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THE FINE SPECTRA OF THE CESÀRO OPERATOR C_1 OVER THE SEQUENCE SPACE bv_p, $(1 \leq p < \infty)$

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Abstract. The sequence space bv_p consisting of all sequences (x_k) such that $(x_k - x_{k-1})$ in the sequence space ℓ_p has recently been introduced by Başar and Altay [Ukrainian Math. J. **55**(1)(2003), 136–147]; where $1 \leq p \leq \infty$. In the present paper, the norm of the Cesàro operator C_1 acting on the sequence space bv_p has been found and the fine spectrum of the Cesàro operator C_1 over the sequence space bv_p has been determined, where $1 \leq p < \infty$.

1. Preliminaries, Background and Notation

Let X and Y be the Banach spaces and $T : X \to Y$ also be a bounded linear operator. By $R(T)$, we denote the *range* of T, i.e.,

$$R(T) = \{y \in Y : y = Tx,\ x \in X\}.$$

By $B(X)$, we also denote the set of all bounded linear operators on X into itself. If X is any Banach space and $T \in B(X)$ then the adjoint T^* of T is a bounded linear operator on the dual X^* of X defined by $(T^*f)(x) = f(Tx)$ for all $f \in X^*$ and $x \in X$ with $\|T\| = \|T^*\|$. Also by $Ker(T)$, we denote the *kernel* of a bounded linear operator T.

Let $X \neq \{\theta\}$ be a non trivial complex normed space and $T : \mathcal{D}(T) \to X$ a linear operator defined on a subspace $\mathcal{D}(T) \subseteq X$. We do not assume that $D(T)$ is dense in X, or that T has a closed graph $\{(x, Tx) : x \in D(T)\} \subseteq X \times X$. We mean by the expression "T is *invertible*" that there exists a bounded linear operator $S : R(T) \to X$ for which $ST = I$ on $D(T)$ and $\overline{R(T)} = X$; such that $S = T^{-1}$ is necessarily uniquely determined, and linear; the boundedness of S means that T must be *bounded below*, in the sense that there is $k > 0$ for which $\|Tx\| \geq k\|x\|$ for all $x \in D(T)$. Associated with each complex number α is the perturbed operator

$$T_\alpha = T - \alpha I,$$

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defined on the same domain $D(T)$ as T. The *spectrum* $\sigma(T,X)$ consists of those $\alpha \in \mathbb{C}$ for which T_α is not invertible, and the *resolvent* is the mapping from the complement $\sigma(T,X)$ of the spectrum into the algebra of bounded linear operators on X defined by $\alpha \mapsto T_\alpha^{-1}$.

The name *resolvent* is appropriate, since T_α^{-1} helps to solve the equation $T_\alpha x = y$. Thus, $x = T_\alpha^{-1} y$ provided T_α^{-1} exists. More important, the investigation of properties of T_α^{-1} will be basic for an understanding of the operator T itself. Naturally, many properties of T_α and T_α^{-1} depend on α, and spectral theory is concerned with those properties. For instance, we shall be interested in the set of all α's in the complex plane such that T_α^{-1} exists. Boundedness of T_α^{-1} is another property that will be essential. We shall also ask for what α's the domain of T_α^{-1} is dense in X, to name just a few aspects. For our investigation of T, T_α and T_α^{-1}, we need some basic concepts in spectral theory which are given as follows (see [11, pp. 370-371]):

By a *regular value* α of a linear operator $T : \mathcal{D}(T) \to X$ is meant a complex number such that

(R1) T_α^{-1} exists,
(R2) T_α^{-1} is bounded,
(R3) T_α^{-1} is defined on a set which is dense in X.

