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SOME QUESTIONS ON THE IDEAL CLASS GROUP OF IMAGINARY ABELIAN FIELDS

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Abstract. Let k be an imaginary quadratic field. Assume that the class number of k is exactly an odd prime number p, and p splits into two distinct primes in k. Then it is known that a prime ideal lying above p is not principal. In the present paper, we shall consider a question whether a similar result holds when the class number of k is $2p$. We also consider an analogous question for the case that k is an imaginary quartic abelian field.

1. Questions

At first, we shall introduce the following:

Theorem A. *Let k be an imaginary quadratic field. Assume that the class number of k is exactly an odd prime number p which splits into two distinct primes $\mathfrak{p}$ and $\mathfrak{p}'$ in k. Then $\mathfrak{p}$ is not principal.*

This result is mentioned in the proof of [2, Proposition 2.4]. In the present paper, we try to generalize the above result. In particular, we shall consider the following two questions:

Question 1.1. *Let k be an imaginary quadratic field. Assume that the class number of k is exactly $2p$ with an odd prime p which splits into two distinct primes $\mathfrak{p}$ and $\mathfrak{p}'$ in k. Then is $\mathfrak{p}^2$ not principal?*

Question 1.2. *Let k be an imaginary quartic abelian field. Assume that both of the class number and the relative class number of k are exactly an odd prime number p, and p splits completely in k. Let $\mathfrak{p}$ be a prime ideal of k lying above p. Then is $\mathfrak{p}$ not principal?*

The assertion in the above questions (and Theorem A) implies that all classes in the Sylow p-subgroup of the ideal class group contain a power of $\mathfrak{p}$. Of course, it is not satisfied in general (see Remark 2.9).

Question 1.1 has originally arisen from a question on Iwasawa theory. Under the assumption of Theorem A, it is known that both of the Iwasawa λ-

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and μ-invariants of the "$\mathfrak{p}$-ramified" $\mathbb{Z}_p$-extension of k are zero ([2, Proposition 2.4]). If Question 1.1 has a positive answer, we can obtain a similar result (see section 4).

The author expects that at least Question 1.1 always has a positive answer.

We shall consider Question 1.1 in section 2. We will show that Question 1.1 has a positive answer for many imaginary quadratic fields. Especially, if the absolute value of the discriminant of k is "small", then Question 1.1 has a positive answer for k (Corollary 2.5). Moreover, if a rational prime which is smaller than 1525 ramifies in k, then Question 1.1 has a positive answer for k (Corollary 2.8).

In section 3, we shall consider Question 1.2. We will show that if k is a bicyclic biquadratic field, then Question 1.2 has a positive answer for k (Proposition 3.1). However, little is known in the case that k is a cyclic quartic field.

In section 4, we will give an application to Iwasawa theory.

We will use the following notations throughout the present paper. We denote by $\left(\frac{\cdot}{\cdot}\right)$ the quadratic residue symbol. Let k be an algebraic number field. We denote by $\mathrm{Cl}(k)$ the ideal class group of k, $h(k)$ the class number of k, and $d(k)$ the absolute value of the discriminant of k. For a fractional ideal $\mathfrak{a}$ of k, we denote by $c(\mathfrak{a})$ the ideal class of k which contains $\mathfrak{a}$. For a finite extension k'/k of algebraic number fields, we denote by $N_{k'/k}$ the norm mapping from k' to k. If k is a CM-field, then we denote by k^+ the maximal real subfield of k and $h^-(k) = h(k)/h(k^+)$ the relative class number.

2. Consideration for Question 1.1

First, we shall briefly recall the proof of Theorem A which is stated in [2]. Let k be an imaginary quadratic field such that $h(k) = p$ with an odd prime p which splits into two distinct primes $\mathfrak{p}$ and $\mathfrak{p}'$ in k. Since $h(k)$ is odd, we may write $k = \mathbb{Q}(\sqrt{-q})$ with an odd prime number q which satisfies $q \equiv 3 \pmod 4$. If $\mathfrak{p}$ is principal, then we have an inequality $p \geq q/4$ by taking the norm of a generator of $\mathfrak{p}$ to $\mathbb{Q}$. However, we can see $p = h(k) < q/4$ by using Dirichlet's class number formula. It is a contradiction.

