Math. J. Okayama Univ. **51** (2009), 71–81

LINKAGE AND DUALITY OF MODULES

Kenji NISHIDA

Abstract. Martsinkovsky and Strooker [13] recently introduced module theoretic linkage using syzygy and transpose. This generalization brings possibility of much application of linkage, especially, to homological theory of modules. In the present paper, we connect linkage of modules to certain duality of modules. We deal with Gorenstein dimension, Cohen-Macaulay modules over a Gorenstein local ring using linkage and generalize the results to non-commutative algebras.

1. Introduction

The theory of linkage, devised by Peskine and Szpiro[16], is recently generalized to module theoretic version by Martsinkovsky and Strooker in [13]. They generalize the theory for wide class of rings, including non-commutative semiperfect Noetherian rings. The use of composition of two sort of functors, syzygy and transpose, enables them to extend the definition of linkage. These functors are fundamental for homological theory of Noetherian rings. There appears a relation to the duality theory introduced and studied by Auslander and Bridger[1] and Iyama[11], that is, vanishing of certain extension group is equivalent to existence of linkage for a dual of a module.

In section two, we shall apply this fact to Gorenstein dimension and Cohen-Macaulayness of modues, mainly over commutative Gorenstein local ring, and obtain characterization of these notions by linkage. A duality studied by Iyama[11] can be applicable to the full subcategory of Cohen-Macaulay modules of codimension $k > 0$ over a Gorenstein local ring. We decide the image of this subcategory and give a characterization of a Cohen-Macaulay module of codimension $k > 0$ (Theorem 2.6, Corollary 2.7).

In section three, we make an attempt to gerenalize the results of section two to non-commutative Noetherian ring. It is desirable that a candidate satisfies the equations (2.1) and (2.2) in section two. A module-finite Gorenstein algebra studied in [8] satisfies them. Therefore, we treat such algebras and show that these equations hold. We find that similar proof works for module-finite Gorenstein algebras and get the same results as in section two. The equation (2.1), called Auslander-Bridger Formula, is proved in

2000 *Mathematics Subject Classification.* 13C14, 13C40, 16E65
keywords: Linkage, Duality, Gorenstein dimension
Research of the author is supported by Grant-in-Aid for Scientific Researches C(2) in Japan

many ways. Among others, we use the original version of [1] to prove it. To do so, we dissolve a question, proposed by [14], about the proof of this formula in [1] (see Remark 3.4.1).

2. Linkage and duality

Let Λ be a left and right Noetherian ring. Let modΛ (respectively, modΛ^{op}) be the category of all finitely generated left (respectively, right) Λ-modules. Throughout the paper, all modules are finitely generated and left modules (if the ring is non-commutative) and right modules are considered as Λ^{op}-modules. We denote the stable category by $\underline{\mathrm{mod}}\Lambda$, the syzygy functor by $\Omega : \underline{\mathrm{mod}}\Lambda \to \underline{\mathrm{mod}}\Lambda$, and the transpose functor by $\mathrm{Tr} : \underline{\mathrm{mod}}\Lambda \to \underline{\mathrm{mod}}\Lambda^{\mathrm{op}}$. Recall the definition of the functor Tr ([1], Chapter two, section 1 where it is denoted by D, or [2], Chapter Ⅳ, section 1). Let $P_1 \xrightarrow{f} P_0 \to M \to 0$ be a projective resolution of a module $M \in \mathrm{mod}\Lambda$. Then the transpose of M, $\mathrm{Tr}M \in \mathrm{mod}\Lambda^{\mathrm{op}}$, is equal to $\mathrm{Cok}f^*$, where $(-)^* : \mathrm{mod}\Lambda \to \mathrm{mod}\Lambda^{\mathrm{op}}$ is defined by $M^* := \mathrm{Hom}_\Lambda(M, \Lambda)$.

In order to relate linkage and duality, we combine the above functors. One is the functor $T_k := \mathrm{Tr}\Omega^{k-1}$ for $k > 0$ which was firstly introduced by Auslander and Bridger [1] to define and study the (homological) torsion-freeness and reflexivity, and recently used by Iyama [11] to define a duality between some full subcategories given by grade. An another tool, an operator $\lambda := \Omega\mathrm{Tr}$, is introduced in the nice article by Martsinkovsky and Strooker [13]. Using the operator λ, they defined the notion of linkage of modules.

Let Λ be a semiperfect ring and M, N are finitely generated Λ-modules having no nonzero projective direct summands. Then $M \cong N$ in modΛ if and only if $\underline{M} \cong \underline{N}$ in $\underline{\mathrm{mod}}\Lambda$ (e.g., [6], 25.1.5 Corollary). Then λM is determined up to isomorphism for a finitely gererated Λ-module M.

