Math. J. Okayama Univ. **49** (2007), 139–147

GENERALIZED DERIVATIONS WITH COMMUTATIVITY AND ANTI-COMMUTATIVITY CONDITIONS

Howard E. BELL and Nadeem-Ur REHMAN

Abstract. Let R be a prime ring with 1, with $\mathrm{char}(R) \neq 2$; and let $F : R \longrightarrow R$ be a generalized derivation. We determine when one of the following holds for all $x, y \in R$: (i) $[F(x), F(y)] = 0$; (ii) $F(x) \circ F(y) = 0$; (iii) $F(x) \circ F(y) = x \circ y$.

1. Introduction

Let R be an associative ring with center $Z = Z(R)$. For each $x, y \in R$ denote the commutator $xy - yx$ by $[x, y]$ and the anti-commutator $xy + yx$ by $x \circ y$. Recall that a ring R is prime if for any $a, b \in R$, $aRb = \{0\}$ implies that $a = 0$ or $b = 0$. An additive mapping $d : R \longrightarrow R$ is called a derivation if $d(xy) = d(x)y + xd(y)$ for all $x, y \in R$. In particular, for fixed $a \in R$, the mapping $I_a : R \longrightarrow R$ given by $I_a(x) = [x, a]$ is a derivation called an inner derivation.

An additive function $F : R \longrightarrow R$ is called a generalized inner derivation if $F(x) = ax + xb$ for fixed $a, b \in R$. For such a mapping F, it is easy to see that

$$F(xy) = F(x)y + x[y, b] = F(x)y + xI_b(y) \text{ for all } x, y \in R.$$

This observation leads to the following definition, given in [6]: an additive mapping $F : R \longrightarrow R$ is called a generalized derivation with associated derivation d if

$$F(xy) = F(x)y + xd(y) \text{ for all } x, y \in R.$$

Familiar examples of generalized derivations are derivations and generalized inner derivations, and the later include left multipliers and right multipliers. Since the sum of two generalized derivations is a generalized derivation, every map of the form $F(x) = cx + d(x)$, where c is a fixed element of R and d is a derivation, is a generalized derivation; and if R has 1, all generalized derivations have this form.

Our primary purpose is to determine when a generalized derivation F satisfies $[F(x), F(y)] = 0$ for all $x, y \in R$, where R is a prime ring with 1 for

---

Mathematics Subject Classification. 16W25, 16N60, 16U80.

Key words and phrases. prime rings, generalized derivations and commutativity.

The first author was supported by the Natural Sciences and Engineering Research Council of Canada, Grant 3961.

which char$(R) \neq 2$; and we also study the conditions $F(x) \circ F(y) = 0$ and $F(x) \circ F(y) = x \circ y$. Our results extend known results for derivations.

2. Preliminary results

We shall use without explicit mention the following basic identities:

$$[xy, z] = x[y, z] + [x, z]y$$
$$[x, yz] = y[x, z] + [x, y]z$$
$$x \circ (yz) = (x \circ y)z - y[x, z] = y(x \circ z) + [x, y]z$$
$$(xy) \circ z = x(y \circ z) - [x, z]y = (x \circ z)y + x[y, z]$$

We shall also use the elementary fact that if R is prime and d is a nonzero derivation, then $xd(R) = \{0\}$ or $d(R)x = \{0\}$ implies $x = 0$.

We shall require several lemmas, all but two of which are known.

Lemma 2.1. Let R be a prime ring and d a nonzero derivation of R.

(a) ([4, Theorem 2]). If char$(R) \neq 2$ and $[d(x), d(y)] = 0$ for all $x, y \in R$, then R is commutative.

(b) ([5, Theorem 2]). If char$(R) \neq 2$ and $[a, d(R)] = \{0\}$, then $a \in Z$.

Lemma 2.2 ([8, Corollary 3.2]). Let R be a prime ring. If R admits a nonzero generalized derivation F with associated derivation $d \neq 0$, such that $[F(x), x] = 0$ for all $x \in R$, then R is commutative.

Lemma 2.3 ([1, Theorem 4.3]). Let R be a prime ring with char$(R) \neq 2$, and let I be a nonzero ideal of R. If R admits a nonzero derivation d such that $d(x) \circ d(y) = 0$ for all $x, y \in I$, then R is commutative.

