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A LOWER BOUND FOR THE RATIONAL LS-CATEGORY OF A COFORMAL ELLIPTIC SPACE

Dedicated to the memory of Professor Shiroshi SAITO

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Abstract. We give a lower bound for the rational LS-category of certain spaces, including the coformal elliptic ones, in terms of the dimension of its total rational cohomology.

1. Introduction

The Lusternik Schnirelmann category of a space X, $\operatorname{cat} X$, is the least integer m such that X can be covered by $m+1$ open sets, each of which is contractible in X. Let X be a simply connected CW complex with rational cohomology of finite type. Denote its Sullivan minimal model [3] as $(\Lambda V, d)$. The rational LS category, $\operatorname{cat}_0(X)$, is the least integer n such that X has the rational homotopy type of Y with $\operatorname{cat}(Y) = n$. Then $\operatorname{cat}_0(X) \leq \operatorname{cat}(X)$. A minimal model $(\Lambda V, d)$ is called elliptic if $\dim H^*(\Lambda V, d) < \infty$ and $\dim V < \infty$ [3]. The rational Toomer invariant $e_0(\Lambda V, d)$ is given by the biggest integer s for which there is a non trivial cohomology class in $H^*(\Lambda V, d)$ represented by a cycle in $\Lambda^{\geq s} V$. If $(\Lambda V, d)$ is elliptic, it is proved that $e_0(\Lambda V, d) = \operatorname{cat}_0(X)$ [2, Theorem 3]. An elliptic model $(\Lambda V, d)$ is called an F_0-model if its cohomology is concentrated in even degrees. It is equivalent to the condition that $\dim V^{even} = \dim V^{odd}$ [4]. If $(\Lambda V, d)$ is elliptic, is $\dim H^*(\Lambda V, d) \leq 2^{e_0(\Lambda V, d)}$? ([7]) The answer is affirmative if the model is formal, in particular if it is an F_0-model [5]. Since $e_0(X \times Y) = e_0(X) + e_0(Y)$ [3, p.391], the proposed inequality is closed under products, that is, if X and Y satisfy the inequality, then so too does $X \times Y$.

Let k be the biggest integer for which we may write $d = \sum_{i \geq k} d_i$ with $d_i(V) \subset \Lambda^i V$. Thus d_k induces a differential in ΛV. In this paper, we prove:

Theorem. *If both $(\Lambda V, d)$ and $(\Lambda V, d_k)$ are elliptic, then*

$$\dim H^*(\Lambda V, d) \leq 2^{e_0(\Lambda V, d)}.$$

Recall that $(\Lambda V, d)$ is coformal if and only if $d(V) \subset \Lambda^2 V$. We then have

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Corollary. *If an elliptic space X is coformal, then*

$$\dim H^*(X;\mathbb{Q}) \leq 2^{\mathrm{cat}_0(X)}.$$

In view of [1, Proposition 10.6], we have: If an elliptic model $(\Lambda V, d)$ is coformal, $\dim H^*(\Lambda V, d) \leq 2^{e_0(\Lambda V, d)} = 2^{\dim V^{odd}}$.

2. Proof

We first observe that the two numerical invariants of a model "$\dim H^*(\Lambda V, d)$" and "$e_0(\Lambda V, d)$" do not depend on the gradation of $V = \oplus_i V^i$:

Lemma 1. *If $(\Lambda V, d_V) \cong (\Lambda W, d_W)$ as differential (generally not graded) algebras, then $\dim H^*(\Lambda V, d_V) = \dim H^*(\Lambda W, d_W)$ and $e_0(\Lambda V, d_V) = e_0(\Lambda W, d_W)$.*

Lemma 2. [5] *If the cohomology of an F_0-space X is given by $H^*(X;\mathbb{Q}) \cong \mathbb{Q}[x_1, \ldots, x_p]/(f_1, \ldots, f_p)$, then*

$$\dim H^*(X;\mathbb{Q}) \leq 2^{\deg f_1 + \cdots + \deg f_p - p}.$$

Here $\deg f$ means the degree of a polynomial f in $\mathbb{Q}[x_1, \ldots, x_p]$, not the degree defined by each one of x_i in the graded algebra $\mathbb{Q}[x_1, \ldots, x_p]$. For a given model $(\Lambda V, d)$, we denote by $(\Lambda V, d_\sigma)$ its associated pure model, i.e., d_σ is the component of d which satisfies $d_\sigma V^{even} = 0$ and $d_\sigma V^{odd} \subset \Lambda V^{even}$ [4, page 181, the first paragraph].

Proof of Theorem. Put

$$(\Lambda V, d) = (\Lambda(x_1, \cdots, x_n, y_1, \cdots, y_m), d)$$

with $|x_i|$ even and $|y_i|$ odd. Then $n \leq m$ [4, Theorem 1']. For the associated pure model $(\Lambda V, d_{k\sigma})$ of $(\Lambda V, d_k)$, $\dim H^*(\Lambda V, d_{k\sigma}) < \infty$ from our assumption and [4, Proposition 1]. This is isomorphic as differential algebras to

$$(\Lambda W, d_W) = (\Lambda(u_1, \cdots, u_n, v_1, \cdots, v_m), d_W)$$

with $|u_i| = 2$, $|v_i| = 2k - 1$ and $d_W = d_{k\sigma}$, i.e., d_W is given by $\phi d_{k\sigma} \phi^{-1}$ for the isomorphism $\phi : \Lambda V \to \Lambda W$ with $\phi(x_i) = u_i$ and $\phi(y_i) = v_i$. ¿From Lemma 1, $\dim H^*(\Lambda W, d_W) < \infty$. By [4, Lemma 8], there is a subspace W' of W for which

$$(\Lambda W', d_W) = (\Lambda(u_1, \cdots, u_n, v'_1, \cdots, v'_n), d_W)$$

and such that $f_1 = d_W(v'_1), \cdots, f_n = d_W(v'_n)$ is a regular sequence in $\mathbb{Q}[u_1, \cdots, u_n]$, i.e., $(\Lambda W', d_W)$ is an F_0-model. Thus, there is a KS extension

$$(\Lambda W', d_W) \to (\Lambda W, d_W) \to (\Lambda(v'_{n+1}, \cdots, v'_m), 0) \tag{1}$$

where $v'_{n+1}, \cdots, v'_m$ is a basis of a complement of ${W'}^{2k-1}$ in $\mathbb{Q}\langle v_1, \cdots, v_m\rangle$. Then the following inequalities are combined to establish our statement.

$$\begin{aligned}
\dim H^*(\Lambda V, d) &\leq_{(a)} \dim H^*(\Lambda V, d_k) \\
&\leq_{(b)} \dim H^*(\Lambda V, d_{k\sigma}) \\
&=_{(c)} \dim H^*(\Lambda W, d_W) \\
&\leq_{(d)} 2^{m-n} \dim H^*(\Lambda W', d_W) \\
&\leq_{(e)} 2^{m-n} \cdot 2^{\deg f_1 + \cdots + \deg f_n - n} \\
&= 2^{m-n} \cdot 2^{kn-n} \\
&= 2^{n(k-2)+m} \\
&=_{(f)} 2^{e_0(\Lambda V, d)},
\end{aligned}$$

in which we have (a) from the Milnor-Moore spectral sequence, (b) from the odd spectral sequence [4, Section 5], (c) from Lemma 1, (d) from the Serre spectral sequence applied to (1), (e) from Lemma 2 and (f) from [6, Theorem 1.1]. □

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