

ON ϕ -RECURRENT $N(k)$ -CONTACT METRIC MANIFOLDS

Dedicated to PROFESSOR DAVID E. BLAIR

UDAY CHAND DE AND ABOUL KALAM GAZI

ABSTRACT. In this paper we prove that a ϕ -recurrent $N(k)$ -contact metric manifold is an η -Einstein manifold with constant coefficients. Next, we prove that a 3-dimensional ϕ -recurrent $N(k)$ -contact metric manifold is of constant curvature. The existence of a ϕ -recurrent $N(k)$ -contact metric manifold is also proved.

1. Introduction

The notion of local symmetry of a Riemannian manifold has been weakened by many authors in several ways to a different extent. As a weaker version of local symmetry, T. Takahashi [1] introduced the notion of local ϕ -symmetry on a Sasakian manifold. Generalizing the notion of local ϕ -symmetry, one of the authors, De, [2] introduced the notion of ϕ -recurrent Sasakian manifold. In the context of contact geometry the notion of ϕ -symmetry is introduced and studied by Boeckx, Bueken and Vanhecke [3] with several examples.

In the present paper we study ϕ -recurrent $N(k)$ -contact metric manifold which generalizes the result of De, Shaikh and Biswas [2]. The paper is organized as follows:

Section 2 contains necessary details about contact metric manifolds, some preliminaries and a brief account of (k, μ) manifolds and the basic results. In Section 3, it is proved that a ϕ -recurrent $N(k)$ -contact metric manifold is a special type of η -Einstein manifold. Also it is shown that the characteristic vector field of the $N(k)$ -contact metric manifold and the vector field associated to the 1-form of recurrence are co-directional. In Section 4, it is also proved that a 3-dimensional ϕ -recurrent $N(k)$ -contact metric manifold is of constant curvature. The last section provides the existence of the ϕ -recurrent $N(k)$ -contact metric manifold by an example which is neither symmetric nor locally ϕ -symmetric.

Mathematics Subject Classification. Primary 53C15; Secondary 53C40.

Key words and phrases. $N(k)$ -contact metric manifolds, η -Einstein manifold, ϕ -recurrent $N(k)$ -contact metric manifolds.

The authors are thankful to the referee for valuable suggestions towards the improvement of this paper.

2. Contact Metric Manifolds

A $(2n+1)$ -dimensional manifold M^{2n+1} is said to admit an almost contact structure if it admits a tensor field ϕ of type $(1, 1)$, a vector field ξ and a 1-form η satisfying

$$(2.1) \quad (a) \quad \phi^2 = -I + \eta \otimes \xi, \quad (b) \quad \eta(\xi) = 1, \quad (c) \quad \phi\xi = 0, \quad (d) \quad \eta \circ \phi = 0.$$

An almost contact metric structure is said to be normal if the induced almost complex structure J on the product manifold $M^{2n+1} \times \mathbf{R}$ defined by

$$J(X, f \frac{d}{dt}) = (\phi X - f\xi, \eta(X) \frac{d}{dt})$$

is integrable, where X is tangent to M , t is the coordinate of \mathbf{R} and f is a smooth function on $M \times \mathbf{R}$. Let g be a compatible Riemannian metric with almost contact structure (ϕ, ξ, η) , that is,

$$(2.2) \quad g(\phi X, \phi Y) = g(X, Y) - \eta(X)\eta(Y).$$

Then M becomes an almost contact metric manifold equipped with an almost contact metric structure (ϕ, ξ, η, g) . From (2.1) it can be easily seen that

$$(2.3) \quad (a)g(X, \phi Y) = -g(\phi X, Y), \quad (b)g(X, \xi) = \eta(X),$$

for all vector fields X, Y . An almost contact metric structure becomes a contact metric structure if

$$(2.4) \quad g(X, \phi Y) = d\eta(X, Y),$$

for all vector fields X, Y . The 1-form η is then a contact form and ξ is its characteristic vector field. We define a $(1, 1)$ tensor field h by $h = \frac{1}{2}\mathcal{L}_\xi\phi$, where \mathcal{L} denotes the Lie-differentiation. Then h is symmetric and satisfies $h\phi = -\phi h$. We have $Tr.h = Tr.\phi h = 0$ and $h\xi = 0$. Also,