Throughout the paper, we assume that the underlying rings are always semiperfect. It is well-known that a commutative local ring ([17], Theorem 4.46) and a module-finite algebra over a commutative Noetherian complete local ring ([5], p.132) are semiperfect.

Following [13], we define

2.1. Definition. A finitely generated Λ-module M and a Λ^{op}-module N are said to be *horizontally linked* if $M \cong \lambda N$ and $N \cong \lambda M$, in other words, M is horizontally linked (to λM) if and only if $M \cong \lambda^2 M$.

A rather different definition of linkage of modules is proposed by Yoshino and Isogawa [18]. However, both definitions coincide for Cohen-Macaulay modules over a commutative Gorenstein ring (see [13], section 3).

It is noted in [13], that a projective module is horizontally linked if and only if it is isomorphic to the zero module (the statement after [13], Definition 3). We also note that, for a finitely generated Λ-module M having no nonzero projective direct summands, if the projective dimension of $\mathrm{Tr}M$ equals one, i.e. $M^* = 0$, then M is horizontally linked if and only if M is isomorphic to the zero module.

Let us start with the following simple observation which connect duality with linkage.

2.2. Proposition. *Let $k > 0$. Then T_kM is horizontally linked if and only if* $\mathrm{Ext}^k_\Lambda(M, \Lambda) = 0$.

Proof. This is a direct consequence of the definition of linkage and the following exact sequence:

$$0 \to \mathrm{Ext}^k_\Lambda(M, \Lambda) \to T_kM \to \lambda^2 T_kM \to 0,$$

which is given in [11], section 2 or [10], the proof of Lemma 2.1. □

We prepare the facts about Gorenstein dimension from [1] and [4], Chapter 1. A Λ-module M is said to have *G-dimension zero*, denoted by $\text{G-dim}_\Lambda M = 0$, if $M^{**} \cong M$ and $\mathrm{Ext}^k_\Lambda(M, \Lambda) = \mathrm{Ext}^k_{\Lambda^{\mathrm{op}}}(M^*, \Lambda) = 0$ for $k > 0$. This is equivalent to '$\mathrm{Ext}^k_\Lambda(M, \Lambda) = \mathrm{Ext}^k_{\Lambda^{\mathrm{op}}}(\mathrm{Tr}M, \Lambda) = 0$ for $k > 0$' ([1], Proposition 3.8). For a positive integer k, we say that M has G-dimension less than or equal to k, denoted by $\text{G-dim}_\Lambda M \leq k$, if there exists an exact sequence $0 \to G_k \to \cdots \to G_0 \to M \to 0$ with $\text{G-dim}_\Lambda G_i = 0$ for $(0 \leq i \leq k)$. It follows from [1], Theorem 3.13 that $\text{G-dim}_\Lambda M \leq k$ if and only if $\text{G-dim}_\Lambda \Omega^k M = 0$. If $\text{G-dim}_\Lambda M < \infty$, then $\text{G-dim}_\Lambda M = \sup\{k : \mathrm{Ext}^k_\Lambda(M, \Lambda) \neq 0\}$([1], p. 95 or [4], (1.2.7)).

Invariance of G-dimension under linkage is studied in [13].

2.3. Theorem([13], Theorem 1). *Let Λ be a semiperfect right and left Noetherian ring and M a Λ-module having no nonzero projective direct summands. Then the following conditions are equivalent.*

(1) $\text{G-dim}_\Lambda M = 0$,

(2) $\text{G-dim}_{\Lambda^{\mathrm{op}}} \lambda M = 0$ and M is horizontally linked.

In the rest of this section, we consider a commutative ring case and apply the above results to Cohen-Macaulay modules over a commutative Gorenstein local ring. See [3] for Cohen-Macaulay rings and modules and Gorenstein rings.

Let R be a (commutative) Gorenstein local ring and M a finitely generated R-module. Then there are the following useful equalities:

$$\text{(2.1)}\quad \text{G-dim}_R M + \text{depth} M = \dim R$$

$$\text{(2.2)}\quad \text{grade}_R M + \dim M = \dim R,$$

where $\text{grade}_R M := \inf\{k \geq 0 : \text{Ext}^k_R(M, R) \neq 0\}$. The first equality is due to [1], Theorems 4.13 and 4.20 and the second one to [15], Lemma 4.8. (See also [8], Proposition 4.11.)

The combination of linkage and duality produces the following characterization of a maximal Cohen-Macaulay module which improves [13], Proposition 8.