Let k be an imaginary quadratic field such that $h(k) = 2p$ with an odd prime p which splits into two distinct primes $\mathfrak{p}$ and $\mathfrak{p}'$ in k. We shall apply the above method for Question 1.1. Assume that $\mathfrak{p}^2$ is principal. Then $p^2 \geq d(k)/4$. Since $h(k) = 2p$, if $h(k) < \sqrt{d(k)}$ then Question 1.1 has a positive answer. However, the Brauer-Siegel theorem implies that

$$\frac{\log h(k)}{\log \sqrt{d(k)}} \to 1, \quad (d(k) \to \infty).$$

Hence it seems difficult to solve Question 1.1 by applying this method directly. If we remove the restriction on the class number, an imaginary quadratic field k which satisfies $h(k) > \sqrt{d(k)}$ really exists.

We begin a more detailed consideration for Question 1.1. Since $h(k) = 2p$, we may assume that k is one of the following:

(a) $k = \mathbb{Q}(\sqrt{-q})$ with an odd prime q satisfying $q \equiv 5 \pmod 8$,
(b) $k = \mathbb{Q}(\sqrt{-2q})$ with an odd prime q satisfying $q \equiv 3, 5 \pmod 8$,
(c) $k = \mathbb{Q}(\sqrt{-lq})$ with odd primes l, q satisfying $l \equiv 1, q \equiv 3 \pmod 4$ and $\left(\frac{l}{q}\right) = -1$.

Proposition 2.1. *Assume that k is of the form (a) or (b), that is, the prime 2 ramifies in k. Then Question 1.1 has a positive answer for k.*

Proof. We shall only show for the case that k is of the form (b). The rest case can be proven similarly.

Let $\mathfrak{p}$ be a prime ideal in k lying above p. Assume that $\mathfrak{p}^2$ is principal. We take a generator $a + b\sqrt{-2q}$ of $\mathfrak{p}^2$, where a and b are integers. Since p splits in k, we see $ab \neq 0$. We may assume that $a > 0$. By taking the norm of $a + b\sqrt{-2q}$ to $\mathbb{Q}$, we see $p^2 = a^2 + 4b^2q$. Hence $(p - a)(p + a) = 4b^2q$. Note that the right hand side is positive and then $p > a$. Since q is a prime number, q divides $p + a$ or $p - a$. If q divides $p - a$, then $p > p - a \geq q$. Otherwise, $2p > p + a \geq q$. Consequently we have the inequality $2p > q$.

On the other hand, by using a modified version of Dirichlet's class number formula (see, e.g., [8, Theorem 9.7.7]), we can see

$$2p = h(k) = \sum_{i=0}^{2q} \chi_k(i),$$

where χ_k is the Dirichlet character corresponding to k. Since $\chi_k(i) = 0$ for even i, the right hand side is less than or equal to q. Hence we see that $2p \leq q$. It is a contradiction. $\square$

In the rest of this section, we assume that k is of the form (c). In this case, Question 1.1 has not been solved yet. However, we can see that Question 1.1 has a positive answer for many cases.

Proposition 2.2. *Assume that k is of the form (c) and $h(k) = 2p$ with an odd prime p which splits in k. If $\left(\frac{l}{p}\right) = 1$, then Question 1.1 has a positive answer for k.*

Proof. Assume that $\left(\frac{l}{p}\right) = 1$. We note that $H := \mathbb{Q}(\sqrt{l}, \sqrt{-q})$ is the Hilbert 2-class field of k. By the assumption, the prime $\mathfrak{p}$ lying above p splits in H/k. Hence, the order of the ideal class $c(\mathfrak{p})$ containing $\mathfrak{p}$ is 1 or p. If the order of $c(\mathfrak{p})$ is 1, then we can see $p > lq/4$ by taking the norm of a generator of $\mathfrak{p}$ to $\mathbb{Q}$. Hence $h(k) = 2p > lq/2$. However, we can easily see that $h(k) < lq/2$ by using Dirichlet's class number formula. It is a contradiction. Then the order of $c(\mathfrak{p})$ is p, and this implies that $\mathfrak{p}^2$ is not principal. $\square$

We will show that if $d(k)$ is "small" then Question 1.1 has a positive answer. First, we shall prove the following lemma.