$$(2.5) \quad \nabla_X \xi = -\phi X - \phi hX,$$

holds in a contact metric manifold. A normal contact metric manifold is a Sasakian manifold. An almost contact metric manifold is Sasakian if and only if

$$(2.6) \quad (\nabla_X \phi)(Y) = g(X, Y)\xi - \eta(Y)X, \quad X, Y \in TM,$$

where ∇ is the Levi-Civita connection of the Riemannian metric g . A contact metric manifold $M^{2n+1}(\phi, \xi, \eta, g)$ for which ξ is a Killing vector is said to be a K -contact manifold. A Sasakian manifold is K -contact but not conversely. However a 3-dimensional K -contact manifold is Sasakian [4]. It is well known that the tangent sphere bundle of a flat Riemannian manifold admits a

contact metric structure satisfying $R(X, Y)\xi = 0$ ([5]). On the other hand, on a Sasakian manifold the following holds:

$$(2.7) \quad R(X, Y)\xi = \eta(Y)X - \eta(X)Y.$$

As a generalization of both $R(X, Y)\xi = 0$ and the Sasakian case; D. Blair, T. Koufogiorgos and B. J. Papantoniou [6] considered the (k, μ) -nullity condition on a contact metric manifold and gave several reasons for studying it. The (k, μ) -nullity distribution $N(k, \mu)$ ([6], [7]) of a contact metric manifold M is defined by

$$\begin{aligned} & N(k, \mu) : p \longrightarrow N_p(k, \mu) \\ & = \{W \in T_p M : R(X, Y)W = (kI + \mu h)(g(Y, W)X - g(X, W)Y)\}, \end{aligned}$$

for all $X, Y \in TM$, where $(k, \mu) \in \mathbf{R}^2$. A contact metric manifold M^{2n+1} with $\xi \in N(k, \mu)$ is called a (k, μ) -manifold. In particular on a (k, μ) -manifold, we have

$$(2.8) \quad R(X, Y)\xi = k[\eta(Y)X - \eta(X)Y] + \mu[\eta(Y)hX - \eta(X)hY].$$

On a (k, μ) -manifold $k \leq 1$. If $k = 1$, the structure is Sasakian ($h = 0$ and μ is indeterminate) and if $k < 1$, the (k, μ) -nullity condition determines the curvature of M^{2n+1} completely [6]. Infact, for a (k, μ) -manifold, the condition of being a Sasakian manifold, a K -contact manifold, $k = 1$ and $h = 0$ are all equivalent.

In a (k, μ) -manifold the following relations hold ([6], [8]):

$$(2.9) \quad h^2 = (k - 1)\phi^2, \quad k \leq 1,$$

$$(2.10) \quad (\nabla_X \phi)(Y) = g(X + hX, Y)\xi - \eta(Y)(X + hX),$$

$$(2.11) \quad R(\xi, X)Y = k[g(X, Y)\xi - \eta(Y)X] + \mu[g(hX, Y)\xi - \eta(Y)hX],$$

$$(2.12) \quad S(X, \xi) = 2nk\eta(X),$$

$$(2.13) \quad \begin{aligned} S(X, Y) = & [2(n - 1) - n\mu]g(X, Y) + [2(n - 1) + \mu]g(hX, Y) \\ & + [2(1 - n) + n(2k + \mu)]\eta(X)\eta(Y), \quad n \geq 1, \end{aligned}$$

$$(2.14) \quad r = 2n(2n - 2 + k - n\mu),$$

$$(2.15) \quad S(\phi X, \phi Y) = S(X, Y) - 2nk\eta(X)\eta(Y) - 2(2n - 2 + \mu)g(hX, Y),$$

where S is the Ricci tensor of type $(0, 2)$, Q is the Ricci-operator, that is, $g(QX, Y) = S(X, Y)$ and r is the scalar curvature of the manifold. From (2.5), it follows that

$$(2.16) \quad (\nabla_X \eta)(Y) = g(X + hX, \phi Y).$$

Also in a (k, μ) -manifold

$$(2.17) \quad \eta(R(X, Y)Z) = k[g(Y, Z)\eta(X) - g(X, Z)\eta(Y)] \\ + \mu[g(hY, Z)\eta(X) - g(hX, Z)\eta(Y)]$$

holds.