The *resolvent set* $\rho(T,\ X)$ of T is the set of all regular values α of T. Its complement $\sigma(T,\ X) = \mathbb{C}\backslash\rho(T,\ X)$ in the complex plane $\mathbb{C}$ is called the *spectrum* of T. Furthermore, the spectrum $\sigma(T,\ X)$ is partitioned into the following three disjoint sets:

The *point (discrete) spectrum* $\sigma_p(T,\ X)$ is the set such that T_α^{-1} does not exist. Any such $\alpha \in \sigma_p(T,\ X)$ is called an *eigenvalue* of T.

The *continuous spectrum* $\sigma_c(T,\ X)$ is the set such that T_α^{-1} exists and satisfies (R3) but not (R2); that is T_α^{-1} is unbounded.

The *residual spectrum* $\sigma_r(T,\ X)$ is the set such that T_α^{-1} exists (and may be bounded or not) but not satisfy (R3); that is the domain of T_α^{-1} is not dense in X.

To avoid trivial misunderstandings, let us say that some of the sets defined above may be empty. This is an existence problem which we are going to discuss. Indeed, it is well-known that in the finite dimensional case one has $\sigma_c(T,\ X) = \sigma_r(T,\ X) = \emptyset$ and the spectrum $\sigma(T,\ X)$ coincides with the set $\sigma_p(T,\ X)$.

By a *sequence space*, we understand a linear subspace of the space $w = \mathbb{C}^{\mathbb{N}}$ of all complex sequences which contains ϕ, the set of all finitely non-zero sequences, where $\mathbb{N} = \{0, 1, 2, \ldots\}$. We write ℓ_∞, c and c_0 for the sequence spaces of all bounded, convergent and null sequences, respectively. Also by ℓ_p, we denote the spaces of all p-absolutely summable sequences, respectively; where $1 \leq p < \infty$. bv is the space consisting of all sequences

(x_k) such that $(x_k - x_{k+1})$ in ℓ_1 and bv_0 is the intersection of the spaces bv and c_0.

Let $n, k \in \mathbb{N}$ and $A = (a_{nk})$ be an infinite matrix of complex numbers a_{nk}, and write

$$(Ax)_n = \sum_k a_{nk} x_k \ , (n \in \mathbb{N}, x \in D_{00}(A)), \tag{1.1}$$

where $D_{00}(A)$ denotes the subspace of w consisting of $x \in w$ for which the sum on the right side of (1.1) exists as a finite sum. For simplicity in notation, here and in what follows, the summation without limits runs from 0 to ∞. More generally if μ is a normed sequence space, we can write $D_\mu(A)$ for the $x \in w$ for which the sum in (1.1) converges in the norm of μ. We shall write

$$(\lambda : \mu) = \{A : \lambda \subseteq D_\mu(A)\}$$

for the space of those matrices which send the whole of the sequence space λ into the sequence space μ in this sense. A sequence x is said to be A-summable to α if Ax converges to α which is called as the A-limit of x. We shall assume throughout unless stated otherwise that $p,\ q > 1$ with $p^{-1} + q^{-1} = 1$ and use the convention that any term with negative subscript is equal to naught.

We summarize the knowledge in the existing literature concerning with the spectrum and the fine spectrum of the linear operators defined by some particular limitation matrices over some sequence spaces. Wenger [18] examined the fine spectrum of the integer power of the Cesàro operator in c. Reade [16] worked with the spectrum of the Cesàro operator in the sequence space c_0. Gonzàlez [10] studied the fine spectrum of the Cesàro operator in the sequence space ℓ_p. Okutoyi [15] computed the spectrum of the Cesàro operator on the sequence space bv. Recently, Yıldırım [19] worked with the fine spectrum of the Rhally operators acting on the sequence spaces c_0 and c. Lately, Coşkun [8] studied the spectrum and fine spectrum for p-Cesàro operator acting on the space c_0. Akhmedov and Başar [1, 2] have recently determined, independently than that of Gonzàlez [10], the fine spectrum of the Cesàro operator in the sequence spaces c_0, ℓ_∞ and ℓ_p, by the different way; respectively, where $1 < p < \infty$. Quite recently, de Malafosse [14] and Altay and Başar [5] have respectively studied the spectrum and the fine spectrum of the difference operator on the sequence spaces s_r and c_0, c; where s_r denotes the Banach space of all sequences $x = (x_k)$ normed by