Lemma 2.3. *Assume that k is of the form (c) and $h(k) = 2p$ with an odd prime p which splits two distinct primes $\mathfrak{p}$ and $\mathfrak{p}'$ in k. Moreover, assume that $\left(\frac{l}{p}\right) = -1$. If $\mathfrak{p}^2$ is principal, then there are non-zero integers b' and c' such that*

$$p = \frac{(b')^2 l + (c')^2 q}{4}$$

and $b' \equiv c' \pmod 2$.

Proof. Under the assumptions, we can see that the order of $c(\mathfrak{p})$ is exactly 2 by using the argument given in the proof of Proposition 2.2. By Hilbert 94 or Tannaka-Terada's principal ideal theorem, we see that $\mathfrak{p}$ becomes principal in $\mathbb{Q}(\sqrt{l}, \sqrt{-q})$. We put $H = \mathbb{Q}(\sqrt{l}, \sqrt{-q})$ and denote by $\mathcal{O}_H$ the ring of algebraic integers in H. Let $\alpha \in \mathcal{O}_H$ be a generator of $\mathfrak{p}\mathcal{O}_H$. We can write

$$\alpha = \frac{a + b\sqrt{l} + c\sqrt{-q} + d\sqrt{-lq}}{4}$$

with some integers a, b, c, and d.

We note that $N_{H/\mathbb{Q}(\sqrt{l})}\alpha$ is a totally positive integer of $\mathbb{Q}(\sqrt{l})$. Since $N_{H/\mathbb{Q}(\sqrt{l})}\alpha$ generates the unique prime ideal of $\mathbb{Q}(\sqrt{l})$ lying above p, we can write $N_{H/\mathbb{Q}(\sqrt{l})}\alpha = p\varepsilon$ with a totally positive unit ε of $\mathbb{Q}(\sqrt{l})$. We note that the norm of the fundamental unit of $\mathbb{Q}(\sqrt{l})$ to $\mathbb{Q}$ is -1. From this, we can take α which satisfies $N_{H/\mathbb{Q}(\sqrt{l})}\alpha = p$ by multiplying some power of the fundamental unit. Hence we have the equation

$$N_{H/\mathbb{Q}(\sqrt{l})}\alpha = \frac{(a^2 + b^2 l + c^2 q + d^2 lq) + (2ab + 2cdq)\sqrt{l}}{16} = p.$$

This implies that $2ab + 2cdq = 0$, and then

$$p = \frac{a^2 + b^2 l + c^2 q + d^2 lq}{16}.$$

Next, we shall take the norm of α to $\mathbb{Q}(\sqrt{-q})$. If $q > 3$, we see $N_{H/\mathbb{Q}(\sqrt{-q})}\alpha = \pm p$. If $q = 3$, we can take α which satisfies $N_{H/\mathbb{Q}(\sqrt{l})}\alpha = p$

and $N_{H/\mathbb{Q}(\sqrt{-q})}\alpha = \pm p$ by multiplying some third root of unity. Hence we have the equation

$$N_{H/\mathbb{Q}(\sqrt{-q})}\alpha = \frac{(a^2 - b^2 l - c^2 q + d^2 lq) + (2ac - 2bdl)\sqrt{-q}}{16} = \pm p.$$

This implies that $2ac - 2bdl = 0$, and then

$$\pm p = \frac{a^2 - b^2 l - c^2 q + d^2 lq}{16}.$$

If $N_{H/\mathbb{Q}(\sqrt{-q})}\alpha = p$, then $b = c = 0$. In this case, we can see that both of a and d are even by writing α explicitly with the following integral basis:

$$\left\{1, \frac{1+\sqrt{l}}{2}, \frac{1+\sqrt{-q}}{2}, \frac{1+\sqrt{l}+\sqrt{-q}+\sqrt{-lq}}{4}\right\}.$$

Hence we can write $\alpha = (a' + d'\sqrt{-lq})/2$ with integers $a' = a/2$ and $d' = d/2$. This implies that $\mathfrak{p}$ is already principal in k. However, it contradicts to the fact that the order of $c(\mathfrak{p})$ is 2. Then we see $N_{H/\mathbb{Q}(\sqrt{-q})}\alpha = -p$, and hence $a = d = 0$. We can see that both of b and c are even and $b \equiv c \pmod 4$ by writing α explicitly with the above integral basis. Hence we can write $\alpha = (b'\sqrt{l} + c'\sqrt{-q})/2$ with integers $b' = b/2$ and $c' = c/2$ and they satisfy $b' \equiv c' \pmod 2$.

