GENERALIZED DERIVATIONS WITH COMMUTATIVITY AND ANTI-COMMUTATIVITY CONDITIONS

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Abstract. Let $R$ be a prime ring with 1, with char($R$) $\neq 2$; and let $F : R \longrightarrow R$ be a generalized derivation. We determine when one of the following holds for all $x, y \in R$: (i) $[F(x), F(y)] = 0$; (ii) $F(x) \circ F(y) = 0$; (iii) $F(x) \circ F(y) = x \circ y$.

1. Introduction

Let $R$ be an associative ring with center $Z = Z(R)$. For each $x, y \in R$ denote the commutator $xy - yx$ by $[x, y]$ and the anti-commutator $xy + yx$ by $x \circ y$. Recall that a ring $R$ is prime if for any $a, b \in R$, $aRb = \{0\}$ implies that $a = 0$ or $b = 0$. An additive mapping $d : R \longrightarrow R$ is called a derivation if $d(xy) = d(x)y + xd(y)$ for all $x, y \in R$. In particular, for fixed $a \in R$, the mapping $I_a : R \longrightarrow R$ given by $I_a(x) = [x, a]$ is a derivation called an inner derivation.

An additive function $F : R \longrightarrow R$ is called a generalized inner derivation if $F(x) = ax + xb$ for fixed $a, b \in R$. For such a mapping $F$, it is easy to see that

$$F(xy) = F(x)y + x[y, b] = F(x)y + xI_b(y) \text{ for all } x, y \in R.$$ 

This observation leads to the following definition, given in [6]: an additive mapping $F : R \longrightarrow R$ is called a generalized derivation with associated derivation $d$ if

$$F(xy) = F(x)y + xd(y) \text{ for all } x, y \in R.$$ 

Familiar examples of generalized derivations are derivations and generalized inner derivations, and the later include left multipliers and right multipliers. Since the sum of two generalized derivations is a generalized derivation, every map of the form $F(x) = cx + d(x)$, where $c$ is a fixed element of $R$ and $d$ is a derivation, is a generalized derivation; and if $R$ has 1, all generalized derivations have this form.

Our primary purpose is to determine when a generalized derivation $F$ satisfies $[F(x), F(y)] = 0$ for all $x, y \in R$, where $R$ is a prime ring with 1 for

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which char($R$) $\neq 2$; and we also study the conditions $F(x) \circ F(y) = 0$ and $F(x) \circ F(y) = x \circ y$. Our results extend known results for derivations.

2. Preliminary results

We shall use without explicit mention the following basic identities:

$$[xy, z] = x[y, z] + [x, z]y$$
$$[x, yz] = y[x, z] + [x, y]z$$
$$x \circ (yz) = (x \circ y)z - y[x, z] = y(x \circ z) + [x, y]z$$
$$ (xy) \circ z = x(y \circ z) - [x, z]y = (x \circ z)y + x[y, z]$$

We shall also use the elementary fact that if $R$ is prime and $d$ is a nonzero derivation, then $xd(R) = \{0\}$ or $d(R)x = \{0\}$ implies $x = 0$.

We shall require several lemmas, all but two of which are known.

**Lemma 2.1.** Let $R$ be a prime ring and $d$ a nonzero derivation of $R$.

(a) ([4, Theorem 2]). If char$(R) \neq 2$ and $[d(x), d(y)] = 0$ for all $x, y \in R$, then $R$ is commutative.

(b) ([5, Theorem 2]). If char$(R) \neq 2$ and $[a, d(R)] = \{0\}$, then $a \in Z$.

**Lemma 2.2** ([8, Corollary 3.2]). Let $R$ be a prime ring. If $R$ admits a nonzero generalized derivation $F$ with associated derivation $d \neq 0$, such that $[F(x), x] = 0$ for all $x \in R$, then $R$ is commutative.

**Lemma 2.3** ([1, Theorem 4.3]). Let $R$ be a prime ring with char$(R) \neq 2$, and let $I$ be a nonzero ideal of $R$. If $R$ admits a nonzero derivation $d$ such that $d(x) \circ d(y) = 0$ for all $x, y \in I$, then $R$ is commutative.