The k -nullity distribution $N(k)$ of a Riemannian manifold M [9] is defined by

$$N(k) : p \longrightarrow N_p(k) = \{Z \in T_pM : R(X, Y)Z = g(Y, Z)X - g(X, Z)Y\},$$

k being a constant. If the characteristic vector field $\xi \in N(k)$, then we call a contact metric manifold an $N(k)$ -contact metric manifold [10]. If $k = 1$, then $N(k)$ -contact metric manifold is Sasakian and if $k = 0$, then $N(k)$ -contact metric manifold is locally isometric to the product $E^{n+1} \times S^n(4)$ for $n > 1$ and flat for $n = 1$. If $k < 1$, the scalar curvature is $r = 2n(2n - 2 + k)$. If $\mu = 0$, then a (k, μ) -contact metric manifold reduces to a $N(k)$ -contact metric manifold.

In [11], $N(k)$ -contact metric manifold were studied in some detail. For more details we refer to [12] [13].

In $N(k)$ -contact metric manifold the following relations hold:

$$(2.18) \quad h^2 = (k - 1)\phi^2, \quad k \leq 1,$$

$$(2.19) \quad (\nabla_X \phi)(Y) = g(X + hX, Y)\xi - \eta(Y)(X + hX),$$

$$(2.20) \quad R(\xi, X)Y = k[g(X, Y)\xi - \eta(Y)X],$$

$$(2.21) \quad S(X, \xi) = 2nk\eta(X),$$

$$(2.22) \quad S(X, Y) = 2(n - 1)g(X, Y) + 2(n - 1)g(hX, Y)$$

$$(2.23) \quad + [2(1 - n) + 2nk]\eta(X)\eta(Y), \quad n \geq 1,$$

$$(2.24) \quad r = 2n(2n - 2 + k),$$

$$(2.25) \quad S(\phi X, \phi Y) = S(X, Y) - 2nk\eta(X)\eta(Y) - 4(n - 1)g(hX, Y),$$

$$(2.26) \quad (\nabla_X \eta)(Y) = g(X + hX, \phi Y),$$

$$(2.27) \quad R(X, Y)\xi = k[\eta(Y)X - \eta(X)Y],$$

$$(2.28) \quad \eta(R(X, Y)Z) = k[g(Y, Z)\eta(X) - g(X, Z)\eta(Y)].$$

3. ϕ -recurrent $N(k)$ -contact metric manifolds

Definition 1. ([1]) A Sasakian manifold is said to be locally ϕ -symmetric if the relation

$$\phi^2((\nabla_W R)(X, Y)Z) = 0$$

holds for all vector fields X, Y, Z, W orthogonal to ξ .

Definition 2. ([2]) A $N(k)$ -contact metric manifold is said to be ϕ -recurrent if and only if there exists a non-zero 1-form A such that

$$(3.1) \quad \phi^2((\nabla_W R)(X, Y)Z) = A(W)R(X, Y)Z,$$

for all vector fields X, Y, Z, W . Here X, Y, Z, W are arbitrary vector fields which are not necessarily orthogonal to ξ .

If the 1-form A vanishes identically, then the manifold is said to be a locally ϕ -symmetric manifold.

Definition 3. ([6]) A contact manifold is said to be η -Einstein if the Ricci tensor S of type $(0, 2)$ satisfies the condition

$$(3.2) \quad S(X, Y) = ag(X, Y) + b\eta(X)\eta(Y),$$

where a and b are smooth functions on M^{2n+1} .

Now we prove the main theorem of the paper.

Theorem 3.1. *A ϕ -recurrent $N(k)$ -contact metric manifold is an η -Einstein manifold with constant coefficients.*

Proof. By virtue of (2.1)(a) and (3.1) we have

$$(3.3) \quad -(\nabla_W R)(X, Y)Z + \eta((\nabla_W R)(X, Y)Z)\xi = A(W)R(X, Y)Z,$$

from which it follows that

$$(3.4) \quad \begin{aligned} & -g((\nabla_W R)(X, Y)Z, U) + \eta((\nabla_W R)(X, Y)Z)\eta(U) \\ & = A(W)g(R(X, Y)Z, U). \end{aligned}$$

Let $\{e_i\}$, $i = 1, 2, 3, \dots, 2n + 1$, be an orthonormal basis of the tangent space at any point of the manifold. Putting $X = U = \{e_i\}$ in (3.4) and taking summation over i , $1 \leq i \leq 2n + 1$, we get

$$(3.5) \quad -(\nabla_W S)(Y, Z) + \sum_{i=1}^{2n+1} \eta((\nabla_W R)(e_i, Y)Z)\eta(e_i) = A(W)S(Y, Z).$$

The second term of (3.5) by putting $Z = \xi$ takes the form $g((\nabla_W R)(e_i, Y)\xi, \xi)g(e_i, \xi)$, which is denoted by E . In this case E vanishes.