2.4. Theorem. *Let R be a Gorenstein local ring and M a finitely generated R-module having no nonzero projective direct summands. Then the following are equivalent*

(1) M is a maximal Cohen-Macaulay module,

(2) $T_k M$ is horizontally linked for $k > 0$,

(3) λM is a maximal Cohen-Macaulay module and M is horizontally linked.

Proof. (1) $\Leftrightarrow$ (2): Since R is Gorenstein, M is maximal Cohen-Macaulay if and only if $\text{G-dim}_R M = 0$. This is equivalent to $\text{Ext}^k_R(M, R) = 0$ for $k \neq 0$ by [1], Theorem 4.20. Thus (1) $\Leftrightarrow$ (2) follows from Proposition 2.2.
(1) $\Leftrightarrow$ (3):This is a direct consequence of Theorem 2.3. □

By the above theorem, G-dimension is described using linkage.

2.5. Proposition. *Let R be a Gorenstein local ring and M a finitely generated module. Then $\text{G-dim}_R M \leq k$ if and only if $T_{i+k} M$ is horizontally linked for $i > 0$.*

Proof. $T_{i+k} M$ is horizontally linked for $i > 0 \Leftrightarrow T_i \Omega^k M$ is horizontally linked for $i > 0 \Leftrightarrow \text{G-dim}_R \Omega^k M = 0 \Leftrightarrow \text{G-dim}_R M \leq k$. □

We apply duality theory on a non-commutative Noetherian ring due to Iyama [11] to the category of Cohen-Macaulay modules. Suppose that Λ is a right and left Noetherian ring. Then the functor T_k gives a duality between the categories $\{X \in \underline{\text{mod}}\Lambda : \text{grade}_\Lambda X \geq k\}$ and $\{Y \in \underline{\text{mod}}\Lambda^{\text{op}} : \text{rgrade}_{\Lambda^{\text{op}}} Y \geq k \text{ and } \text{pd}_{\Lambda^{\text{op}}} Y \leq k\}$ [11], 2.1.(1), where $\text{rgrade}_{\Lambda^{\text{op}}} Y := \{k > 0 : \text{Ext}^k_{\Lambda^{\text{op}}}(Y, \Lambda) \neq 0\}$ stands for a reduced grade of Y [9]. Returning to our case, we consider a commutative Noetherian local ring R and a finitely generated R-module M. Then it holds that $\text{G-dim}_R M \geq \text{grade}_R M$, in general.

Moreover, if G-dim$_R M < \infty$, then M is a Cohen-Macaulay module if and only if G-dim$_R M =$ grade$_R M$ by the equations (2.1) and (2.2). Thus we can apply the above duality to the category of Cohen-Macaulay R-modules.

A finitely generated module M over a Cohen-Macaulay local ring R is called a Cohen-Macaulay module of codimension k, if depth$M = \dim M = \dim R - k$. Put the full subcategory of mod$R$

$$\mathcal{C}_k := \{M \in \mathrm{mod}R : M \text{ is a Cohen-Macaulay } R\text{-module of codimension } k\}.$$

In order to give a duality, we need a counterpart of the category $\mathcal{C}_k$. We put the full subcategory of mod$R$

$$\mathcal{C}'_k := \left\{ N \in \mathrm{mod}R : \begin{array}{c} \mathrm{rgrade}_R N = \mathrm{pd}_R N = k \text{ and} \\ \lambda^2 N \text{ is a maximal Cohen-Macaulay module} \end{array} \right\},$$

where pd$_R N$ stands for a projective dimension of N. Let $\underline{\mathcal{C}}_k$, respectively $\underline{\mathcal{C}}'_k$, be the full subcategory of $\underline{\mathrm{mod}}R$ induced from $\mathcal{C}_k$, respectively $\mathcal{C}'_k$.

2.6. Theorem. *Let R be a Gorenstein local ring. Let $k > 0$. Then the functor T_k gives a duality between full subcategories $\underline{\mathcal{C}}_k$ and $\underline{\mathcal{C}}'_k$ such that $T_k \circ T_k$ is isomorphic to the identity funtor.*

Proof. It follows from [11], 2.1. (1) that

$$M \in \mathcal{C}_k \Rightarrow \mathrm{rgrade}_R T_k M = \mathrm{pd}_R T_k M = k \text{ and } T_k T_k M \cong M,$$

$$N \in \mathcal{C}'_k \Rightarrow \mathrm{grade}_R T_k N \geq k \text{ and } T_k T_k N \cong N.$$