$$\|x\|_{s_r} = \sup_{k \in \mathbb{N}} \frac{|x_k|}{r^k}\ ,\ (r > 0).$$

Also, Akhmedov and Başar [3, 4], and Altay and Başar [6] have determined the fine spectrum with respect to Goldberg's classification of the difference operator Δ and the generalized difference operator $B(r,s)$ over the sequence spaces ℓ_p, bv_p and c_0, c; respectively.

In this work, our purpose is to find the norm of the Cesàro operator $C_1 \in B(bv_p)$ and to investigate the fine spectrum of the Cesàro operator C_1 on the sequence space bv_p which is the natural continuation of Akhmedov and Başar [4], and Altay and Başar [5, 6].

2. The Space bv_p of Sequences of p-bounded Variation

We wish to give some required knowledge about the sequence space bv_p. In [7], the sequence space bv_p is defined by

$$bv_p = \left\{ x = (x_k) \in w : \sum_k \left| x_k - x_{k-1} \right|^p < \infty \right\}, \ (1 \leq p < \infty).$$

It was proved that bv_p is a BK-space which is linearly isomorphic to the space ℓ_p and the inclusion $bv_p \supset \ell_p$ strictly holds. The α-, β- and γ-duals of the space bv_p are determined together with the fact that bv_2 is the only Hilbert space among the spaces bv_p. The continuous dual of the space bv_p is determined and given by the following lemma which is needed in proving Theorem 3.2, below:

Lemma 2.1. [4, Theorem 2.3] *Define the spaces d_1 and d_q consisting of all sequences $a = (a_k)$ normed by*

$$\|a\|_{d_1} = \sup_{k,n \in \mathbb{N}} \left| \sum_{j=k}^{n} a_j \right| < \infty$$

and

$$\|a\|_{d_q} = \left(\sum_k \left| \sum_{j=k}^{\infty} a_j \right|^q \right)^{1/q} < \infty \ , \ (1 < q < \infty).$$

Then, bv_1^* and bv_p^* are isometrically isomorphic to d_1 and d_q, respectively.

The basis of the space bv_p is also constructed and given by the following lemma:

Lemma 2.2. [7, Theorem 3.1] *Define the sequence $b^{(k)} = \{b_n^{(k)}\}_{n \in \mathbb{N}}$ of the elements of the space bv_p for every fixed $k \in \mathbb{N}$ by*

$$b_n^{(k)} = \begin{cases} 0 & , \quad (n < k) \\ 1 & , \quad (n \geq k) \end{cases} .$$

Then the sequence $\{b^{(k)}\}_{k\in\mathbb{N}}$ *is a basis for the space* bv_p *and any* $x \in bv_p$ *has a unique representation of the form*

$$x = \sum_k \lambda_k b^{(k)},$$

where $\lambda_k = x_k - x_{k-1}$ *for all* $k \in \mathbb{N}$.

3. The Fine Spectra of the Cesàro Operator C_1 Over the Sequence Space bv_p

In this section, the fine spectra of the Cesàro operator C_1 over the sequence space bv_p has been examined. We shall begin with giving the basic result concerning with the norm of Cesàro operator C_1 on the space bv_p. The Cesàro operator C_1 is represented by the matrix

$$C_1 = \begin{bmatrix} 1 & 0 & 0 & \dots & 0 & \dots \\ \frac{1}{2} & \frac{1}{2} & 0 & \dots & 0 & \dots \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & \dots & 0 & \dots \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ \frac{1}{n+1} & \frac{1}{n+1} & \frac{1}{n+1} & \dots & \frac{1}{n+1} & \dots \\ \vdots & \vdots & \vdots & \dots & \vdots & \ddots \end{bmatrix}.$$

Theorem 3.1. $C_1 \in B(bv_p)$ *with the norm* $\|C_1\|_{(bv_p:bv_p)} = 1$.