**Lemma 2.4.** Let $R$ be a prime ring with 1. Let $F$ be a generalized derivation with associated derivation $d \neq 0$, such that $d(F(x)) = 0$ for all $x \in R$; and let $c = F(1)$. Then $cd(x) + d(x)c = 0$ for all $x \in R$. Moreover, if char$(R) \neq 2$, $c^2 \in Z$; and if $c \in Z$, then $c = 0$ and $F = d$.

**Proof.** We have

$$(2.1) \quad d(F(x)) = 0 \quad \text{for all} \quad x \in R.$$  

Replacing $x$ by $xy$ in (2.1) and using (2.1), we get

$$(2.2) \quad F(x)d(y) + d(x)d(y) + xd^2(y) = 0 \quad \text{for all} \quad x, y \in R.$$  

Applying $d$ again on (2.2) and using (2.1), we have

$$(2.3) \quad F(x)d^2(y) + d^2(x)d(y) + d(x)d^2(y) + d(x)d^2(y) + xd^3(y) = 0 \quad \text{for all} \quad x, y \in R.$$
But replacing $y$ by $d(y)$ in (2.2), we get

$$F(x)d^2(y) + d(x)d^2(y) + xd^3(y) = 0;$$

and combining (2.3) and (2.4), we find that

$$d(x)d^2(y) + d^2(x)d(y) = 0 \text{ for all } x, y \in R.$$

Since $R$ has $1$,

$$F(x) = F(1x) = F(1)x + 1d(x) = cx + d(x) \text{ for all } x \in R.$$

Using the hypothesis that $d(F(R)) = \{0\}$, we get $d(c) = 0$; and by applying $d$ to (2.6), we obtain

$$cd(x) + d^2(x) = 0 \text{ for all } x \in R.$$

Using this fact to substitute for $d^2(x)$ and $d^2(y)$ in (2.5), we get

$$d(x)(-cd(y)) + (-cd(x))d(y) = 0;$$

hence

$$(d(x)c + cd(x))d(y) = 0 \text{ for all } x, y \in R.$$

Thus,

$$cd(x) + d(x)c = 0 \text{ for all } x \in R.$$

Suppose now that char($R$) $\neq 2$. It follows from (2.8) that $[c^2, d(R)] = \{0\}$, so that $c^2 \in Z$ by Lemma 2.1(b). If $c \in Z$, then (2.8) yields $2cd(R) = \{0\}$; hence $2c = 0 = c$ and $F = d$. □

Henceforth, except in our final section, $R$ will always be a prime ring with extended centroid $C$ and central closure $S = RC$. (For definitions and basic properties of $C$ and $S$, see [7, Section 2] or [3, Chapter 1, Section 3]). Note that if $R$ has $1$, then $C$ is the center $Z(S)$ of $S$.

**Lemma 2.5** [2, Theorem 2.1]. Let $R$ be a prime ring and let $d, g, h$ be derivations on $R$ for which there exist $a, b \in R \setminus Z$ such that $d(x) = ag(x) + h(x)b$ for all $x \in R$. Then there exists $\lambda \in C$ such that $d(x) = [\lambda ab, x]$, $g(x) = [\lambda b, x]$ and $h(x) = [\lambda a, x]$ for all $x \in R$.

**Lemma 2.6**. Let $R$ be prime ring with $1$. let $F$ be a generalized derivation with associated derivation $d \neq 0$, such that $d(F(x)) = 0$ for all $x \in R$; and suppose $c = F(1) \notin Z$. Then

(i) there exists $\lambda \in C$ such that $d(x) = [\lambda c, x]$ for all $x \in R$;

(ii) $F$ can be extended to a generalized derivation $\hat{F}$ on $S$;

(iii) if $[F(x), F(y)] = 0$ for all $x, y \in R$, then $[\hat{F}(x), \hat{F}(y)] = 0$ for all $x, y \in S$. 

Proof. (i) By Lemma 2.4, $cd(x) + d(x)c = 0$ for all $x \in R$; hence by Lemma 2.5, there exists $\lambda \in C$ such that $d(x) = [\lambda c, x]$ for all $x \in R$.

(ii) Define $\hat{F}(x) = cx + [\lambda c, x]$ for all $x \in S$.