Since p is prime to l and q, we see that $b'c' \neq 0$. The lemma follows. $\square$

By using this, we can obtain the following:

Proposition 2.4. *Assume that k is of the form (c) and $h(k) = 2p$ with an odd prime p which splits in k. If $h(k) < (l+q)/2$, then Question 1.1 has a positive answer for k. Moreover, if $lq \equiv 7 \pmod 8$ and $h(k) < \min\{2l + 8q, 8l + 2q\}$, then Question 1.1 has a positive answer for k.*

Proof. Throughout the proof, we may suppose that $\left(\frac{l}{p}\right) = -1$ by Proposition 2.1.

Assume that $\mathfrak{p}^2$ is principal. Then by Lemma 2.3, we can write $p = ((b')^2 l + (c')^2 q)/4$ with some non-zero integers b' and c'. Hence we see $p \geq (l+q)/4$. Since $h(k) = 2p$, the former part follows.

Assume that $lq \equiv 7 \pmod 8$ and $\mathfrak{p}^2$ is principal. Similarly, we can write $4p = (b')^2 l + (c')^2 q$. Since p is odd, we see $4p \equiv 4 \pmod 8$. Recall that $b' \equiv c' \pmod 2$. If both of b' and c' are odd, then

$$(b')^2 l + (c')^2 q \equiv l + q \equiv 0 \pmod 8$$

from the assumption that $lq \equiv 7 \pmod 8$. Hence both of b' and c' must be even, and $p = (b'')^2 l + (c'')^2 q$ with $b'' = b'/2$ and $c'' = c'/2$. Moreover, either b'' or c'' must be even because p is odd. Hence we see that $p \geq \min\{l + 4q, 4l + q\}$. The latter part follows. □

Next, we shall quote the following:

Theorem B (Ramaré [13]). *Let χ be a primitive Dirichlet character of conductor f. Assume that $\chi(-1) = -1$ and f is odd. Then*

$$\left|\left(1 - \frac{\chi(2)}{2}\right) L(1,\chi)\right| \leq \frac{1}{4}\left(\log f + 5 - 2\log\frac{3}{2}\right),$$

where $L(s,\chi)$ is the Dirichlet L-function.

Let k be an imaginary quadratic field which is of the form (c). From the above theorem, we obtain the following upper bound:

$$h(k) \leq \frac{\sqrt{lq}}{2(2 - \chi_k(2))\pi}\left(\log lq + 5 - 2\log\frac{3}{2}\right) \tag{1}$$

by using the analytic class number formula, where χ_k is the Dirichlet character corresponding to k. We mentioned at the beginning of this section that if $h(k) < \sqrt{lq}$ then Question 1.1 has a positive answer. Moreover, if $lq \equiv 7 \pmod 8$ and $h(k) < 8\sqrt{lq}$ then Question 1.1 has a positive answer by Proposition 2.4. Connecting the above upper bound of $h(k)$, we obtain the following:

Corollary 2.5. *Assume that k is of the form (c) and $h(k) = 2p$ with an odd prime p which splits in k.*

- *If $lq \equiv 3 \pmod 8$ and*

$$lq < \frac{9}{4}\exp(6\pi - 5) = 2327920.965\ldots,$$

then Question 1.1 has a positive answer.

- *If $lq \equiv 7 \pmod 8$ and*

$$lq < \frac{9}{4}\exp(16\pi - 5) = 102501865638106235900.902\ldots,$$

then Question 1.1 has a positive answer.