Proof. Since the linearity and the boundedness of the operator $C_1 : bv_p \to bv_p$ is obvious, we omit the detail. Let us take any $x = (x_k) \in bv_p$. Then, since one can observe that

$$\begin{aligned} & \left| \frac{x_0 + x_1 + \dots + x_k}{k+1} - \frac{x_0 + x_1 + \dots + x_{k-1}}{k} \right|^p \\ & = \left| \frac{(x_k - x_0) + (x_k - x_1) + \dots + (x_k - x_{k-1})}{k(k+1)} \right|^p \end{aligned}$$

and the inequalities

$$\begin{aligned} |x_k - x_0| &\leq |x_k - x_{k-1}| + |x_{k-1} - x_{k-2}| + \dots + |x_2 - x_1| + |x_1 - x_0| \\ |x_k - x_1| &\leq |x_k - x_{k-1}| + |x_{k-1} - x_{k-2}| + \dots + |x_2 - x_1| \\ &\vdots \\ |x_k - x_{k-2}| &\leq |x_k - x_{k-1}| + |x_{k-1} - x_{k-2}| \end{aligned}$$

hold we see by using the following known inequalities

$$\left(\sum_{n=1}^{k} |a_n| \right)^p \leq k^{p-1} \sum_{n=1}^{k} |a_n|^p \;,\; (p \geq 1),$$

and

$$\frac{n^p}{(k+1)^p} \leq \frac{k}{k+1} \ ; \ (1 \leq n \leq k \ , \ p \geq 1)$$

that

$$\begin{aligned}
&\left| \frac{x_0 + x_1 + \cdots + x_k}{k+1} - \frac{x_0 + x_1 + \cdots + x_{k-1}}{k} \right|^p \\
&\leq \frac{(k|x_k - x_{k-1}| + (k-1)|x_{k-1} - x_{k-2}| + \cdots + 2|x_2 - x_1| + |x_1 - x_0|)^p}{k^p(k+1)^p} \\
&\leq \frac{k^{p-1}(k^p|x_k - x_{k-1}|^p + (k-1)^p|x_{k-1} - x_{k-2}|^p + \cdots + 2^p|x_2 - x_1|^p + |x_1 - x_0|^p)}{k^p(k+1)^p} \\
&= \frac{k^p|x_k - x_{k-1}|^p + (k-1)^p|x_{k-1} - x_{k-2}|^p + \cdots + 2^p|x_2 - x_1|^p + |x_1 - x_0|^p}{k(k+1)^p} \\
&\leq \frac{k|x_k - x_{k-1}|^p + (k-1)|x_{k-1} - x_{k-2}|^p + \cdots + 2|x_2 - x_1|^p + |x_1 - x_0|^p}{k(k+1)}.
\end{aligned}$$

Now, we obtain by applying the Application 1 of Knopp [12, p. 143] that

$$\begin{aligned}
\|C_1 x\|_{bv_p}^p &\leq |x_0|^p + \sum_{k=1}^{\infty} \frac{\sum_{j=1}^{k} j|x_j - x_{j-1}|^p}{k(k+1)} \\
&= \sum_k |x_k - x_{k-1}|^p = \|x\|_{bv_p}^p.
\end{aligned}$$