(iii) Let $x \in S$, and write $x = \sum_{i=1}^{n} r_{i}u_{i}$, where $r_{i} \in R$, $u_{i} \in C$. Then using the fact that $u_{i} \in Z(S)$, we get $\hat{F}(x) = \sum cr_{i}u_{i} + \sum [\lambda c, r_{i}u_{i}] = \sum u_{i}(cr_{i} + [\lambda c, r_{i}]) = \sum u_{i}F(r_{i})$; and (iii) follows at once.

\begin{lemma} [7, Theorem 3] \end{lemma}

If $R$ is prime and $S$ satisfies a generalized polynomial identity over $C$, then $S$ is primitive.

3. The condition $[F(x), F(y)] = 0$

In view of Lemma 2.1(a), it is natural to conjecture that if a prime ring $R$ of characteristic different from 2 admits a nonzero generalized derivation $F$ such that $[F(x), F(y)] = 0$ for all $x, y \in R$, then $R$ is commutative. However, this is not the case.

Example 3.1. Let $R$ be either the ring $H$ of real quaternions or the subring $K$ of $H$ consisting of all elements $a + bi + cj + dk$ where $a, b, c, d$ are integers. Define $F(x) = ix + xi$ for all $x \in R$. Then $R$ is a noncommutative prime ring, and $F$ is a generalized derivation such that $[F(x), F(y)] = 0$ for all $x, y \in R$.

Example 3.2. Let $K$ be any field, and let $R$ be the ring $M_{2}(K)$ of $2 \times 2$ matrices over $K$. Define $F(x) = cx + xc$, where $c$ is either $e_{11} - e_{22}$ or $e_{12}$. It is easy to verify that, in either case, $[F(x), F(y)] = 0$ for all $x, y \in R$.

Example 3.3. Let $R$ be the noncommutative prime ring $M_{2}(\mathbb{Z})$; and for arbitrary $x = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in R$, define $d(x) = \begin{bmatrix} 0 & -b \\ c & 0 \end{bmatrix}$. Define $F : R \longrightarrow R$ by $F(x) = (e_{11} - e_{22})x + d(x)$. It is easily verified that $d$ is a derivation on $R$, so that $F$ is a generalized derivation; and since $F\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = \begin{bmatrix} a & 0 \\ 0 & -d \end{bmatrix}$, $[F(x), F(y)] = 0$ for all $x, y \in R$. Note that $F$ is the restriction to $R$ of the map $\hat{F} : M_{2}(Q) \longrightarrow M_{2}(Q)$ given by $\hat{F}(x) = cx + xc$, where $c = \frac{1}{2}e_{11} - \frac{1}{2}e_{22}$.

In fact, for 2-torsion free prime rings with 1, these examples illustrate all possibilities, as our next (and principal) theorem shows.
Theorem 3.4. Let $R$ be a prime ring with 1 such that $\text{char}(R) \neq 2$. If $R$ admits a nonzero generalized derivation $F$ such that $[F(x), F(y)] = 0$ for all $x, y \in R$, then one of the following holds:

(i) $R$ is commutative;
(ii) $R$ is a noncommutative subring of a division ring $\Delta$, and there exists $\delta \in \Delta$ such that $F(x) = \delta x + x\delta$ for all $x \in R$;
(iii) $R$ is a noncommutative subring of a $2 \times 2$ total matrix ring $M$ over a field, and there exists $m \in M$ such that $F(x) = mx + xm$ for all $x \in R$.

The following lemma is the first step in the proof.

Lemma 3.5. Let $R$ be a noncommutative prime ring with 1 and with $\text{char}(R) \neq 2$. Let $F$ be a nonzero generalized derivation with associated derivation $d$, such that $[F(x), F(y)] = 0$ for all $x, y \in R$. Then

(i) $d(F(x)) = 0$ for all $x \in R$;
(ii) $c = F(1) \notin Z$ and $c^2 \in Z$;
(iii) $S$ is primitive and there exists $s \in S$ such that $s^2 \in Z(S)$ and $\hat{F}(x) = sx + xs$ for all $x \in S$.

Proof. (i) If $d = 0$, then $c \neq 0$ and $[cx, cy] = 0$ for all $x, y \in R$; thus $cR$ is a nonzero commutative right ideal. But a noncommutative prime ring cannot have such a right ideal, hence $d \neq 0$.