Remark 2.6. By using the method which is given in the proof of Proposition 2.4, we can see that if $lq \equiv 3 \pmod 8$, $(l + q)/4$ is not a prime number, and $h(k) < \min\{(l + 9q)/2, (9l + q)/2\}$, then Question 1.1 has a positive answer for k. In particular, if $lq \equiv 3 \pmod 8$, $(l + q)/4$ is not a prime number, and

$$lq < \frac{9}{4}\exp(18\pi - 5) = 548888937249265038410446.318\ldots,$$

then Question 1.1 has a positive answer.

Next, we will show that if a "small" prime ramifies in k, then Question 1.1 has a positive answer. In the following, we use slightly different notations. We put $k = \mathbb{Q}(\sqrt{-rs})$ with rational primes r, s which satisfy $rs \equiv 3 \pmod 4$ and $\left(\frac{r}{s}\right) = -1$. Fix an odd prime s, and put

$$f_s(x) = \frac{x+s}{2} - \frac{\sqrt{xs}}{6\pi}\left(\log sx + 5 - 2\log\frac{3}{2}\right).$$

Assume that $h(k) = 2p$ with an odd prime p which splits in k. By Proposition 2.4 and (1), if $f_s(r) > 0$, then Question 1.1 has a positive answer for k.

We put $\kappa = (9/4)\exp(6\pi - 5)$. If $r < \kappa/s$, then $f_s(r) > 0$. Moreover, if $f_s'(\kappa/s) > 0$, then we see that $f_s(r) > 0$ for all r. We note that if

$$s < \frac{9\pi \exp\left(\frac{6\pi-5}{2}\right)}{6\pi + 2} = 1379.394\ldots,$$

then $f_s'(\kappa/s) > 0$. This implies:

Proposition 2.7. *We put $k = \mathbb{Q}(\sqrt{-rs})$ with rational primes r, s which satisfy $rs \equiv 3 \pmod 4$ and $\left(\frac{r}{s}\right) = -1$. Assume that $h(k) = 2p$ with an odd prime p which splits in k. If $s \leq 1379$, then Question 1.1 has a positive answer for k.*

Moreover, if we fix a prime $s > 1379$, then at most finitely many primes r satisfy $f_s(r) < 0$. Hence we can check whether Question 1.1 has a positive answer for all r. For example, we put $s = 1523$. There are only 23 primes r which satisfies $rs \geq \kappa$, $rs \equiv 3 \pmod 4$, $\left(\frac{r}{s}\right) = -1$, and $f_s(r) < 0$. These are 1609, 1621, 1637, 1693, 1733, 1741, 1777, 1801, 1861, 1913, 1933, 1973, 2053, 2069, 2089, 2113, 2153, 2161, 2237, 2269, 2281, 2297, 2309. All primes r in this list satisfy $rs < 10^{20}$. Hence by Corollary 2.5 and Remark 2.6, if $rs \equiv 7 \pmod 8$ or $(r+s)/4$ is not a prime, then Question 1.1 has a positive answer. From this, we see that the primes r for which we must check the class number of $\mathbb{Q}(\sqrt{-rs})$ are 1913 and 2153. We find that $h(\mathbb{Q}(\sqrt{-1523 \times 1913})) = 310$ and $h(\mathbb{Q}(\sqrt{-1523 \times 2153})) = 350$. Both fields do not satisfy the assumption of Question 1.1. Hence if $s = 1523$, then Question 1.1 has a positive answer for all r. Similarly, we checked that Question 1.1 has a positive answer if $1379 < s < 1525$. (We note that $\sqrt{\kappa} = 1525.752\ldots$.) As a consequence, we have the following:

Corollary 2.8. *Let k be an imaginary quadratic field. Assume that $h(k) = 2p$ with an odd prime p which splits in k. If a rational prime which is smaller than* 1525 *ramifies in k, then Question 1.1 has a positive answer for k.*

Remark 2.9. We can also consider the following question: *if $h(k) = 3p$ and p splits in k, then is the cube of a prime lying above p not principal?* However, this question has a negative answer. We put $k = \mathbb{Q}(\sqrt{-15391})$. Then $h(k) = 3 \times 31$ and the rational prime 31 splits in k. Let $\mathfrak{p}$ be a prime in k lying above 31. Then $\mathfrak{p}^3$ is principal because $31^3 = (120+\sqrt{-15391})(120-\sqrt{-15391})$.