So,

$$\|C_1 x\|_{bv_p} \leq \|x\|_{bv_p},$$

which leads us to the consequence

$$\|C_1\|_{(bv_p:bv_p)} \leq 1. \tag{3.1}$$

Now, let us consider the element

$$b^{(0)} = (1, \ 1, \ 1, \ldots).$$

It is clear that

$$C_1 b^{(0)} = b^{(0)}$$

and $\|b^{(0)}\|_{bv_p} = 1$. Hence,

$$\|C_1\|_{(bv_p:bv_p)} \geq \|C_1 b^{(0)}\|_{bv_p} = \|b^{(0)}\|_{bv_p} = 1,$$

which yields the fact that

$$\|C_1\|_{(bv_p:bv_p)} \geq 1. \tag{3.2}$$

The inequality (3.1) together with the inequality (3.2) show that

$$\|C_1\|_{(bv_p:bv_p)} = 1. \tag{3.3}$$

Thus, (3.3) completes the proof. □

Theorem 3.2. *$C_1^* \in B(d_q)$ with the norm $\|C_1^*\|_{(d_q:d_q)} = 1$; where $1 < q \leq \infty$.*

Proof. Since $d_1 \cong bv_1^*$ and $d_p \cong bv_p^*$ by Lemma 2.1 and $\|C_1^*\|_{(d_q:d_q)} = \|C_1\|_{(bv_p:bv_p)}$, this is immediate by Theorem 3.1, where $1 < p < \infty$. □

Theorem 3.3. $\sigma(C_1,\ bv_p) = \left\{\alpha \in \mathbb{C} : \left|\alpha - \frac{1}{2}\right| \leq \frac{1}{2}\right\}$.

Proof. To prove the theorem, it is enough to show that $(C_1 - \alpha I)^{-1}$ is bounded for all α's such that $|\alpha - 1/2| > 1/2$. Suppose $y = (y_k) \in bv_p$. Solving the equation $(C_1 - \alpha I)x = y$ for x in terms of y, we derive that

$$\begin{aligned}
x_0 &= \frac{1}{1-\alpha} y_0 \\
x_1 &= \frac{-1}{(1-\alpha)(1-2\alpha)} y_0 + \frac{2}{1-2\alpha} y_1 \\
x_2 &= \frac{2\alpha}{\prod_{k=1}^{3} 1 - k\alpha} y_0 - \frac{2}{\prod_{k=2}^{3} 1 - k\alpha} y_1 + \frac{3}{1-3\alpha} y_2 \\
&\vdots \\
x_n &= \frac{1}{n+1} \sum_{k=0}^{n-1} (-1)^{n-k} \left(\prod_{j=k+1}^{n+1} \frac{j}{1-j\alpha} \right) \alpha^{n-k-1} y_k + \frac{n+1}{1-(n+1)\alpha} y_n
\end{aligned}$$

Therefore, we have $(C_1 - \alpha I)^{-1} = (e_{nk})$ defined by

$$e_{nk} = \begin{cases} \frac{(-1)^{n-k}}{n+1} \left(\prod_{j=k+1}^{n+1} \frac{j}{1-j\alpha} \right) \alpha^{n-k-1} & , \quad (0 \leq k \leq n-1) \\ \frac{k+1}{1-(k+1)\alpha} & , \quad (k = n) \\ 0 & , \quad (k > n) \end{cases}$$

Thus, it is seen by [17] that

$$\|C_1 - \alpha I\|_{(bv_1:bv_1)} < \infty,$$

if $|\alpha - 1/2| > 1/2$ which is equivalent to the fact that $Re(1/\alpha) < 1$.

Furthermore, if $p > 1$ then

$$|x_n - x_{n-1}|^p = |x_n - x_{n-1}|^{p-1} \cdot |x_n - x_{n-1}|\,. \tag{3.4}$$

We can show that

$$|x_n - x_{n-1}| \to 0\ ;\ (n \to \infty), \tag{3.5}$$

if $Re(1/\alpha) < 1$. Indeed,

$$x_n \quad - \quad x_{n-1} = \frac{(-1)^n}{n+1} \prod_{k=1}^{n+1} \frac{k}{1-k\alpha} \alpha^{n-1} y_0 +$$

$$+ \left[\frac{(-1)^{n+1}}{n+1}\prod_{k=2}^{n+1}\frac{k}{1-k\alpha}\alpha^{n-2}y_1 - \frac{(-1)^{n-1}}{n}\prod_{k=1}^{n}\frac{k}{1-k\alpha}\alpha^{n-2}y_0\right] +$$
$$+ \quad \cdots + \left[\frac{n+1}{1-(n+1)\alpha}y_n - \frac{n}{1-n\alpha}y_{n-1}\right].$$