Lemma 2.4. Let R be a prime ring with 1. Let F be a generalized derivation with associated derivation $d \neq 0$, such that $d(F(x)) = 0$ for all $x \in R$; and let $c = F(1)$. Then $cd(x)+d(x)c = 0$ for all $x \in R$. Moreover, if char$(R) \neq 2$, $c^2 \in Z$; and if $c \in Z$, then $c = 0$ and $F = d$.

Proof. We have

$$d(F(x)) = 0 \text{ for all } x \in R. \tag{2.1}$$

Replacing x by xy in (2.1) and using (2.1), we get

$$F(x)d(y) + d(x)d(y) + xd^2(y) = 0 \quad \text{for all } x, y \in R. \tag{2.2}$$

Applying d again on (2.2) and using (2.1), we have

$$\begin{aligned} F(x)d^2(y) + d^2(x)d(y) + d(x)d^2(y) + d(x)d^2(y) \\ +xd^3(y) = 0 \qquad \text{for all } x, y \in R. \end{aligned} \tag{2.3}$$

But replacing y by $d(y)$ in (2.2), we get

$$F(x)d^2(y) + d(x)d^2(y) + xd^3(y) = 0; \tag{2.4}$$

and combining (2.3) and (2.4), we find that

$$d(x)d^2(y) + d^2(x)d(y) = 0 \text{ for all } x, y \in R. \tag{2.5}$$

Since R has 1,

$$F(x) = F(1x) = F(1)x + 1d(x) = cx + d(x) \text{ for all } x \in R. \tag{2.6}$$

Using the hypothesis that $d(F(R)) = \{0\}$, we get $d(c) = 0$; and by applying d to (2.6), we obtain

$$cd(x) + d^2(x) = 0 \text{ for all } x \in R. \tag{2.7}$$

Using this fact to substitute for $d^2(x)$ and $d^2(y)$ in (2.5), we get

$$d(x)(-cd(y)) + (-cd(x))d(y) = 0;$$

hence

$$(d(x)c + cd(x))d(y) = 0 \text{ for all } x, y \in R.$$

Thus,

$$cd(x) + d(x)c = 0 \text{ for all } x \in R. \tag{2.8}$$

Suppose now that $\mathrm{char}(R) \neq 2$. It follows from (2.8) that $[c^2, d(R)] = \{0\}$, so that $c^2 \in Z$ by Lemma 2.1(b). If $c \in Z$, then (2.8) yields $2cd(R) = \{0\}$; hence $2c = 0 = c$ and $F = d$. $\square$

Henceforth, except in our final section, R will always be a prime ring with extended centroid C and central closure $S = RC$. (For definitions and basic properties of C and S, see [7, Section 2] or [3, Chapter 1, Section 3]). Note that if R has 1, then C is the center $Z(S)$ of S.

Lemma 2.5 [2, Theorem 2.1]. Let R be a prime ring and let d, g, h be derivations on R for which there exist $a, b \in R \backslash Z$ such that $d(x) = ag(x) + h(x)b$ for all $x \in R$. Then there exists $\lambda \in C$ such that $d(x) = [\lambda ab, x]$, $g(x) = [\lambda b, x]$ and $h(x) = [\lambda a, x]$ for all $x \in R$.

Lemma 2.6. Let R be prime ring with 1. let F be a generalized derivation with associated derivation $d \neq 0$, such that $d(F(x)) = 0$ for all $x \in R$; and suppose $c = F(1) \notin Z$. Then

(*i*) there exists $\lambda \in C$ such that $d(x) = [\lambda c, x]$ for all $x \in R$;
(*ii*) F can be extended to a generalized derivation $\hat{F}$ on S;
(*iii*) if $[F(x), F(y)] = 0$ for all $x, y \in R$, then $[\hat{F}(x), \hat{F}(y)] = 0$ for all $x, y \in S$.

Proof. (i) By Lemma 2.4, $cd(x) + d(x)c = 0$ for all $x \in R$; hence by Lemma 2.5, there exists $\lambda \in C$ such that $d(x) = [\lambda c, x]$ for all $x \in R$.

