CUT LOCI AND DISTANCE FUNCTIONS

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1. Introduction

Let \((M, g)\) be a compact Riemannian manifold and \(d(p, q)\) the distance between \(p, q \in M\) induced from the metric \(g\). Then the distance function \(f := d_p, d_p(x) := d(p, x)\), to a point \(p \in M\) plays a fundamental role in Riemannian geometry. Recall that \(d_p\) is directionally differentiable at any \(q \neq p\), namely for any unit tangent vector \(\xi \in U_q M\) we have the first variation formula

\[
f'_q(\xi) = - \cos \alpha,
\]

where \(\alpha\) denotes the infimum of angles between \(\xi\) and the initial directions of minimal geodesics from \(q\) to \(p\).

The behavior of the distance function \(d_p\) is closely related to the structure of the cut locus of \(p\). Recall that the cut locus \(C(p)\) of \(p \in M\) is given as follows: for any unit speed geodesic \(u = \gamma_u(0) \in U_p M\), there exists the last parameter value \(i_p(u)\) up to which \(\gamma_u\) is a minimal geodesic segment, namely \(\gamma_u|[0, t]\) realizes the distance \(d(p, \gamma_u(t))\) for \(0 < t \leq i_p(u)\). We call \(\gamma_u(i_p(u))\) the cut point of \(p\) and \(i_p(u)\) the cut distance to \(p\) along \(\gamma_u\). Then the cut locus \(C(p)\) of \(p\) is defined as

\[
C(p) := \{\gamma_u(i_p(u)) \mid u \in U_p M\}.
\]

Recall that \(q\) is a cut point of \(p\) along \(\gamma_u\) if and only if either there exists another minimal geodesic \(\gamma_v, v \in U_p M, \ v \neq u\), emanating from \(p\) with \(q = \gamma_u(i_p(u)) = \gamma_v(i_p(u))\), or \(q\) is a (first) conjugate point to \(p\) along \(\gamma_u\), which means that there exists a nontrivial Jacobi field \(Y(t)\) along \(\gamma_u\) with \(Y(0) = Y(i_p(u)) = 0\) (see e.g., [16]). The set \(\check{C}(p) := \{i_p(u)u \mid u \in U_p M\}\) is called the tangent cut locus of \(p\).

Then if \(q(\neq p)\) does not belong to the cut locus \(C(p)\), \(d_p\) is differentiable at \(q\) and its gradient vector \(\nabla d_p(q)\) is given by \(\dot{\gamma}(l)\), where \(\gamma\) is a unique minimal

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geodesic parameterized by arclength joining $p$ to $q$ and we set $l = d(p, q)$. Note that $\|\nabla d_p(q)\| = 1$. On the other hand, if $q \in C(p)$ then $d_p$ is in general not differentiable at $q$.

Now, $q(\neq p)$ is said to be noncritical for $f = d_p$, if there exists a $\xi \in U_q M$ such that $f_q'(\xi) > 0$. Namely, $q(\neq p)$ is a critical point of $d_p$ if for any $\xi \in U_q M$ there exists a minimal geodesic $\gamma$ from $q$ to $p$ whose initial direction makes an angle $\alpha \leq \pi/2$ with $\xi$. Note that a critical point $q(\neq p)$ is a cut point of $p$, where $d_p$ is not differentiable. We consider $p$ itself a critical point of $f$, since it is a unique minimum point. If $f$ assumes a local maximum at $q$, then $q$ is a critical point in the above sense. The notion of critical points was first considered by K. Grove and K. Shiohama, and then by M. Gromov ([9], [6]). If $q$ is noncritical, then constructing a gradient-like vector field for $-f$ we may put a neighborhood of $q$ nearer to $p$ by an isotopy of $M$ (isotopy lemma). Therefore, we have an analogy of Morse theory for the case without critical points, and this idea has played an essential role in problems on curvature and topology of Riemannian manifolds ([9], [6], [7]).

Now since distance function is the most fundamental function on Riemannian manifold, we ask the behavior of the levels of $d_p$ when it passes a critical value. Namely, we ask how to define the notion of index of $d_p$ at a critical point $q$ and how we can get a normal form of $d_p$ around $q$ under some nondegeneracy condition. As mentioned above, $d_p$ is not differentiable at its critical points, and the structure of the cut locus $C(p)$ of $p$ is related to the behavior of $d_p$ around critical points.

In this note we are concerned with the above problem under the assumption that $C(p)$ has rather nice structure, and we discuss an analogy of Morse theory for distance function. We give an application (Corollary 3.12), and hope to give further applications with the present approach ( [12] is our motivation for the present work). Indeed, V. Gerschikovich and H. Rubinstein have studied Morse theory for generic distance functions from the view point of min-type functions ([3], [4], [5]), and got results closely related to what we will discuss in the following. Especially, they studied the surface case in detail. Here we try to take a more geometric and direct approach.