If we use Lemma 7 in [16, p. 266] with the last relation then we have (3.5). Thus, (3.4) and (3.5) yield that

$$\left\|(C_1 - \alpha I)^{-1}\right\|_{(bv_p:bv_p)} < \infty \ ,$$

if $Re(1/\alpha) < 1$. This completes the proof. □

We should remark the reader from now on that the index p has different meanings in the notation of the sequence spaces bv_p, bv_p^* and in the point spectrums $\sigma_p(\Delta,\ bv_p)$, $\sigma_p(\Delta^*,\ bv_p^*)$ which occur in the following two theorems.

Theorem 3.4. $\sigma_p(C_1,\ bv_p) = \{1\}$.

Proof. Suppose that $C_1x = \alpha x$ for $x \neq \theta = (0,\ 0,\ 0, \ldots)$ in bv_p. Consider the system of the linear equations

$$\left.\begin{array}{rcl} x_0 & = & \alpha x_0 \\ \frac{1}{2}x_0 + \frac{1}{2}x_1 & = & \alpha x_1 \\ \frac{1}{3}x_0 + \frac{1}{3}x_1 + \frac{1}{3}x_2 & = & \alpha x_2 \\ & \vdots & \\ \frac{1}{k+1}x_0 + \frac{1}{k+1}x_1 + \frac{1}{k+1}x_2 + \cdots + \frac{1}{k+1}x_k & = & \alpha x_k \\ & \vdots & \end{array}\right\}. \tag{3.6}$$

If x_0 is the non-zero term of the sequence $x = (x_n)$, then $\alpha = 1$ and we obtain from (3.6) that $x_k = x_0$ for any $k \geq 1$. Hence, $x = (x_k) \in bv_p$ such that $x \neq \theta$ for $p \geq 1$.

If x_{n_0} is the first non-zero entry of the sequence $x = (x_n)$, then we find that

$$\frac{1}{n_0+1}x_{n_0} = \alpha x_{n_0},$$

which yields the fact $\alpha = 1/(n_0+1)$. Therefore, we also get by (3.6) that

$$x_{n_0+k} \quad = \quad \frac{(n_0+1)(n_0+2)\cdots(n_0+k)}{k!}x_{n_0}$$

for any $k \geq 1$. Furthermore,

$$|x_{n_0+k} - x_{n_0+k-1}|^p \quad = \quad \frac{n_0^p(n_0+1)^p\cdots(n_0+k-1)^p}{(k!)^p}\,|x_{n_0}|^p$$

$$= \frac{1}{[(n_0-1)!]^p}(k+1)^p(k+2)^p\cdots(n_0+k-1)^p\,|x_{n_0}|^p\,,$$

which shows that $x \notin bv_p$ and this completes the proof. $\square$

Prior to giving Theorem 3.6 we shall quote a lemma which is needed in proving.

Lemma 3.5. [13, p. 115] *All harmonic series* $\sum_n n^{-\alpha}$ *for* $\alpha \leq 1$ *are divergent, and for* $\alpha > 1$*, convergent.*

Theorem 3.6. $\sigma_p(C_1^*,\ bv_p^*) = \left\{\alpha \in \mathbb{C} : |\alpha - \frac{1}{2}| < \frac{1}{2}\right\} \cup \{1\}$.