It suffices to prove that $T_k M \in \mathcal{C}'_k$ (respectively, $\mathcal{C}_k$) whenever $M \in \mathcal{C}_k$ (respectively, $\mathcal{C}'_k$). Let $M \in \mathcal{C}_k$ and $\cdots \to P_i \to \cdots \to P_0 \to M \to 0$ a minimal free resolution of M. Then we have an exact sequence $0 \to (\Omega^k M)^* \to P_k^* \to P_{k+1}^* \to \mathrm{Tr}\Omega^k M \to 0$, which gives an exact sequence

$$0 \to (\Omega^k M)^* \to P_k^* \to \lambda^2 T_k M \to 0,$$

by definition of T_k. From an exact sequence $0 \to \Omega^{k+1} M \to P_k \to \Omega^k M \to 0$ and the assumption that G-dim$_R M = k$, we have an exact sequence

$$0 \to (\Omega^k M)^* \to P_k^* \to (\Omega^{k+1} M)^* \to 0.$$

Hence $\lambda^2 T_k M \cong (\Omega^{k+1} M)^*$. Since G-dim$_R M = k$, we have G-dim$_R \Omega^{k+1} M = 0$, so that G-dim$_R (\Omega^{k+1} M)^* = 0$, hence G-dim$_R \lambda^2 T_k M = 0$ which implies that $\lambda^2 T_k M$ is a maximal Cohen-Macaulay module.

Conversely, let $N \in \mathcal{C}'_k$. It suffices to prove that G-dim$_R T_k N \leq k$. Since $\Omega^k T_k N$ is a syzygy module, the canonical homomorphism $\varphi : \Omega^k T_k N \to (\Omega^k T_k N)^{**}$ is monic. From Auslander formula [1], Proposition 2.6, Cok$\varphi =$ Ext$^2_R(\mathrm{Tr}\Omega^k T_k N, R)$, whose right hand side is equal to Ext$^1_R(\lambda^2 T_k T_k N, R) =$ Ext$^1_R(\lambda^2 N, R) = 0$ by assumption. Hence $\Omega^k T_k N \cong (\Omega^k T_k N)^{**}$. From a

minimal free resolution $\cdots \to Q_i \to \cdots \to Q_0 \to T_k N \to 0$, we get an exact sequence

$$0 \to (\Omega^k T_k N)^* \to Q_k^* \to \lambda^2 N \to 0.$$

Since $\lambda^2 N$ is a maximal Cohen-Macaulay module, $(\Omega^k T_k N)^*$ is also a maximal Cohen-Macaulay module, so that $(\Omega^k T_k N)^{**} \cong \Omega^k T_k N$ is a maximal Cohen-Macaulay module. This implies that $\text{G-dim}_R \Omega^k T_k N = 0$, and then $\text{G-dim}_R T_k N \leq k$. □

2.7. Corollary. *Let M be a finitely generated R-module of $\text{grade}_R M = k > 0$. Then the following are equivalent*

(1) M is a Cohen-Macaulay module of codimension k,

(2) $\text{rgrade}_R T_k M = \text{pd}_R T_k M = k$ and $\lambda^2 T_k M$ is a maximal Cohen-Macaulay module.

3. A generalization to non-commutative Gorenstein algebras

We shall generalize the results of section two concerning Gorenstein rings to module finite (non-commutative) Gorenstein algebras over a commutative Noetherian complete local ring. Such algebras are extensively studied in [8].

Throughout this section, let $(R, \mathfrak{m})$ be a commutative Noetherian complete local ring. Let Λ be a module-finite R-algebra, that is, Λ is an R-algebra which is finitely generated as an R-module. As is noted in section 2, Λ is semiperfect. We define a Gorenstein algebra (cf. [8], Lemma 4.7).

3.1. Definition. Let Λ be a module-finite R-algebra. We call Λ a *Gorenstein* R-algebra, if Λ is a Cohen-Macaulay R-module with $\text{id}_\Lambda \Lambda = \dim_R \Lambda$, where $\text{id}_\Lambda X$ stands for an injective dimension of a Λ-module X.

It follows from [8], Corollary 4.8 that this definition is left-right symmetric. In order to characterize a Gorenstein algebra, we need the following 'homogeneity condition' studied in [7].

(h) $\text{Ext}^t_\Lambda(S, \Lambda) \neq 0$ for every simple Λ-module S, where $t = \text{depth}_R \Lambda$.

We denote by (h^{op}), when we consider Λ^{op}-modules. Then we state a theorem which is essential for generalizing the results in §2 to non-commutative algebras.