We have

\[(3.1)\quad [F(x), F(y)] = 0 \quad \text{for all} \quad x, y \in R.\]

Replacing $y$ by $yz$ in (3.1) and using (3.1), we get

\[(3.2)\quad F(y)[F(x), z] + y[F(x), d(z)] + [F(x), y]d(z) = 0 \quad \text{for all} \quad x, y, z \in R.\]

Now replacing $y$ by $ry$ in (3.2) gives

\[F(r)y[F(x), z] + rd(y)[F(x), z] + ry[F(x), d(z)] + r[F(x), y]d(z) + [F(x), r]yd(z) = 0 \quad \text{for all} \quad x, y, z, r \in R;\]

and hence application of (3.2) yields

\[F(r)y[F(x), z] + rd(y)[F(x), z] + [F(x), r]yd(z) - rF(y)[F(x), z] = 0.\]

Letting $z = F(x)$, we obtain $[F(x), r]yd(F(x)) = 0$ for all $x, y, r \in R$ - i.e. $[F(x), r]Rd(F(x)) = \{0\}$ for all $x, r \in R$.

Since $R$ is prime, for each $x \in R$, either $F(x) \in Z$ or $d(F(x)) = 0$. The sets of $x \in R$ for which these alternatives hold are additive subgroups whose union is $R$; therefore, either $F(R) \subseteq Z$ or $d(F(x)) = 0$ for all $x \in R$. But by Lemma 2.2, $F(R) \subseteq Z$ would force $R$ to be a commutative; hence
\[ d(F(R)) = \{0\}. \]

(ii) Since \( R \) is not commutative, it follows from Lemmas 2.4 and 2.1 (a) that \( c \notin Z \) and \( c^2 \in Z \).

(iii) By Lemma 2.4 we now have \( cd(x) + d(x)c = 0 \) for all \( x \in R \); and since \( c \notin Z \), it follows by Lemma 2.6 that there exists \( \lambda \in C \) such that \( d(x) = [\lambda c, x] = \lambda[c, x] \) for all \( x \in R \). Therefore \( \hat{F}(x) = cx + \lambda[c, x] \) for all \( x \in S \). Since \( \hat{F}(1) = c \) and \( [\hat{F}(x), \hat{F}(y)] = 0 \) for all \( x, y \in S \), we have

\[
(3.3) \quad c(cx + \lambda[c, x]) = (cx + \lambda[c, x])c \text{ for all } x \in S.
\]

Now \( c^2 \in Z(S) \) by Lemma 2.4, so (3.3) can be written as

\[
(2\lambda + 1)(c^2x - cxc) = 0 \text{ for all } x \in S,
\]

from which it follows that

\[
(3.4) \quad 2\lambda + 1 = 0 \text{ or } c^2x - cxc = 0 \text{ for all } x \in S.
\]

Since \( c^2 \in Z(S) \), either \( c \) is regular or \( c^2 = 0 \). In the first case we see from (3.4) that \( 2\lambda + 1 \neq 0 \) contradicts the fact that \( c \notin Z \); in the second case \( 2\lambda + 1 \neq 0 \) yields \( c = 0 \), contrary to our observation that \( c \notin Z \). Therefore \( \lambda = -\frac{1}{2} \) and for each \( x \in S \), \( \hat{F}(x) = cx - \frac{1}{2}(cx - xc) = sx + xs \), where \( s = \frac{c}{2} \).

Recalling that \( [\hat{F}(x), \hat{F}(y)] = 0 \) for all \( x, y \in S \), we see that \( S \) satisfies the generalized polynomial identity \( \frac{1}{4}(cx + xc)(cy + yc) = \frac{1}{4}(cy + yc)(cx + xc) \over C \); hence \( S \) is primitive by Lemma 2.7.

\[ \square \]

Proof of Theorem 3.4. In view of Lemma 3.5 and Jacobson’s density theorem, we may assume that \( R \) is a noncommutative dense ring of linear transformations on a vector space \( V \) over a division ring \( \Delta \), and that there exists \( k \in R \setminus \{0\} \) such that \( k^2 \in Z \) and \( F(x) = kx + xk \) for all \( x \in R \). We need only show that \( \dim(V) \leq 2 \) and that in the case \( \dim(V) = 2 \), \( \Delta \) is a field. For any subset \( W \subseteq V \), we denote by \( < W > \) the subspace generated by \( W \).