Namely we have

$$\begin{aligned} g((\nabla_W R)(e_i, Y)\xi, \xi) &= g(\nabla_W R(e_i, Y)\xi, \xi) - g(R(\nabla_W e_i, Y)\xi, \xi) \\ &\quad - g(R(e_i, \nabla_W Y)\xi, \xi) - g(R(e_i, Y)\nabla_W \xi, \xi) \end{aligned}$$

at $p \in M$. Using (2.3)(b) and (2.27) we obtain

$$\begin{aligned} g(R(e_i, \nabla_W Y)\xi, \xi) &= g(k[\eta(\nabla_W Y)e_i - \eta(e_i)\nabla_W Y], \xi) \\ &= k[\eta(\nabla_W Y)\eta(e_i) - \eta(e_i)\eta(\nabla_W Y)] = 0. \end{aligned}$$

Thus we obtain

$$g((\nabla_W R)(e_i, Y)\xi, \xi) = g(\nabla_W R(e_i, Y)\xi, \xi) - g(R(e_i, Y)\nabla_W \xi, \xi).$$

In virtue of $g(R(e_i, Y)\xi, \xi) = g(R(\xi, \xi)e_i, Y) = 0$, we have

$$g(\nabla_W R(e_i, Y)\xi, \xi) + g(R(e_i, Y)\xi, \nabla_W \xi) = 0, \quad \text{since } (\nabla_W g) = 0,$$

which implies

$$g((\nabla_W R)(e_i, Y)\xi, \xi) = -g(R(e_i, Y)\xi, \nabla_W \xi) - g(R(e_i, Y)\nabla_W \xi, \xi) = 0.$$

Using (2.5) and applying skew-symmetry of R we get

$$\begin{aligned} g((\nabla_W R)(e_i, Y)\xi, \xi) &= g(R(e_i, Y)\xi, \phi W + \phi hW) + g(R(e_i, Y)(\phi W + \phi hW), \xi) \\ &= g(R(\phi W + \phi hW, \xi)Y, e_i) + g(R(\xi, \phi W + \phi hW)Y, e_i). \end{aligned}$$

Hence we obtain

$$\begin{aligned} E &= \sum_{i=1}^{2n+1} \left[g(R(\phi W + \phi hW, \xi)Y, e_i)g(\xi, e_i) \right. \\ &\quad \left. + g(R(\xi, \phi W + \phi hW)Y, e_i)g(\xi, e_i) \right] \\ &= g(R(\phi W + \phi hW, \xi)Y, \xi) + g(R(\xi, \phi W + \phi hW)Y, \xi) = 0. \end{aligned}$$

Replacing Z by ξ in (3.5) and using (2.21) we have

$$(3.6) \quad -(\nabla_W S)(Y, \xi) = 2nkA(W)\eta(Y).$$

Now we have

$$(\nabla_W S)(Y, \xi) = \nabla_W S(Y, \xi) - S(\nabla_W Y, \xi) - S(Y, \nabla_W \xi).$$

Using (2.21) and (2.5) in the above relation, it follows that

$$(3.7) \quad (\nabla_W S)(Y, \xi) = 2nk(\nabla_W \eta)(Y) + S(Y, \phi W + \phi hW).$$

In virtue of (3.7), (2.26) and (2.3)(a) we get

$$(3.8) \quad (\nabla_W S)(Y, \xi) = -2nkg(\phi W + \phi hW, Y) + S(Y, \phi W + \phi hW).$$

By (3.6) and (3.8) we have

$$(3.9) \quad 2nkg(\phi W + \phi hW, Y) - S(Y, \phi W + \phi hW) = 2nkA(W)\eta(Y).$$