(ii) Define $\hat{F}(x) = cx + [\lambda c, x]$ for all $x \in S$.

(iii) Let $x \in S$, and write $x = \sum_{i=1}^{n} r_i u_i$, where $r_i \in R$, $u_i \in C$. Then using the fact that $u_i \in Z(S)$, we get $\hat{F}(x) = \sum cr_i u_i + \sum [\lambda c, r_i u_i] = \sum u_i(cr_i + [\lambda c, r_i]) = \sum u_i F(r_i)$; and ($iii$) follows at once. $\square$

Lemma 2.7 [7, Theorem 3]. If R is prime and S satisfies a generalized polynomial identity over C, then S is primitive.

3. The condition $[F(x), F(y)] = 0$

In view of Lemma 2.1(a), it is natural to conjecture that if a prime ring R of characteristic different from 2 admits a nonzero generalized derivation F such that $[F(x), F(y)] = 0$ for all $x, y \in R$, then R is commutative. However, this is not the case.

Example 3.1. Let R be either the ring H of real quaternions or the subring K of H consisting of all elements $a + bi + cj + dk$ where a, b, c, d are integers. Define $F(x) = ix + xi$ for all $x \in R$. Then R is a noncommutative prime ring, and F is a generalized derivation such that $[F(x), F(y)] = 0$ for all $x, y \in R$.

Example 3.2. Let K be any field, and let R be the ring $M_2(K)$ of 2×2 matrices over K. Define $F(x) = cx + xc$, where c is either $e_{11} - e_{22}$ or e_{12}. It is easy to verify that, in either case, $[F(x), F(y)] = 0$ for all $x, y \in R$.

Example 3.3. Let R be the noncommutative prime ring $M_2(\mathbb{Z})$; and for arbitrary $x = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in R$, define $d(x) = \begin{bmatrix} 0 & -b \\ c & 0 \end{bmatrix}$. Define $F : R \longrightarrow R$ by $F(x) = (e_{11} - e_{22})x + d(x)$. It is easily verified that d is a derivation on R, so that F is a generalized derivation; and since $F\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = \begin{bmatrix} a & 0 \\ 0 & -d \end{bmatrix}$, $[F(x), F(y)] = 0$ for all $x, y \in R$. Note that F is the restriction to R of the map $\hat{F} : M_2(Q) \longrightarrow M_2(Q)$ given by $\hat{F}(x) = cx + xc$, where $c = \frac{1}{2}e_{11} - \frac{1}{2}e_{22}$.

In fact, for 2-torsion free prime rings with 1, these examples illustrate all possibilities, as our next (and principal) theorem shows.

Theorem 3.4. Let R be a prime ring with 1 such that $\text{char}(R) \neq 2$. If R admits a nonzero generalized derivation F such that $[F(x), F(y)] = 0$ for all $x, y \in R$, then one of the following holds:

(i) R is commutative;

(ii) R is a noncommutative subring of a division ring Δ, and there exists $\delta \in \Delta$ such that $F(x) = \delta x + x\delta$ for all $x \in R$;

(iii) R is a noncommutative subring of a 2×2 total matrix ring M over a field, and there exists $m \in M$ such that $F(x) = mx + xm$ for all $x \in R$.

The following lemma is the first step in the proof.

Lemma 3.5. Let R be a noncommutative prime ring with 1 and with $\text{char}(R) \neq 2$. Let F be a nonzero generalized derivation with associated derivation d, such that $[F(x), F(y)] = 0$ for all $x, y \in R$. Then

(i) $d(F(x)) = 0$ for all $x \in R$;

(ii) $c = F(1) \notin Z$ and $c^2 \in Z$;

(iii) S is primitive and there exists $s \in S$ such that $s^2 \in Z(S)$ and $\hat{F}(x) = sx + xs$ for all $x \in S$.

Proof. (i) If $d = 0$, then $c \neq 0$ and $[cx, cy] = 0$ for all $x, y \in R$; thus cR is a nonzero commutative right ideal. But a noncommutative prime ring cannot have such a right ideal, hence $d \neq 0$.