3. Consideration for Question 1.2

In this section, let k be an imaginary quartic abelian field. In this case, k is a bicyclic biquadratic field or a cyclic quartic field. Assume that k satisfies $h(k) = h^-(k) = p$ with an odd prime p which splits completely in k.

First, we shall show the following:

Proposition 3.1. *If k is a bicyclic biquadratic field, then Question 1.2 has a positive answer.*

Proof. Since $h^-(k) = p$, there is a unique imaginary quadratic subfield k' of k which satisfies $h(k') = p$ or $h(k') = 2p$. Let $A(k)$ (resp. $A(k')$) be the Sylow p-subgroup of $\mathrm{Cl}(k)$ (resp. $\mathrm{Cl}(k')$). Let $\mathfrak{p}$ be a prime of k' lying above p. By Theorem A, Proposition 2.1, and Proposition 2.2, we can see that $A(k')$ is generated by $c(\mathfrak{p})$. Let $\mathfrak{P}$ be a prime of k lying above $\mathfrak{p}$. Since $\mathfrak{p}$ is not principal and $\mathfrak{p} = N_{k/k'}\mathfrak{P}$, it follows that $\mathfrak{P}$ is not principal. (We can also show this by using the following method. We denote by $\sigma_{\mathfrak{P}}$ (resp. $\sigma_{\mathfrak{p}}$) the Frobenius element of $\mathrm{Gal}(H(k)/k)$ (resp. $\mathrm{Gal}(H(k')/k')$) corresponding to $\mathfrak{P}$ (resp. $\mathfrak{p}$), where $H(k)$ (resp. $H(k')$) is the Hilbert class field of k (resp. k'). Since the restriction $\sigma_{\mathfrak{P}}|_{H(k')}$ coincides with $\sigma_{\mathfrak{p}}$ and the order of $\sigma_{\mathfrak{p}}$ is divisible by p, we see that the order of $\sigma_{\mathfrak{P}}$ is exactly p.) The proposition follows. $\square$

We assume that k is a cyclic quartic field. If $h^-(k)$ is an odd prime, then we can see that the conductor of k is an odd prime q by [3, Theorem 3']. Moreover, we see $q \equiv 5 \pmod 8$ because k is an imaginary cyclic quartic field. By specializing the method given in the proof of [9, Theorem D], we can obtain the following:

Lemma 3.2. *Let q be an odd prime which satisfies $q \equiv 5 \pmod 8$, and k the imaginary cyclic quartic field of conductor q. Let p be a rational prime which splits completely in k, and $\mathfrak{P}$ a prime of k lying above p. If $\mathfrak{P}$ is principal, then $p > q/8$.*

Proof. Let ε be the fundamental unit of $k^+ = \mathbb{Q}(\sqrt{q})$. Since $h(k^+)$ is odd, we can see that k/k^+ has a relative integral basis (see, e.g., [6]).

Assume that, $\mathfrak{P}$ is principal. We claim that

$$\mathfrak{P} = \left(\frac{\alpha + \beta\sqrt{-\varepsilon\sqrt{q}}}{2} \right)$$

with non-zero algebraic integers α, β in k^+. It is known that $k = \mathbb{Q}(\sqrt{-(q + b\sqrt{q})})$ with an even integer b (see, e.g., [10]). Since k/k^+ has a relative integral basis, we can write $k = k^+(\sqrt{-\varepsilon\sqrt{q}})$ by using [6, Lemma 2]. Moreover, we can apply Theorem 2 of [6]. From this theorem, every algebraic integer of k is written in the form $\frac{\alpha+\beta\sqrt{-\varepsilon\sqrt{q}}}{2}$ with algebraic integers α, β in k^+. Hence we can take an generator of $\mathfrak{P}$ written in the above form. Since p splits completely in k, both of α and β must be non-zero. The claim follows.