Replacing Y by ϕY in (3.9) and using (2.1)(d), (2.2), (2.25) we get

$$2nkg(\phi W + \phi hW, \phi Y) - S(\phi Y, \phi W + \phi hW) = 0$$

or,

$$2nk[g(W + hW, Y) - \eta(W + hW)\eta(Y)] - S(Y, W + hW) + 2nk\eta(W + hW)\eta(Y) + 4(n-1)g(hY, W + hW) = 0$$

or,

$$2nkg(Y, W) + 2nkg(Y, hW) - S(Y, W) - S(Y, hW) + 4(n-1)g(Y, hW) + 4(n-1)g(Y, h^2W) = 0$$

since, $g(X, hY) = g(hX, Y)$. Now by (2.23), (2.18) and (2.1)(a) this implies

$$S(Y, W) + S(Y, hW) = 2nkg(Y, W) + [2nk + 4(n-1)]g(Y, hW) + 4(n-1)(k-1)g(Y, -W + \eta(W)\xi)$$

or,

$$S(Y, W) + 2(n-1)g(Y, hW) - 2(n-1)(k-1)g(Y, W) + 2(n-1)(k-1)\eta(Y)\eta(W) = [2nk - 4(n-1)(k-1)]g(Y, W) + [2nk + 4(n-1)]g(Y, hW) + 4(n-1)(k-1)\eta(Y)\eta(W),$$

which implies,

$$(3.10) \quad S(Y, W) = 2(n+k-1)g(Y, W) + 2(nk+n-1)g(Y, hW) + 2(n-1)(k-1)\eta(Y)\eta(W).$$

Replacing W by hW and using (2.23), (2.18) and (2.1)(a) we get from (3.10)

$$-2kg(Y, hW) = -2nk(k-1)g(Y, W) + 2nk(k-1)\eta(Y)\eta(W).$$

Since we may assume that $k \neq 0$, this implies

$$(3.11) \quad g(Y, hW) = n(k-1)g(Y, W) - n(k-1)\eta(Y)\eta(W).$$

From (3.10) and (3.11) we get

$$S(Y, W) = 2[(n+k-1) + n(k-1)(nk+n-1)]g(Y, W) + 2[(n-1)(k-1) - n(k-1)(nk+n-1)]\eta(Y)\eta(W)$$

or,

$$(3.12) \quad S(Y, W) = ag(Y, W) + b\eta(Y)\eta(W),$$

where $a = 2[(n+k-1) + n(k-1)(nk+n-1)]$, $b = 2[(n-1)(k-1) - n(k-1)(nk+n-1)]$ are constant. So, the manifold is an η -Einstein manifold with constant coefficients. Hence the theorem is proved. \square

Now, from (3.3) we have

$$(3.13) \quad (\nabla_W R)(X, Y)Z = \eta((\nabla_W R)(X, Y)Z)\xi - A(W)R(X, Y)Z.$$

From (3.13) and the second Bianchi identity we get

$$(3.14) \quad A(W)\eta(R(X, Y)Z) + A(X)\eta(R(Y, W)Z) + A(Y)\eta(R(W, X)Z) = 0.$$

Using (2.28), we get from (3.14)

$$(3.15) \quad k[A(W)(g(Y, Z)\eta(X) - g(X, Z)\eta(Y)) + A(X)(g(W, Z)\eta(Y) - g(Y, Z)\eta(W)) + A(Y)(g(X, Z)\eta(W) - g(W, Z)\eta(X))] = 0.$$

Putting $Y = Z = \{e_i\}$ in (3.15) and taking summation over i , $1 \leq i \leq 2n+1$, we get

$$k(2n - 1)[A(W)\eta(X) - A(X)\eta(W)] = 0,$$

which implies that

$$(3.16) \quad A(W)\eta(X) = A(X)\eta(W).$$

Replacing X by ξ in (3.16), it follows that

$$(3.17) \quad A(W) = \eta(\rho)\eta(W),$$

for any vector field W , where $A(\xi) = g(\xi, \rho) = \eta(\rho)$, ρ being the vector field associated to the 1-form A , that is, $g(X, \rho) = A(X)$. Hence we can state the following theorem:

Theorem 3.2. *In a ϕ -recurrent $N(k)$ -contact metric manifold (M^{2n+1}, g) , $n > 1$, the characteristic vector field ξ and the vector field ρ associated to the 1-form A are co-directional and the 1-form A is given by (3.17).*