We have

$$[F(x), F(y)] = 0 \text{ for all } x, y \in R. \tag{3.1}$$

Replacing y by yz in (3.1) and using (3.1), we get

$$F(y)[F(x), z] + y[F(x), d(z)] + [F(x), y]d(z) = 0 \text{ for all } x, y, z \in R. \tag{3.2}$$

Now replacing y by ry in (3.2) gives

$$F(r)y[F(x), z] + rd(y)[F(x), z] + ry[F(x), d(z)] + r[F(x), y]d(z)$$
$$+[F(x), r]yd(z) = 0 \qquad \text{for all } x, y, z, r \in R;$$

and hence application of (3.2) yields

$$F(r)y[F(x), z] + rd(y)[F(x), z] + [F(x), r]yd(z) - rF(y)[F(x), z] = 0.$$

Letting $z = F(x)$, we obtain $[F(x), r]yd(F(x)) = 0$ for all $x, y, r \in R$ -i.e.

$$[F(x), r]Rd(F(x)) = \{0\} \text{ for all } x, r \in R.$$

Since R is prime, for each $x \in R$, either $F(x) \in Z$ or $d(F(x)) = 0$. The sets of $x \in R$ for which these alternatives hold are additive subgroups whose union is R; therefore, either $F(R) \subseteq Z$ or $d(F(x)) = 0$ for all $x \in R$. But by Lemma 2.2, $F(R) \subseteq Z$ would force R to be a commutative; hence

$d(F(R)) = \{0\}$.

(*ii*) Since R is not commutative, it follows from Lemmas 2.4 and 2.1 (a) that $c \notin Z$ and $c^2 \in Z$.

(*iii*) By Lemma 2.4 we now have $cd(x) + d(x)c = 0$ for all $x \in R$; and since $c \notin Z$, it follows by Lemma 2.6 that there exists $\lambda \in C$ such that $d(x) = [\lambda c, x] = \lambda[c, x]$ for all $x \in R$. Therefore $\hat{F}(x) = cx + \lambda[c, x]$ for all $x \in S$. Since $\hat{F}(1) = c$ and $[\hat{F}(x), \hat{F}(y)] = 0$ for all $x, y \in S$, we have

$$c(cx + \lambda[c, x]) = (cx + \lambda[c, x])c \text{ for all } x \in S. \tag{3.3}$$

Now $c^2 \in Z(S)$ by Lemma 2.4, so (3.3) can be written as

$$(2\lambda + 1)(c^2x - cxc) = 0 \text{ for all } x \in S,$$

from which it follows that

$$2\lambda + 1 = 0 \text{ or } c^2x - cxc = 0 \text{ for all } x \in S. \tag{3.4}$$

Since $c^2 \in Z(S)$, either c is regular or $c^2 = 0$. In the first case we see from (3.4) that $2\lambda + 1 \neq 0$ contradicts the fact that $c \notin Z$; in the second case $2\lambda + 1 \neq 0$ yields $c = 0$, contrary to our observation that $c \notin Z$. Therefore $\lambda = -\frac{1}{2}$ and for each $x \in S$, $\hat{F}(x) = cx - \frac{1}{2}(cx - xc) = sx + xs$, where $s = \frac{c}{2}$. Recalling that $[\hat{F}(x), \hat{F}(y)] = 0$ for all $x, y \in S$, we see that S satisfies the generalized polynomial identity $\frac{1}{4}(cx + xc)(cy + yc) = \frac{1}{4}(cy + yc)(cx + xc)$ over C; hence S is primitive by Lemma 2.7. $\square$

Proof of Theorem 3.4. In view of Lemma 3.5 and Jacobson's density theorem, we may assume that R is a noncommutative dense ring of linear transformations on a vector space V over a division ring Δ, and that there exists $k \in R \setminus \{0\}$ such that $k^2 \in Z$ and $F(x) = kx + xk$ for all $x \in R$. We need only show that $\dim(V) \leq 2$ and that in the case $\dim(V) = 2$, Δ is a field. For any subset $W \subseteq V$, we denote by $< W >$ the subspace generated by W.