By taking the norm of the above generator to $\mathbb{Q}$, we obtain the following:

$$p = \frac{1}{16} \left\{ (\alpha\alpha^\sigma)^2 + (\beta\beta^\sigma)^2 q + \sqrt{q}((\alpha^\sigma)^2\beta^2\varepsilon - \alpha^2(\beta^\sigma)^2\varepsilon^\sigma) \right\},$$

where σ is the nontrivial automorphism of $\mathrm{Gal}(k^+/\mathbb{Q})$. We note that $\sqrt{q}((\alpha^\sigma)^2\beta^2\varepsilon - \alpha^2(\beta^\sigma)^2\varepsilon^\sigma)$ is a positive rational integer and divisible by q. Hence we see $p > (q + q)/16 = q/8$. $\square$

As a conclusion of the above lemma, if $h^-(k) < q/8$ then Question 1.2 has a positive answer. By using Theorem B, if $q > 5$ then we have the following upper bound:

$$h^-(k) \leq \frac{q}{40\pi^2} \left(\log q + 5 - 2\log\frac{3}{2} \right)^2 \tag{2}$$

(see also Corollary 11 of [11]). Unfortunately, the above lemma is not useful to deduce that Question 1.2 has a positive answer for all k. In fact, if we remove the restriction on the class number, there exist imaginary cyclic quartic fields k of conductor q which satisfy $h(k) > q/8$ (see [10]).

We note that if an odd prime p divides $h(k)$ and p does not divide $h(k^+)$, then the p-rank of the Sylow p-subgroup of $\mathrm{Cl}(k)$ is greater than or equal to the order of p in $(\mathbb{Z}/4\mathbb{Z})^\times$ (see, e.g., [14, Theorem 10.8]). Hence we see that if $h(k) = h^-(k) = p$, then $p \equiv 1 \pmod 4$. On the other hand, we can obtain the following result. It is also considered as an analog of Theorem A.

Proposition 3.3. *Let q be an odd prime which satisfies $q \equiv 5 \pmod 8$, and k the imaginary cyclic quartic field of conductor q. Assume that k satisfies $h(k) = h^-(k) = p^2$ with an odd prime $p \equiv 3 \pmod 4$ which splits completely in k. Then $\mathrm{Cl}(k)$ is generated by the classes containing a prime ideal lying above p.*

Proof. We may assume that $q \geq 13$. Let $\mathfrak{P}$ be a prime of k lying above p. By Lemma 3.2, we see that if $h(k) < q^2/64$, then $\mathfrak{P}$ is not principal. We note that

$$\frac{q}{40\pi^2}\left(\log q + 5 - 2\log\frac{3}{2}\right)^2 < \frac{q^2}{64}$$

holds if $q \geq 13$. Hence $\mathfrak{P}$ is not principal by (2).

Let D be the subgroup of $\mathrm{Cl}(k)$ generated by the classes containing a prime ideal lying above p. Since $\mathfrak{P}$ is not principal, D is a nontrivial p-group. We note that $\mathrm{Gal}(k/\mathbb{Q})$ acts on D. By using the same argument given in the proof of [14, Theorem 10.8], we can see that the p-rank of D is greater than or equal to 2. Since $\mathrm{Cl}(k) \cong (\mathbb{Z}/p\mathbb{Z})^2$, the assertion follows. □

4. Application to Iwasawa theory

Our questions relate to a question on the Iwasawa invariants of certain non-cyclotomic $\mathbb{Z}_p$-extensions. Let N be an algebraic number field and p a rational prime. For a $\mathbb{Z}_p$-extension M/N, we denote by $\lambda(M/N)$, $\mu(M/N)$, and $\nu(M/N)$ the Iwasawa λ-, μ-, and ν-invariants of M/N, respectively.

4.1. Let k be an imaginary quadratic field and p an odd prime which splits into two distinct primes $\mathfrak{p}$ and $\mathfrak{p}'$ in k. By class field theory, there exists a unique $\mathbb{Z}_p$-extension K/k which is unramified outside $\mathfrak{p}$. As an analog of Greenberg's conjecture, there is a question (cf. [2]): *are the invariants $\lambda(K/k)$ and $\mu(K/k)$ always zero?*.