3.2. Theorem. (cf. [1], Theorem 4.20) *Let Λ be a module-finite R-algebra and $t = \text{depth}_R \Lambda$. Then the following are equivalent.*

(1) Λ is a Gorenstein R-algebra,

(2) Λ satisfies the condition (h) and $\text{id}_\Lambda \Lambda < \infty$,

(2′) Λ *satisfies the condition* (h^{op}) *and* $\mathrm{id}_{\Lambda^{\mathrm{op}}}\Lambda < \infty$,

(3) Λ *satisfies the condition* (h) *and* $\text{G-dim}_{\Lambda}M < \infty$ *for any finitely generated* Λ*-module* M,

(3′) Λ *satisfies the condition* (h^{op}) *and* $\text{G-dim}_{\Lambda^{\mathrm{op}}}M < \infty$ *for any finitely generated* Λ^{op}*-module* M

When this is the case, it holds that $\mathrm{id}_{\Lambda}\Lambda = \mathrm{id}_{\Lambda^{\mathrm{op}}}\Lambda = \dim_R \Lambda = t$.

The proof is done after generalizing several facts in [1], Chapter four to our case.

3.3. Proposition. (cf. [1], Proposition 4.12) *Assume that* Λ *satisfies the condition* (h) *and* M *is a finitely generated* Λ*-module with* $\text{G-dim}_{\Lambda}M < \infty$. *Then the following are equivalent*

(1) $\text{G-dim}_{\Lambda}M = 0$,

(2) $\mathrm{depth}_R M \geq \mathrm{depth}_R \Lambda$,

(3) $\mathrm{depth}_R M = \mathrm{depth}_R \Lambda$,

Proof. Let $\mathrm{depth}_R\Lambda = t$.
(1) $\Rightarrow$ (2): Let $x_1, \cdots, x_i$ be a Λ-sequence in $\mathfrak{m}$. We show that $x_1, \cdots, x_i$ is an M-sequence by induction on i. Let $i = 1$. Since $M \cong M^{**}$ and x_1 is Λ-regular, x_1 is M-regular.

Suppose that $i > 1$ and the assertion holds for $i-1$. Then $x_1, \cdots, x_{i-1}$ is an M-sequence. Put $I = (x_1, \cdots, x_{i-1})$, $\overline{R} = R/I$, $\overline{\Lambda} = \Lambda/I\Lambda$, $\overline{M} = M/IM$. Then $(\overline{R}, \mathfrak{m}/I)$ is a commutative Noetherian complete local ring. By [1], Lemma 4.9 or [4], Corollary 1.4.6, $\text{G-dim}_{\overline{\Lambda}}\overline{M} = \text{G-dim}_{\Lambda}M = 0$. Since $\overline{x}_i \in \overline{R}$ is a $\overline{\Lambda}$-regular element, $\overline{x}_i$ is $\overline{M}$-regular, hence $x_1, \cdots, x_i$ is an M-sequence. Therefore, $\mathrm{depth}M \geq \mathrm{depth}\Lambda$.
(2) $\Rightarrow$ (3): Assume that $t = 0$. We prove that if $\text{G-dim}M < \infty$, then $\text{G-dim}M = 0$. The condition (h) implies $\mathrm{Hom}_{\Lambda}(S, \Lambda) \neq 0$ for every simple Λ-module S. Hence, for a finitely generated Λ-module M, if $M^* = 0$ then $M = 0$.

Suppose that $\text{G-dim}_{\Lambda}M \leq 1$. We have an exact sequence $0 \to L_1 \to L_0 \to M \to 0$ with $\text{G-dim}_{\Lambda}L_i = 0$ $(i = 0, 1)$. Hence we have an exact sequence

$$0 \to M^* \to L_0^* \to L_1^* \to \mathrm{Ext}^1_{\Lambda}(M, \Lambda) \to 0$$

and $\mathrm{Ext}^i_{\Lambda}(M, \Lambda) = 0$ for $i > 1$. By this sequence, we have an exact sequence

$$0 \to \mathrm{Ext}^1_{\Lambda}(M, \Lambda)^* \to L_1 \to L_0,$$

where $L_1 \to L_0$ is monic. Thus $\mathrm{Ext}^1_{\Lambda}(M, \Lambda)^* = 0$, and so $\mathrm{Ext}^1_{\Lambda}(M, \Lambda) = 0$ by the previous paragraph. Since $\text{G-dim}_{\Lambda}M < \infty$, this proves $\text{G-dim}_{\Lambda}M = 0$.

Suppose that G-dim$_\Lambda M \leq n$. Let $0 \to L_n \xrightarrow{f_n} \cdots \xrightarrow{f_1} L_0 \to M \to 0$ be exact with G-dim$_\Lambda L_i = 0$ $(0 \leq i \leq n)$. Since G-dim$_\Lambda(\mathrm{Im}f_{n-1}) \leq 1$, we have G-dim$_\Lambda(\mathrm{Im}f_{n-1}) = 0$ by the above argument. Repeating this process, we get G-dim$_\Lambda M = 0$.