By a standard argument it follows that if $\dim(V) > 1$ and $k(u) \in < u >$ for each $u \in V$, then there exists $\beta \in \Delta \setminus \{0\}$ such that $k(u) = \beta u$ for all $u \in V$. But in this case we have $(kx + xk)(ky + yk)(u) = 4\beta^2 xy(u)$ and $(ky+yk)(kx+xk)(u) = 4\beta^2 yx(u)$ for all $u \in V$, contradicting our hypothesis that R is not commutative.

Assume that $\dim(V) \geq 3$, and choose $u \in V$ such that $k(u) = v \notin < u >$. Since $k^2 \in Z(R)$, there exists $\alpha \in Z(\Delta)$ such that $k^2(w) = \alpha w$ for all $w \in V$; therefore $k(v) = \alpha u$.

Suppose that $k(V) \not\subseteq < u, v >$, in which case there exists $z \in V \setminus < u, v >$ and $w \in V$ such that $k(w) = z$. Then $\{u, v, w\}$ is a linearly independent subset of V and there exist $a, b \in R$ such that $a(u) = v, a(v) = w, a(w) = u, b(u) = u, b(v) = 0$ and $b(w) = 0$. It is readily verified that the condition $(ka+ak)(kb+bk)(u) = (kb+bk)(ka+ak)(u)$ implies that $b(z) = z$. It follows that if $z \in < u, v, w >$, then $z = b(z) \in < u >$, contrary to the fact that $z \notin < u, v >$; therefore $\{u, v, w, z\}$ is a linearly independent subset of V and there exist $a', b' \in R$ such that $(a'(u), a'(v), a'(w), a'(z)) = (v, w, u, 0)$ and $(b'(u), b'(v), b'(w), b'(z)) = (u, 0, 0, 0)$. But the argument given for a and b shows that this is incompatible with the requirement that $[F(a'), F(b')] = 0$; therefore we must have $k(V) \subseteq < u, v >$.

Since $\dim(V) \geq 3$, $\ker(k) \neq \{0\}$ and there exists $t \in V \setminus \{0\}$ such that $k(t) = 0$. Therefore $k^2(t) = \alpha t = 0$, so that $\alpha = 0$, $k(v) = 0$ and $k^2 = 0$. Thus, if $q \in V$ and $k(q) = \gamma u + \delta(v)$, then $0 = k^2(q) = \gamma v$ so $\gamma = 0$. Hence $k(V) \subseteq < v >$ and $\ker(k)$ has dimension at least 2; and since $< u, v > \neq \ker(k)$, there exists $y \in \ker(k) \setminus < u, v >$. Choosing $a, b \in R$ such that $(a(u), a(v), a(y)) = (v, y, u)$ and $(b(u), b(v), b(y)) = (u, u, y)$, we get $(kb + bk)(ka + ak)(u) = 0$ and $(ka + ak)(kb + bk)(u) = y -a$ contradiction. Therefore $\dim(V) < 3$ as required.

Finally, assume $\dim(V) = 2$. As before, we have linearly independent u, v such that $k(u) = v$. Let $\beta, \gamma \in \Delta$ and consider $a, b \in R$ such that $(a(u), a(v)) = (0, \beta u)$ and $(b(u), b(v)) = (0, \gamma u)$. Then $(ka + ak)(u) = \beta u$ and $(kb+bk)(u) = \gamma u$, and the condition $[F(a), F(b)] = 0$ gives $\beta\gamma u = \gamma\beta u$, so that $\beta\gamma = \gamma\beta$. Thus Δ is a field. □

4. Anti-commutativity conditions

In our final section we present some more elementary results, which involve anti-commutativity hypotheses.

Theorem 4.1. Let R be a prime ring with 1 and $\mathrm{char}(R) \neq 2$. If F is a generalized derivation on R such that $F(x) \circ F(y) = 0$ for all $x, y \in R$, then $F = 0$.

Proof. Note that if R is commutative, it is a domain; and the condition $F(x) \circ F(y) = 0$ is just $2F(x)F(y) = 0$. Taking $y = x$ then shows that $F(x) = 0$ for all $x \in R$.

Assume that $F \neq 0$. Then R is not commutative; and since $F(1) \circ F(1) = 0$, we have $c^2 = 0$. Note that we cannot have $d = 0$, for in that case $F(1) \circ F(x) = 0$ becomes $cxc = 0$ for all $x \in R$, which implies that $c = 0$ and hence $F = 0$.