For example, if $h(k)$ is not divisible by p, then $\lambda(K/k) = \mu(K/k) = 0$. Moreover, it is known that if $A(k)$ is generated by a power of $c(\mathfrak{p})$, then $\lambda(K/k) = \mu(K/k) = 0$ (see [12], [2]). Hence, if $h(k) = p$, then $\lambda(K/k) = \mu(K/k) = 0$ by Theorem A ([2]). Similarly, if $h(k) = 2p$ and Question 1.1 has a positive answer for k, then $\lambda(K/k) = \mu(K/k) = 0$.

Moreover, if $A(k)$ is generated by a power of $c(\mathfrak{p})$, then Greenberg's generalized conjecture (GGC) also holds for k and p ([12]). (For the detail of GGC, see [5].)

4.2. Next, let k be an imaginary quartic abelian field and p an odd prime which splits completely in k. Let $\mathfrak{p}_1$ and $\mathfrak{p}_2$ be the distinct primes in k^+ lying above p, and $\mathfrak{P}_1$ (resp. $\mathfrak{P}_2$) be a prime in k lying above $\mathfrak{p}_1$ (resp. $\mathfrak{p}_2$). By class field theory, there exists a unique $\mathbb{Z}_p$-extension K/k which is unramified outside $\mathfrak{P}_1, \mathfrak{P}_2$ (see, e.g., [7, Lemma 2.2]). Let k^+_∞ be the cyclotomic $\mathbb{Z}_p$-extension of k^+. In [7], it is shown that if $h(k)$ is not divisible by p and $\lambda(k^+_\infty/k^+) = \mu(k^+_\infty/k^+) = \nu(k^+_\infty/k^+) = 0$, then $\lambda(K/k) = \mu(K/k) = 0$. Moreover, Goto [4] independently obtained the following (the statement is modified by using the argument given in [7]):

Theorem C (Goto [4]). *If both of $\mathfrak{P}_1$ and $\mathfrak{P}_2$ are totally ramified, $A(k)$ is generated by a power of $c(\mathfrak{P}_1)$ and $c(\mathfrak{P}_2)$, and $\lambda(k_\infty^+/k^+) = \mu(k_\infty^+/k^+) = \nu(k_\infty^+/k^+) = 0$, then $\lambda(K/k) = \mu(K/k) = 0$.*

By using this, we can see the following:

Proposition 4.1. *Assume that $h(k) = p$ and Question 1.2 has a positive answer for k. If $\lambda(k_\infty^+/k^+) = \mu(k_\infty^+/k^+) = \nu(k_\infty^+/k^+) = 0$, then $\lambda(K/k) = \mu(K/k) = 0$.*

Proof. For a positive integer n, let k_n be the n-th layer of K/k. By using the argument given in the proof of [7, Proposition 3.2], we can see that both of $\mathfrak{P}_1$ and $\mathfrak{P}_2$ are totally ramified or unramified in k_1/k. If both of $\mathfrak{P}_1$ and $\mathfrak{P}_2$ are totally ramified, then the assertion follows from Theorem C. Otherwise, we can see that the order of $A(k_n)^{\mathrm{Gal}(k_n/k)}$ is 1 for $n \geq 1$ by using the genus formula. Hence $A(k_n)$ is trivial for all $n \geq 1$. □

Proposition 4.2. *Assume that k is a cyclic quartic field, $h(k) = h^-(k) = p^2$, and $p \equiv 3 \pmod 4$. If both of $\mathfrak{P}_1$ and $\mathfrak{P}_2$ are totally ramified and $\lambda(k_\infty^+/k^+) = \mu(k_\infty^+/k^+) = \nu(k_\infty^+/k^+) = 0$, then $\lambda(K/k) = \mu(K/k) = 0$.*

Proof. Let D be the subgroup of $\mathrm{Cl}(k)$ generated by the classes containing a prime ideal lying above p. By Proposition 3.3, we see that $\mathrm{Cl}(k) = D$. Since $h(k^+) = 1$, D is actually generated by $c(\mathfrak{P}_1)$ and $c(\mathfrak{P}_2)$. Hence we can apply Theorem C. □

By using the argument given in [7] (with some modifications), we can see that if k satisfies the assumption of Proposition 4.1 or Proposition 4.2, then GGC for k and p holds.

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