Next, we prove that depth$_R M = 0$. We only need the reflexivity of M. By the condition (h), there exists an exact sequence $0 \to C \to \Lambda$, where C is a direct sum of all non-isomorphic simple Λ-modules. This sequence gives an exact sequence

$$0 \to \mathrm{Hom}_\Lambda(M^*, C) \to M^{**} \cong M.$$

Since $M^* \neq 0$, we have $\mathrm{Hom}_\Lambda(M^*, C) \neq 0$. Since $\mathfrak{m}\mathrm{Hom}_R(M^*, C) = 0$, we see that $\mathfrak{m}$ has no M-regular element, so that depth$M = 0$. This proves (2) $\Rightarrow$ (3) for the case that depth$_R\Lambda = 0$.

Let $t = \mathrm{depth}\Lambda > 0$. We have depth$M \geq$ depth$\Lambda \geq 1$, so that there is an element $x \in \mathfrak{m}$ which is Λ and M-regular. By [1], Lemma 4.9, we have G-dim$_{\Lambda/x\Lambda} M/xM < \infty$. It follows from [3], Proposition 1.2.10 (d) that

$$\mathrm{depth}_{R/xR} M/xM = \mathrm{depth}_R M - 1 \geq \mathrm{depth}_R\Lambda - 1 = \mathrm{depth}_{R/xR}\Lambda/x\Lambda.$$

Hence, by induction on t, we have depth$_{R/xR} M/xM =$ depth$_{R/xR}\Lambda/x\Lambda$, and then depth$M =$ depthΛ. Therefore, (2) $\Rightarrow$ (3) holds.
(3) $\Rightarrow$ (1): By assumption, it suffices to prove that $\mathrm{Ext}^i_\Lambda(M, \Lambda) = 0$ for $i > 0$. We show the assertion by induction on t.

Let $t = 0$. Then G-dim$_\Lambda M = 0$ by the proof (2) $\Rightarrow$ (3).

Let $t > 0$. Then depth$M =$ depth$\Lambda \geq 1$. We take an element $x \in \mathfrak{m}$ which is Λ and M-regular. Then, by [3], Proposition 1.2.10 (d),

$$\mathrm{depth}_{R/xR} M/xM = \mathrm{depth}_R M - 1 = \mathrm{depth}_R\Lambda - 1 = \mathrm{depth}_{R/xR}\Lambda/x\Lambda.$$

Hence we have $\mathrm{Ext}^i_{\Lambda/x\Lambda}(M/xM, \Lambda/x\Lambda) = 0$ for $i > 0$ by induction. This gives $\mathrm{Ext}^i_\Lambda(M, \Lambda/x\Lambda) = 0$ for $i > 0$ (cf. [12], p.155). From an exact sequence $0 \to \Lambda \xrightarrow{x} \Lambda \to \Lambda/x\Lambda \to 0$, we get an exact sequence

$$\mathrm{Ext}^i_\Lambda(M, \Lambda) \xrightarrow{x} \mathrm{Ext}^i_\Lambda(M, \Lambda) \to \mathrm{Ext}^i_\Lambda(M, \Lambda/x\Lambda) = 0.$$

By Nakayama's Lemma, it holds that $\mathrm{Ext}^i_\Lambda(M, \Lambda) = 0$ for $i > 0$ □

3.4. Proposition. (cf. [1], Theorem 4.13) *Assume that Λ satisfies the condition (h). Let $M \in$ modΛ with G-dim$_\Lambda M < \infty$. Then we have an equality*

$$\text{G-dim}_\Lambda M + \mathrm{depth}_R M = \mathrm{depth}_R\Lambda$$

Proof. If G-dim$M = 0$, we are done by the previous proposition. Suppose that G-dim$M = n > 0$ and the equation holds for $n-1$. Let $0 \to K \to P \to$

$M \to 0$ be exact with P finitely generated projective. Since G-dim$K = n-1$, we have G-dim$K$ + depth$K$ = depthΛ by induction. Suppose that depth$M \geq$ depth$P$. Then depth$M \geq$ depthΛ. Hence G-dim$M = 0$ holds by the previous proposition. This contradicts to G-dim$M > 0$. Hence depth$M <$ depth$P$, so depth$K$ = depth$M + 1$ by, e.g., [3], Proposition 1.2.9. Therefore, $n +$ depth$M$ = depthΛ. □

3.4.1. Remark. Recently, it is pointed out in [14] that the proof of [1], Theorem 4.13 (b) includes a 'serious mistake'. Although we think that there is no 'serious mistake', we might say that the proof needs some supplementary explanation as above to use induction on G-dim$M$.

