q(R) \\
\vee & \vee & \vee \\
\end{array}
\]

However K. Koike informed the author the following examples;

**Example.** Let $Q$ be a local serial ring, and $J(Q) \neq 0, J(Q)^2 = 0$. Then $J(Q) = S(Q)$. We put

\[
R = \left( \begin{array}{cc}
Q & Q \\
J & Q \\
\end{array} \right) / \left( \begin{array}{cc}
0 & J \\
0 & J \\
\end{array} \right),
\]

where $J = J(Q)$. Then $R$ is a serial ring of an admissible sequence $(3,2)$ and so we see that $R = Q(R)$. Also

\[
T_1 = \left( \begin{array}{cc}
Q & Q \\
J & Q \\
\end{array} \right), \quad T_2 = \left( \begin{array}{cc}
Q & Q \\
J & Q \\
\end{array} \right) / \left( \begin{array}{cc}
0 & J \\
0 & 0 \\
\end{array} \right),
\]

\[
Q(T_1) = \left( \begin{array}{cc}
Q & Q \\
Q & Q \\
\end{array} \right), \quad Q(T_2) = T_2.
\]

\[
\begin{pmatrix} J & J \\
J & J \end{pmatrix}
\]

is a unique non-trivial ideal of $Q(T_1)$. Hence there does not exist a surjective ring homomorphism $Q(T_1)$ to $Q(T_2)$. 

Example. We put

\[
T = \begin{pmatrix} K & K & K \\ 0 & K & K \\ 0 & 0 & K \end{pmatrix},\quad I = \begin{pmatrix} 0 & 0 & K \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},
\]

where \( K \) is a field, and \( R = T/I \). Then \( R \) is a serial ring of an admissible sequence \((2,2,1)\) and we have a natural map

\[
T = T_1 \to R.
\]

However the maximal quotient ring \( Q(T) \) of \( T \) is the full matrix algebra with degree 3 over a field \( K \) and \( Q(R) = R \). Since \( Q(T) \) is semisimple, there does not exist a surjective ring homomorphism \( Q(T) \to Q(R) \).

Acknowledgements

The author thanks to Prof. K. Oshiro, Prof. Y. Baba, Prof. T. Sumioka, Prof. K. Koike and Prof. J. Kado for their valuable advice.

References


Kazuaki NONOMURA
DEPARTMENT OF GENERAL SCIENCE
TSURUOKA NATIONAL COLLEGE
OF TECHNOLOGY, 997-8511, JAPAN
e-mail address: nonomura@tsuruoka-nct.ac.jp

(Received October 28, 2006)