Proof. Suppose $C_1^* f = \alpha f$ for $f \neq \theta$ in bv_p^*. Then, by solving the system of the linear equations

$$\left.\begin{array}{rcl} f_0 + \frac{1}{2}f_1 + \frac{1}{3}f_2 + \cdots & = & \alpha f_0 \\ \frac{1}{2}f_1 + \frac{1}{3}f_2 + \cdots & = & \alpha f_1 \\ \frac{1}{3}f_2 + \cdots & = & \alpha f_2 \\ & \vdots & \\ \frac{1}{k+1}f_k + \cdots & = & \alpha f_k \\ & \vdots & \end{array}\right\}$$

we obtain that

$$f_k = \prod_{n=1}^{k}\left(1 - \frac{1}{n\alpha}\right) f_0\ ,\ (k = 1,\ 2, \ldots)$$

if $\alpha \neq 0$. Since $f = (f_0,\ 0,\ 0,\ \ldots) \neq \theta$ in bv_p^* for $\alpha = 1$, it is clear that $1 \in \sigma_p(C_1^*,\ bv_p^*)$. Define the sequence $z = (z_k)$ by

$$z_k = \prod_{n=1}^{k}\left(1 - \frac{1}{n\alpha}\right)\ ,\ (k = 1,\ 2, \ldots).$$

Okutoyi [15, Lemma 1.4] has proved that

$$z_k = A \cdot k^{-1/\alpha} + O\left(k^{-Re(1/\alpha)-1}\right),$$

where A is a constant and the series $\sum z_k$ is bounded if $Re(1/\alpha) > 1$, diverges if $Re(1/\alpha) \leq 1$. Consider the sequence $s = (s_k)$ defined by

$$s_k = \sum_{j=k}^{\infty} \frac{1}{j^{1/\alpha}}\ ,\ (k \in \mathbb{N}).$$

It is known that $|s_1| < \infty$ if and only if $Re(1/\alpha) > 1$. Denote $Re(1/\alpha) = \beta$ and let $\beta > 1$. Therefore, using the fact given by Lemma 3.5 for the convergence of the series

$$s_1 = \sum_{j=1}^{\infty} \frac{1}{j^{\beta}}$$

we obtain that

$$\begin{aligned} s_k \leq & \left[\frac{1}{2^{(m-1)\beta}} + \cdots + \frac{1}{(2^m - 1)^{\beta}}\right] \\ & + \left[\frac{1}{2^{m\beta}} + \cdots + \frac{1}{(2^{m+1} - 1)^{\beta}}\right] + \cdots , \end{aligned} \tag{3.7}$$

where $2^{m-1} \leq k \leq 2^m - 1$. Now, replacing any separate term by the first term in each parenthesis in (3.7) we get that

$$s_j \leq \frac{2^{\beta-1}}{2^{\beta-1} - 1} \cdot \frac{1}{2^{(m-1)(\beta-1)}}. \tag{3.8}$$

It is clear by (3.8) that

$$\left|\sum_{j=k}^{\infty} \frac{1}{j^{1/\alpha}}\right| \leq \frac{2^{\beta-1}}{2^{(m-1)(\beta-1)}\,(2^{\beta-1} - 1)}\ ,\ (k \in \mathbb{N}), \tag{3.9}$$

if $Re(1/\alpha) > 1$. Similarly, one can show that

$$\left|\sum_{j=k}^{\infty} j^{-Re(1/\alpha)-1}\right| \leq B \cdot \frac{1}{2^{(m-1)(\beta-1)}}\ ,\ (k \in \mathbb{N}), \tag{3.10}$$

where B is a positive constant. It follows from (3.9) and (3.10) that

$$\sup_{k,n \in \mathbb{N}} \left|\sum_{j=k}^{n} f_j\right| < \infty. \tag{3.11}$$

If $f_0 \neq 0$ and $Re(1/\alpha) > 1$, then $f \in bv_1^*$ and $f \neq \theta$ whenever $f_0 \neq 0$.