By a standard argument it follows that if \( \dim(V) > 1 \) and \( k(u) \in < u > \) for each \( u \in V \), then there exists \( \beta \in \Delta \setminus \{0\} \) such that \( k(u) = \beta u \) for all \( u \in V \). But in this case we have \( (kx + xk)(ky + yk)(u) = 4\beta^2 xy(u) \) and \( (ky + yk)(kx + xk)(u) = 4\beta^2 yx(u) \) for all \( u \in V \), contradicting our hypothesis that \( R \) is not commutative.

Assume that \( \dim(V) \geq 3 \), and choose \( u \in V \) such that \( k(u) = v \notin < u > \). Since \( k^2 \in Z(R) \), there exists \( \alpha \in Z(\Delta) \) such that \( k^2(w) = \alpha w \) for all \( w \in V \); therefore \( k(v) = \alpha u \).
Suppose that \( k(V) \not\subseteq \langle u, v \rangle \), in which case there exists \( z \in V \setminus \langle u, v \rangle \) and \( w \in V \) such that \( k(w) = z \). Then \( \{u, v, w\} \) is a linearly independent subset of \( V \) and there exist \( a, b \in R \) such that \( a(u) = v, a(v) = w, a(w) = u, b(u) = u, b(v) = 0 \) and \( b(w) = 0 \). It is readily verified that the condition \((ka + ak)(kb + bk)(u) = (kb + bk)(ka + ak)(u)\) implies that \( b(z) = z \). It follows that if \( z \not\in \langle u, v, w \rangle \), then \( z = b(z) \in \langle u \rangle \), contrary to the fact that \( z \not\in \langle u, v \rangle \); therefore \( \{u, v, w, z\} \) is a linearly independent subset of \( V \) and there exist \( a', b' \in R \) such that \((a'(u), a'(v), a'(w), a'(z)) = (v, w, u, 0)\) and \((b'(u), b'(v), b'(w), b'(z)) = (u, 0, 0, 0)\). But the argument given for \( a \) and \( b \) shows that this is incompatible with the requirement that \([F(a'), F(b')] = 0\); therefore we must have \( k(V) \subseteq \langle u, v \rangle \).

Since \( \dim(V) \geq 3 \), \( \ker(k) \neq \{0\} \) and there exists \( t \in V \setminus \{0\} \) such that \( k(t) = 0 \). Therefore \( k^2(t) = \alpha t = 0 \), so that \( \alpha = 0 \), \( k(v) = 0 \) and \( k^2 = 0 \). Thus, if \( q \in V \) and \( k(q) = \gamma u + \delta(v) \), then \( 0 = k^2(q) = \gamma v \) so \( \gamma = 0 \). Hence \( k(V) \subseteq \langle v \rangle \) and \( \ker(k) \) has dimension at least 2; and since \( u, v \not\in \ker(k) \), there exists \( y \in \ker(k) \setminus \langle u, v \rangle \). Choosing \( a, b \in R \) such that \((a(u), a(v), a(y)) = (v, y, u)\) and \((b(u), b(v), b(y)) = (u, y, u)\), we get \((kb + bk)(ka + ak)(u) = 0\) and \((ka + ak)(kb + bk)(u) = y - a contradiction. Therefore \( \dim(V) < 3 \) as required.

Finally, assume \( \dim(V) = 2 \). As before, we have linearly independent \( u, v \) such that \( k(u) = v \). Let \( \beta, \gamma \in \Delta \) and consider \( a, b \in R \) such that \((a(u), a(v)) = (0, \beta u)\) and \((b(u), b(v)) = (0, \gamma u)\). Then \((ka + ak)(u) = \beta u\) and \((kb + bk)(u) = \gamma u\), and the condition \([F(a), F(b)] = 0\) gives \( \beta \gamma u = \gamma \beta u \), so that \( \beta \gamma = \gamma \beta \). Thus \( \Delta \) is a field.

4. Anti-commutativity conditions

In our final section we present some more elementary results, which involve anti-commutativity hypotheses.