Proof of Theorem 3.2. The equivalences (1) ⇔ (2) and (1) ⇔ (2′) are showed by [8], Theorem 3.7 and Corollary 4.10. Then $\mathrm{id}_\Lambda\Lambda = \mathrm{id}_{\Lambda^{\mathrm{op}}}\Lambda = \dim_R \Lambda = t$ is a direct consequence of [8], Proposition 4.11.(5).
(2) ⇒ (3): Since $\mathrm{id}_\Lambda\Lambda = t$, we have $\mathrm{Ext}^i_\Lambda(\Omega^t M, \Lambda) \cong \mathrm{Ext}^{i+t}_\Lambda(M, \Lambda) = 0$ for $i > 0$. Let $\cdots \to P_k \to \cdots \to P_0 \to \Omega^t M \to 0$ be a projective resolution of $\Omega^t M$. Then we have an exact sequence

$$0 \to (\Omega^t M)^* \to P_0^* \to \cdots \to P_k^* \to \cdots .$$

Put $C := \mathrm{Cok}(P_t^* \to P_{t+1}^*)$. Then there is an exact sequence

$$0 \to \mathrm{Tr}\Omega^t M \to P_2^* \to \cdots \to P_t^* \to P_{t+1}^* \to C \to 0.$$

Since $\mathrm{id}_{\Lambda^{\mathrm{op}}}\Lambda = t$, we have

$$\mathrm{Ext}^i_{\Lambda^{\mathrm{op}}}(\mathrm{Tr}\Omega^t M, \Lambda) \cong \mathrm{Ext}^i_{\Lambda^{\mathrm{op}}}(\Omega^t C, \Lambda) \cong \mathrm{Ext}^{i+t}_\Lambda(C, \Lambda) = 0, \text{ for } i > 0.$$

Thus G-dim$_\Lambda \Omega^t M = 0$, and then G-dim$_\Lambda M \leq t$.
(3) ⇒ (2): Suppose that $\mathrm{Ext}^i_\Lambda(M, \Lambda) \neq 0$ holds. Then G-dim$_\Lambda M \geq i$. By Proposition 3.4, we see that $t -$ depth$_R M \geq i$, so $t \geq i$. Therefore, if $t < i$, then $\mathrm{Ext}^i_\Lambda(M, \Lambda) = 0$ for all $M \in \mathrm{mod}\Lambda$. This implies $\mathrm{id}_\Lambda\Lambda \leq t$. □

We assume that Λ is a Gorenstein R-algebra. Let M be a finitely generated Λ-module. Then the following equalities hold true by [8], Proposition 4.11

$$\text{(3.1)} \quad \text{G-dim}_\Lambda M + \text{depth}_R M = \dim_R \Lambda$$

$$\text{(3.2)} \quad \text{grade}_\Lambda M + \dim_R M = \dim_R \Lambda.$$

We consider the functors Ω, Tr, T_k over $\underline{\mathrm{mod}}\Lambda$ or $\underline{\mathrm{mod}}\Lambda^{\mathrm{op}}$, and an operator λ on modΛ or modΛ^{op} then we can define linkage and duality of modules over Λ.

A finitely generated Λ-module is called a *maximal Cohen-Macaulay Λ-module*, respectively a *Cohen-Macaulay Λ-module of codimension k* if it is a maximal Cohen-Macaulay R-module, respectively a Cohen-Macaulay R-module of codimension k. It follows from Theorem 3.2 and equalities (3.1),

(3.2) that $M \in \mathrm{mod}\Lambda$ is a maximal Cohen-Macaulay Λ-module if and only if $\text{G-dim}_\Lambda M = 0$.

Using above arrangement, a generalization of 2.4 - 2.7 in section two are routin task. The proofs are almost the same as in section two. Here, we provide non-commutative version of main theorems, Theorems 2.4 and 2.6.

3.5. Theorem (a non-commutative version of Theorem 2.4) *Let Λ be a Gorenstein R-algebra and M a finitely generated Λ-module having no nonzero projective direct summands. Then the following are equivalent*

(1) M is a maximal Cohen-Macaulay Λ-module,

(2) $T_k M$ is horizontally linked for $k > 0$,

(3) λM is a maximal Cohen-Macaulay Λ-module and M is horizontally linked.