4. 3-dimensional ϕ -recurrent $N(k)$ -contact metric manifolds

In a 3-dimensional Riemannian manifold we have

$$(4.1) \quad R(X, Y)Z = g(Y, Z)QX - g(X, Z)QY + S(Y, Z)X - S(X, Z)Y + \frac{r}{2}[g(X, Z)Y - g(Y, Z)X],$$

where Q is the Ricci-operator, that is, $g(QX, Y) = S(X, Y)$ and r is the scalar curvature of the manifold. Now putting $Z = \xi$ in (4.1) and using (2.3)(b) and (2.21), we get

$$(4.2) \quad R(X, Y)\xi = \eta(Y)QX - \eta(X)QY + 2k[\eta(Y)X - \eta(X)Y] + \frac{r}{2}[\eta(X)Y - \eta(Y)X].$$

Using (2.27) in (4.2), we have

$$(4.3) \quad (k - \frac{r}{2})[\eta(Y)X - \eta(X)Y] = \eta(X)QY - \eta(Y)QX.$$

Putting $Y = \xi$ in (4.3) and using (2.21), we get

$$(4.4) \quad QX = \left(\frac{r}{2} - k\right)X + \left(3k - \frac{r}{2}\right)\eta(X)\xi.$$

Therefore, it follows from (4.4) that

$$(4.5) \quad S(X, Y) = \left(\frac{r}{2} - k\right)g(X, Y) + \left(3k - \frac{r}{2}\right)\eta(X)\eta(Y).$$

Thus from (4.1), (4.4) and (4.5), we get

$$(4.6) \quad \begin{aligned} R(X, Y)Z &= \left(\frac{r}{2} - 2k\right)[g(Y, Z)X - g(X, Z)Y] \\ &\quad + \left(3k - \frac{r}{2}\right)[g(Y, Z)\eta(X)\xi - g(X, Z)\eta(Y)\xi \\ &\quad + \eta(Y)\eta(Z)X - \eta(X)\eta(Z)Y]. \end{aligned}$$

Taking the covariant differentiation to the both sides of the equation (4.6), we get

$$(4.7) \quad \begin{aligned} (\nabla_W R)(X, Y)Z &= \frac{dr(W)}{2}[g(Y, Z)X - g(X, Z)Y - g(Y, Z)\eta(X)\xi \\ &\quad + g(X, Z)\eta(Y)\xi - \eta(Y)\eta(Z)X + \eta(X)\eta(Z)Y] \\ &\quad + \left(3k - \frac{r}{2}\right)[g(Y, Z)\eta(X) - g(X, Z)\eta(Y)]\nabla_W \xi \\ &\quad + \left(3k - \frac{r}{2}\right)[\eta(Y)X - \eta(X)Y](\nabla_W \eta)(Z) \\ &\quad + \left(3k - \frac{r}{2}\right)[g(Y, Z)\xi - \eta(Z)Y](\nabla_W \eta)(X) \\ &\quad - \left(3k - \frac{r}{2}\right)[g(X, Z)\xi - \eta(Z)X](\nabla_W \eta)(Y). \end{aligned}$$

Noting that we may assume that all vector fields X, Y, Z, W are orthogonal to ξ and using (2.1)(b), we get

$$(4.8) \quad \begin{aligned} (\nabla_W R)(X, Y)Z &= \frac{dr(W)}{2}[g(Y, Z)X - g(X, Z)Y] \\ &\quad + \left(3k - \frac{r}{2}\right)[g(Y, Z)(\nabla_W \eta)(X) - g(X, Z)(\nabla_W \eta)(Y)]\xi. \end{aligned}$$

Applying ϕ^2 to the both sides of (4.8) and using (2.1)(a) and (2.1)(c), we get

$$(4.9) \quad \phi^2(\nabla_W R)(X, Y)Z = \frac{dr(W)}{2}[g(X, Z)Y - g(Y, Z)X].$$