It is clear that

$$\left\{\alpha \in \mathbb{C} : Re\left(\frac{1}{\alpha}\right) > 1\right\} = \left\{\alpha \in \mathbb{C} : \left|\alpha - \frac{1}{2}\right| < \frac{1}{2}\right\}.$$

Now, using (3.9) and (3.10) we obtain that

$$\left|\sum_{j=k}^{n} f_j\right|^q \leq M \cdot \frac{1}{2^{(m-1)(\beta-1)q}}, \tag{3.12}$$

where M is a positive constant. Consequently, we derive from (3.12) that

$$\sum_k \left|\sum_{j=k}^{\infty} f_j\right|^q < \infty.$$

This means that $(f_k) \in bv_p^*$ and (3.11) together with (3.12) complete the proof. □

Now, we may give the following lemma requiring in the proof of next theorem:

Lemma 3.7. [9, p. 59] *A linear operator T has a dense range if and only if the adjoint T^* of T is one to one.*

Theorem 3.8. $\sigma_c(C_1,\ bv_p) = \left\{\alpha \in \mathbb{C} : \left|\alpha - \frac{1}{2}\right| = \frac{1}{2}\ ,\ \alpha \neq 1\right\}$.

Proof. It is not hard to show that

$$\left\{\alpha \in \mathbb{C} : \left|\alpha - \frac{1}{2}\right| = \frac{1}{2}\right\} = \left\{\alpha \in \mathbb{C} : Re\left(\frac{1}{\alpha}\right) = 1\right\} \cup \{0\}.$$

Suppose that $\alpha \neq 1$. Then, it follows by Theorem 3.6 that $\alpha \notin \sigma_p(C_1^*,\ bv_p^*)$. Hence, $Ker(C_1^* - \alpha I^*) = \{\theta\}$ for such α's which shows that

$$\overline{R(C_1 - \alpha I)} = bv_p. \tag{3.13}$$

Now, suppose that $\alpha = 0$ and consider the equation

$$C_1 x = \theta.$$

Then, it is easy to see that $x = \theta$, i.e., $Ker(C_1) = \{\theta\}$ and C_1 has an inverse. One can also see that $Ker(C_1^*) = \{\theta\}$ and we thus have

$$\overline{R(C_1)} = bv_p. \tag{3.14}$$

Therefore, we obtain by combining (3.13), (3.14) and Lemma 3.7 that

$$\sigma_c(C_1,\ bv_p) = \{0\} \cup \left\{\alpha \in \mathbb{C} : Re\left(\frac{1}{\alpha}\right) = 1\ ,\ \alpha \neq 1\right\},$$

which completes the proof. □

Theorem 3.9. $\sigma_r(C_1,\ bv_p) = \left\{\alpha \in \mathbb{C} : \left|\alpha - \frac{1}{2}\right| < \frac{1}{2}\right\}$.

Proof. This immediately follows from Theorems 3.3, 3.4 and 3.8, by taking into account the definition of the concept of the spectrum of a bounded linear operator acting in a Banach space. □

Combining Theorems 3.1, 3.3, 3.4 and Theorems 3.8, 3.9; we have the following main theorem:

Theorem 3.10. *(a)* $C_1 \in B(bv_p)$ *with the norm* $\|C_1\|_{(bv_p:bv_p)} = 1$,

(b) $\sigma(C_1,\ bv_p) = \left\{\alpha \in \mathbb{C} : \left|\alpha - \frac{1}{2}\right| \leq \frac{1}{2}\right\}$,

(c) $\sigma_p(C_1,\ bv_p) = \{1\}$,

(d) $\sigma_c(C_1,\ bv_p) = \left\{\alpha \in \mathbb{C} : \left|\alpha - \frac{1}{2}\right| = \frac{1}{2}\ ,\ \alpha \neq 1\right\}$,

(e) $\sigma_r(C_1,\ bv_p) = \left\{\alpha \in \mathbb{C} : \left|\alpha - \frac{1}{2}\right| < \frac{1}{2}\right\}$.

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