Proof. This theorem is proved by the similar way to Theorem 2.4. □

Let

$$\mathcal{C}_k(\Lambda) := \{M \in \mathrm{mod}\Lambda : M \text{ is a Cohen-Macaulay } \Lambda\text{-module of codimension } k\},$$

$$\mathcal{C}'_k(\Lambda) := \left\{ N \in \mathrm{mod}\Lambda : \begin{array}{c} \mathrm{rgrade}_\Lambda N = \mathrm{pd}_\Lambda N = k \text{ and } \lambda^2 N \text{ is} \\ \text{a maximal Cohen-Macaulay } \Lambda\text{-module} \end{array} \right\}$$

and $\underline{\mathcal{C}}_k(\Lambda)$, $\underline{\mathcal{C}}'_k(\Lambda)$ be the full subcategories of $\underline{\mathrm{mod}}\Lambda$ induced from $\mathcal{C}_k(\Lambda)$, $\mathcal{C}'_k(\Lambda)$.

3.6. Theorem. (a non-commutative version of Theorem 2.6) Let Λ be a Gorenstein R-algebra. Let $k > 0$. Then the functor T_k gives a duality between full subcategories $\underline{\mathcal{C}}_k(\Lambda)$ and $\underline{\mathcal{C}}'_k(\Lambda^{\mathrm{op}})$ such that $T_k \circ T_k$ is isomorphic to the identity functor.

Proof. It follows from [8], Theorem 4.12 that if L is a maximal Cohen-Macaulay Λ-module, i.e., $L \in \mathcal{C}_0(\Lambda)$, then $\mathrm{Ext}^i_\Lambda(L, \Lambda) = 0$ for $i > 0$, $\mathrm{Hom}_\Lambda(L, \Lambda) \in \mathcal{C}_0(\Lambda^{\mathrm{op}})$, and $L \cong \mathrm{Hom}_{\Lambda^{\mathrm{op}}}(\mathrm{Hom}_\Lambda(L, \Lambda), \Lambda)$. Then letting $(-)^* := \mathrm{Hom}_\Lambda(-, \Lambda)$, the proof is done by the similar way to Theorem 2.6. □

References

[1] M. Auslander and M. Bridger, *Stable module theory*, Mem. of the AMS 94, Amer. Math. Soc., Providence 1969.

[2] M. Auslander, I. Reiten and S. O. Smalϕ, *Representation Theory of Artin Algebras*, Cambridge Stud. Adv. Math. 36, Cambridge Univ. Press, 1995.

[3] W. Bruns and J. Herzog, *Cohen-Macaulay Rings*, Cambridge Stud. Adv. Math. 39, Cambridge Univ. Press, 1993.

[4] L. W. Christensen, *Gorenstein Dimensions*, Lecture Notes in Mathematics 1747, Springer, 2000.

[5] C. W. Curtis and I. Reiner, *Methods of Representation Theory*, Vol. 1, Wiley-Interscience, 1981.
[6] C. Faith, *Algebra II Ring Theory*, Springer, 1976.
[7] S. Goto and K. Nishida, *Finite modules of finite injective dimension over a Noetherian algebra*, J. London. Math. Soc. 63 (2001) 319-335.
[8] S. Goto and K. Nishida, *Towards a theory of Bass numbers with application to Gorenstein algebras*, Colloq. Math. 91(2) (2002) 191-253.
[9] M. Hoshino, *Syzygies and Gorenstein rings*, Arch. Math. (Basel) 55(4) (1990) 355-360.
[10] M. Hoshino and K. Nishida, *A generalization of the Auslander formula*, in Representations of Algebras and Related Topics. Fields Institute Communications, Vol. 45, 2005, 175-186.
[11] O. Iyama, *Symmetry and duality on n-Gorenstein rings*, J. Algebra 269 (2003) 528-535.
[12] I. Kaplansky, *Commutative Rings*, Allyn and Bacon, Boston, 1970.
[13] A. Martsinkovsky and J. R. Strooker, *Linkage of modules*, J. Algebra 271 (2004) 587-626.
[14] V. Maşek, *Gorenstein dimension and torsion of modules over commutative Noetherian rings*, Comm. in Algebra 28(12) (2000) 5783-5811.
[15] C. Peskine and L. Szpiro, *Dimension projective finie et cohomologie locale*, Publ. Math. I.H.E.S. 42 (1972) 47-119.
[16] C. Peskine and L. Szpiro, *Liaison des variétés algébriques. I*, Invent. Math. 26 (1974) 271-302.
[17] J. J. Rotman, *An Introduction to Homological Algebra*, Academic Press, 1979
[18] Y. Yoshino and S. Isogawa, *Linkage of Cohen-Macaulay modules over a Gorenstein ring*, J. Pure Appl. Algebra 149 (2000) 305-318.

Department of Mathematical Sciences, Shinshu University, Matsumoto, 390-8621, Japan

e-mail address: kenisida@math.shinshu-u.ac.jp

(*Received January 13, 2006*)