By (3.1) the equation (4.9) reduces to

$$(4.10) \quad A(W)R(X, Y)Z = \frac{dr(W)}{2}[g(X, Z)Y - g(Y, Z)X].$$

Putting $W = \{e_i\}$, where $\{e_i\}$, $i = 1, 2, 3$, is an orthonormal basis of the tangent space at any point of the manifold and taking summation over i , $1 \leq i \leq 3$, we obtain

$$(4.11) \quad R(X, Y)Z = \lambda[g(X, Z)Y - g(Y, Z)X],$$

where $\lambda = \frac{dr(e_i)}{2A(e_i)}$ is a scalar, since A is a non-zero 1-form. Then by Schur's theorem λ will be a constant on the manifold. Therefore, M^3 is of constant curvature λ . Thus we get the following theorem:

Theorem 4.1. *A 3-dimensional ϕ -recurrent $N(k)$ -contact metric manifold is of constant curvature.*

5. Existence of ϕ -recurrent $N(k)$ -contact metric manifolds

In this section we give an example of ϕ -recurrent $N(k)$ -contact metric manifold which is neither symmetric nor locally ϕ -symmetric. We take the 3-dimensional manifold $M = \{(x, y, z) \in \mathbf{R}^3 : x \neq 0\}$, where (x, y, z) are the standard coordinates in \mathbf{R}^3 . Let $\{E_1, E_2, E_3\}$ be linearly independent global frame on M given by

$$E_1 = \frac{2}{x} \frac{\partial}{\partial y}, \quad E_2 = 2 \frac{\partial}{\partial x} - \frac{4z}{x} \frac{\partial}{\partial y} + xy \frac{\partial}{\partial z}, \quad E_3 = \frac{\partial}{\partial z}.$$

Let g be the Riemannian metric defined by

$$\begin{aligned} g(E_1, E_3) &= g(E_2, E_3) = g(E_1, E_2) = 0, \\ g(E_1, E_1) &= g(E_2, E_2) = g(E_3, E_3) = 1. \end{aligned}$$

Let η be the 1-form defined by $\eta(U) = g(U, E_3)$ for any $U \in \chi(M)$. Let ϕ be the $(1, 1)$ tensor field defined by $\phi E_1 = E_2$, $\phi E_2 = -E_1$, $\phi E_3 = 0$. Then using the linearity of ϕ and g we have $\eta(E_3) = 1$, $\phi^2 U = -U + \eta(U)E_3$ and $g(\phi U, \phi W) = g(U, W) - \eta(U)\eta(W)$ for any $U, W \in \chi(M)$. Moreover $hE_1 = -E_1$, $hE_2 = E_2$ and $hE_3 = 0$. Thus for $E_3 = \xi$, (ϕ, ξ, η, g) defines a contact metric structure on M . Hence we have $[E_1, E_2] = 2E_3 + \frac{2}{x}E_1$, $[E_1, E_3] = 0$, $[E_2, E_3] = 2E_1$.

The Riemannian connection ∇ of the metric g is given by

$$\begin{aligned} 2g(\nabla_X Y, Z) &= Xg(Y, Z) + Yg(Z, X) - Zg(X, Y) \\ &\quad - g(X, [Y, Z]) - g(Y, [X, Z]) + g(Z, [X, Y]). \end{aligned}$$

Taking $E_3 = \xi$ and using the above formula for Riemannian metric g , it can be easily calculated that

$$\begin{aligned} \nabla_{E_1} E_3 &= 0, \quad \nabla_{E_2} E_3 = 2E_1, \quad \nabla_{E_3} E_3 = 0, \quad \nabla_{E_3} E_1 = 0, \quad \nabla_{E_1} E_2 = \frac{2}{x}E_1, \\ \nabla_{E_2} E_1 &= -2E_3, \quad \nabla_{E_2} E_2 = 0, \quad \nabla_{E_3} E_2 = 0, \quad \nabla_{E_1} E_1 = -\frac{2}{x}E_2. \end{aligned}$$

From the above it can be easily seen that (ϕ, ξ, η, g) is a $N(k)$ -contact metric manifold with $k = -\frac{4}{x} \neq 0$.