**Theorem 4.1.** Let \( R \) be a prime ring with 1 and \( \text{char}(R) \neq 2 \). If \( F \) is a generalized derivation on \( R \) such that \( F(x) \circ F(y) = 0 \) for all \( x, y \in R \), then \( F = 0 \).

**Proof.** Note that if \( R \) is commutative, it is a domain; and the condition \( F(x) \circ F(y) = 0 \) is just \( 2F(x)F(y) = 0 \). Taking \( y = x \) then shows that \( F(x) = 0 \) for all \( x \in R \).

Assume that \( F \neq 0 \). Then \( R \) is not commutative; and since \( F(1) \circ F(1) = 0 \), we have \( c^2 = 0 \). Note that we cannot have \( d = 0 \), for in that case \( F(1) \circ F(x) = 0 \) becomes \( cxc = 0 \) for all \( x \in R \), which implies that \( c = 0 \) and hence \( F = 0 \).

We now have \( d \neq 0 \) and
(4.1) \[ F(x) \circ F(y) = 0 \text{ for all } x, y \in R. \]

Replacing \( y \) by \( yz \) in (4.1) and using (4.1), we get
\[(4.2) \quad (F(x) \circ y)d(z) - F(y)[F(x), z] - y[F(x), d(z)] = 0 \text{ for all } x, y, z \in R.\]

Now replacing \( z \) by \( zF(x) \) in (4.2) and using (4.2), we obtain
\[(4.3) \quad (F(x) \circ y)zd(F(x)) - yz[F(x), d(F(x))] - y[F(x), z]d(F(x)) = 0.\]

Finally, replacing \( y \) by \( ry \) in (4.3) and using (4.3), we conclude that
\[
[F(x), r]yRd(F(x)) = \{0\} \text{ for all } x, y, r \in R.
\]

Again, invoking the primeness of \( R \), we learn that \( F(R) \subseteq Z \) or \( d(F(x)) = 0 \) for all \( x \in R \). But Lemma 2.2 implies that if \( F(R) \subseteq Z \), then \( R \) is commutative, contrary to our assumption that \( F \neq 0 \); therefore \( d(F(x)) = 0 \) for all \( x \in R \), and by Lemma 2.4 \( cd(x) + d(x)c = 0 \) for all \( x \in R \). The condition that \( F(1) \circ F(x) = 0 = c(cx + d(x)) + (cx + d(x))c \) reduces to \( cxc = 0 \); hence \( c = 0 \) and \( R \) is commutative by Lemma 2.3, so we have again contradicted our assumption that \( F \neq 0 \). Therefore, \( F = 0. \) \( \Box \)

**Theorem 4.2.** Let \( R \) be a 2-torsion free ring with 1. If \( F \) is a generalized derivation such that \( F(x) \circ F(y) = x \circ y \) for all \( x, y \in R \), then there exists \( c \) in \( Z \) such that \( c^2 = 1 \) and \( F(x) = cx \) for all \( x \) in \( R \). Thus, if \( R \) is prime, \( F \) is the identity map or its negative.

**Proof.** Since \( F(1) \circ F(1) = 1 \circ 1 \), we have \( c^2 = 1 \). Thus, the condition \( F(x) \circ F(1) = x \circ 1 \) reduces to
\[(4.4) \quad cxc + d(x)c + cd(x) = x.\]

Postmultiplying and premultiplying this equation by \( c \) and comparing the results yields \( d(x) + cd(x)c = 0 \); and premultiplying by \( c \) gives
\[(4.5) \quad cd(x) + d(x)c = 0.\]

It now follows from (4.4) that \( cx = xc \) for all \( x \) in \( R \), so that \( c \) is in \( Z \); and since \( c \) is invertible, (4.5) shows that \( d = 0 \) and hence \( F(x) = cx \) for all \( x \) in \( R \). \( \Box \)

A similar proof yields our final theorem.

**Theorem 4.3.** Let \( R \) be a 2-torsion free ring with 1. If \( F \) is a generalized derivation such that \( F(x) \circ F(y) + x \circ y = 0 \) for all \( x, y \in R \), then there exists \( c \) in \( Z \) such that \( c^2 = -1 \) and \( F(x) = cx \) for all \( x \) in \( R \).
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