We now have $d \neq 0$ and

$$F(x) \circ F(y) = 0 \text{ for all } x, y \in R. \tag{4.1}$$

Replacing y by yz in (4.1) and using (4.1), we get

$$(F(x) \circ y)d(z) - F(y)[F(x), z] - y[F(x), d(z)] = 0 \text{ for all } x, y, z \in R. \tag{4.2}$$

Now replacing z by $zF(x)$ in (4.2) and using (4.2), we obtain

$$(F(x) \circ y)zd(F(x)) - yz[F(x), d(F(x))] - y[F(x), z]d(F(x)) = 0. \tag{4.3}$$

Finally, replacing y by ry in (4.3) and using (4.3), we conclude that

$$[F(x), r]yRd(F(x)) = \{0\} \text{ for all } x, y, r \in R.$$

Again, invoking the primeness of R, we learn that $F(R) \subseteq Z$ or $d(F(x)) = 0$ for all $x \in R$. But Lemma 2.2 implies that if $F(R) \subseteq Z$, then R is commutative, contrary to our assumption that $F \neq 0$; therefore $d(F(x)) = 0$ for all $x \in R$, and by Lemma 2.4 $cd(x) + d(x)c = 0$ for all $x \in R$. The condition that $F(1) \circ F(x) = 0 = c(cx + d(x)) + (cx + d(x))c$ reduces to $cxc = 0$; hence $c = 0$ and R is commutative by Lemma 2.3, so we have again contradicted our assumption that $F \neq 0$. Therefore, $F = 0$. □

Theorem 4.2. Let R be a 2-torsion free ring with 1. If F is a generalized derivation such that $F(x) \circ F(y) = x \circ y$ for all $x, y \in R$, then there exists c in Z such that $c^2 = 1$ and $F(x) = cx$ for all x in R. Thus, if R is prime, F is the identity map or its negative.

Proof. Since $F(1) \circ F(1) = 1 \circ 1$, we have $c^2 = 1$. Thus, the condition $F(x) \circ F(1) = x \circ 1$ reduces to

$$cxc + d(x)c + cd(x) = x. \tag{4.4}$$

Postmultiplying and premultiplying this equation by c and comparing the results yields $d(x) + cd(x)c = 0$; and premultiplying by c gives

$$cd(x) + d(x)c = 0. \tag{4.5}$$

It now follows from (4.4) that $cx = xc$ for all x in R, so that c is in Z; and since c is invertible, (4.5) shows that $d = 0$ and hence $F(x) = cx$ for all x in R. □

A similar proof yields our final theorem.

Theorem 4.3. Let R be a 2-torsion free ring with 1. If F is a generalized derivation such that $F(x) \circ F(y) + x \circ y = 0$ for all $x, y \in R$, then there exists c in Z such that $c^2 = -1$ and $F(x) = cx$ for all x in R.

References

[1] M. Ashraf and N. Rehman, *On commutativity of rings with derivations*, Results Math. **42** (2002) 3–8.
[2] M. Brešar, *Centralizing mappings and derivations in prime rings*, J. Algebra **156** (1993), 385–394.
[3] I. N. Herstein, *Rings with involution*, Univ. Chicago Press, 1976.
[4] I. N. Herstein, *A note on derivations*, Canad. Math. Bull. **21** (1978), 369–370.
[5] I. N. Herstein, *A note on derivations II*, Canad. Math. Bull. **22** (1979), 509–511.
[6] B. Hvala, *Generalized derivations in rings*, Comm. Algebra **26** (1998), 1149–1166.
[7] W. S. Martindale, *Prime rings satisfying a generalized polynomial identity*, J. Algebra **12** (1969), 576–584.
[8] N. Rehman, *On commutativity of rings with generalized derivations*, Math. J. Okayama Univ. **44** (2002) 43–49.

Howard E. BELL
Department of Mathematics, Brock University, St. Catharines, Ontario, Canada L2S 3A1
e-mail address: hbell@brocku.ca

Nadeem-ur REHMAN
Department of Mathematics, Aligarh Muslim University, Aligarh - 202002(U.P.), India
e-mail address: rehman100@gmail.com

(*Received May 3, 2006*)