Using the above relations, we can easily calculate the non-vanishing components of the curvature tensor as follows:

$$R(E_2, E_3)E_2 = -\frac{4}{x}E_1, \quad R(E_2, E_3)E_1 = \frac{4}{x}E_2,$$

and the components which can be obtained from these by symmetry property. We shall now show that in such a $N(k)$ -contact metric manifold the curvature tensor R is ϕ -recurrent. Since $\{E_1, E_2, E_3\}$ form a basis of M^3 , any vector field $X \in \chi(M)$ can be taken as

$$X = a_1E_1 + a_2E_2 + a_3E_3$$

where $a_i \in \mathbf{R}^+$ (= the set of all positive real numbers), $i = 1, 2, 3$. Thus the covariant derivatives of the curvature tensor are given by

$$(\nabla_X R)(E_2, E_3)E_1 = -\frac{8a_2}{x^2}E_2,$$

$$(\nabla_X R)(E_2, E_3)E_2 = \frac{8a_2}{x^2}E_1.$$

Let us now consider the non-vanishing 1-form $A(X) = \frac{2a_2}{x}$, at any point $p \in M$. In our M^3 , (2.1) reduces with the 1-form to the following equations:

$$(5.1) \quad \phi^2((\nabla_X R)(E_2, E_3)E_1) = A(X)R(E_2, E_3)E_1,$$

$$(5.2) \quad \phi^2((\nabla_X R)(E_2, E_3)E_2) = A(X)R(E_2, E_3)E_2.$$

This implies that the manifold under consideration is a ϕ -recurrent $N(k)$ -contact metric manifold, which is neither symmetric nor locally ϕ -symmetric. So, we can state the following:

Theorem 5.1. *There exists a ϕ -recurrent $N(k)$ -contact metric manifold, which is neither symmetric nor locally ϕ -symmetric.*

REFERENCES

- [1] T. Takahashi, Sasakian ϕ -symmetric spaces, Tohoku Math. J., 29(1977), 91-113.
- [2] U. C. De, A. A. shaikh, S. Biswas, On ϕ -recurrent Sasakian manifolds, Novi Sad J.Math., 33(2003), 13-48.
- [3] E. Boeckx, P. Buecken and L. Vanhecke, ϕ -symmetric contact metric spaces, Glasgow Math. J. 41(1999), 409-416.
- [4] Jae-Bok Jun and Un Kyu Kim, On 3-dimensional almost contact metric manifolds, Kyungpook Math. J. 34(1994), 293-301.
- [5] D. E. Blair, Two remarks on contact metric structures, Tohoku Math. J. 29(1977), 319-324.
- [6] D. E. Blair, Th. Koufogiorgos, B. J. papantoniou, Contact metric manifolds satisfying a nullity condition, Israel J. Math. 91(1995), 189-214.

- [7] B. J. Papantoniou, Contact Riemannian manifolds satisfying $R(\xi, X).R = 0$ and $\xi \in (k, \mu)$ -nullity distribution, Yokohama Math. J., 40(1993), 149-161.
- [8] E. Boeckx, A full classification of contact metric (k, μ) -spces, Illinois J. Math. 44(2000), 212-219.
- [9] S. Tano, Ricci curvatures of contact Riemannian manifolds, The Tohoku Mathematical Journal 40(1988), 441-448.
- [10] D. E. Blair, J. S. Kim and M. M. Tripathi, On the concircular curvature tensor of a contact metric manifold, J. Korean Math. Soc. 42(5)2005, 883-892.
- [11] Ch. Baikoussis, D. E. Blair and Th. Koufogiorgos, A decomposition of the curvature tensor of a contact manifold satisfying $R(X, Y)\xi = k(\eta(Y)X - \eta(X)Y)$, Mathematics Technical Report, University of Ioanniana, 1992.
- [12] D. E. Blair, Th. koufogiorgos and R. Sharma, A classification of 3-dimensional contact metric manifolds with $Q\phi = \phi Q$, Kodai Mathetical Journal 13(1990), 391-401.
- [13] D. E. Blair and H. Chen, A classification of 3-dimensional contact metric manifolds with $Q\phi = \phi Q$, II, Bulletin of the Institute of Mathematics Academia Sinica 20(1992), 379-383.

UDAY CHAND DE
DEPARTMENT OF MATHEMATICS
UNIVERSITY OF KALYANI
KALYANI, 741235, WEST BENGAL, INDIA
e-mail address: uc_de@yahoo.com

ABOUL KALAM GAZI
DEPARTMENT OF MATHEMATICS
UNIVERSITY OF KALYANI
KALYANI, 741235, WEST BENGAL, INDIA

(Received November 26, 2006)
(Revised April 13, 2007)