Assume that there is a point $p \in M$ such that the tangent cut locus $\tilde{C}(p)$ of $p$ is disjoint from the first tangent conjugate locus, namely any minimal
geodesic segment emanating from $p$ is conjugate point-free. Then we say that $(M, g)$ satisfies the condition (C) at $p \in M$. We are interested in the structure of the cut locus $C(p)$ of $p$ under the above condition (C). Especially, if $(M, g)$ admits no conjugate points along all geodesics emanating from $p$, then the structure of $C(p)$ may be expressed in terms of the Dirichlet domain of the universal covering space $\tilde{M}$ of $M$ with the induced Riemannian metric $\tilde{g}$ (see [15] for these assertions). If $(M, g)$ is nonpositively curved, then for any point $p \in M$ there appear no conjugate points to $p$ along any geodesic emanating from $p$. On the other hand, A. Weinstein showed that for any compact manifold $M$ with dim $M \geq 2$ except for $S^2$ there exists a Riemannian metric such that there is a point $p \in M$ with the cut locus $C(p)$ disjoint from the first conjugate locus ([18]). In this case $M$ satisfies the condition (C) at $p$.

In §2 we introduce the notion of nondegenerate cut points under the condition (C), and show that the cut locus $C(p)$ admits a nice Whitney stratification if all cut points are nondegenerate. As an application, using this peculiar stratification we give a description of the structure of the cut locus $C(p)$ in a neighborhood of any cut point $q \in C(p)$ in terms of the cone over the cut locus of finitely many unit vectors in general position in the unit sphere $S^{n-1}$ in $T_qM$ (Theorem 2.5). This is also useful to give a normal form of $d_p$ around a critical point under some nondegeneracy condition in §3. We also show that critical points of $d_p$ in the angle sense are critical points of the smooth function, that is the restriction of the distance function $d_p$ to strata containing the critical points, in the usual sense.

In §3, under the above assumption, first we define the notion of index for a critical point of $d_p$ in the angle sense, and we give a normal form of nondegenerate distance function $d_p$ around a critical point by geometric consideration. Then we show that usual procedure of Morse theory works. (Theorem 3.7. Compare [4].) Next, we also consider the condition (F) for $p \in M$, which states that for any unit speed geodesic $\gamma_u$ emanating from $p$, and for any Jacobi field $Y$ along $\gamma_u$ with $Y(0) = 0, \nabla_{\dot{\gamma_u}} Y(0) \neq 0$, we have $\langle Y(t), \nabla_{\dot{\gamma_u}} Y(t) \rangle > 0$ for the parameter value $t > 0$ up to the cut distance $i_p(u)$ to $p$. Note that this implies that the condition (C) holds at $p$, and the condition (F) is satisfied for any $p \in M$ when $(M, g)$ is of nonpositive
sectional curvature. Then under the assumption of condition (F), on any stratum of $C(p)$ of codimension less than $n$, we see that the restriction $f$ of the distance function $d_p$ to the stratum satisfies the following: any critical point $r$ of $d_p$ in the angle sense belonging to the stratum is a strict local minimum of $f$, namely, $r$ is a critical point of $f$ in the usual sense and its Hessian is positive definite. Therefore, in this case we have simpler procedure of Morse theory (see Theorem 3.11).

The structure of cut loci $C(p)$ for generic Riemannian metrics was studied by applying singularity theory to smooth energy integral on the (finite-dimensional approximation of) the space of piecewise smooth paths emanating from $p$ ([1], [2], [17], [19]). However, it is not clear for us whether such structure theorems directly give information on the Riemannian distance function $d_p$, and we take here more geometrical approach under somewhat stronger assumption on the cut locus. We are grateful to M. van Manen for his criticism and pointing out several references including [19]. We are grateful to H. Rubinstein for telling us [3], [4], [5]. We would like to also express our sincere appreciation to the referee for his kind suggestion to make the paper more readable.

2. STRUCTURE OF THE CUT LOCUS DISJOINT FROM THE FIRST CONJUGATE LOCUS

Suppose a compact $n$-dimensional Riemannian manifold $(M, g)$ satisfies the condition (C) at $p$, namely the tangent cut locus $C(p)$ is disjoint from the first tangent conjugate locus of $p$. Let $q \in C(p)$. Then from the assumption there are only finitely many minimal geodesics emanating from $p$ to $q$. We denote by $\{\gamma_0, \cdots, \gamma_k\}(k \geq 1)$ the set of the minimal geodesics parametrized by arclength from $p$ to $q$, where $k + 1$ is called the order of the cut point $q$. Note that we may find open neighborhoods $U \ni q$ in $M$ and $V_i \ni l\gamma_i(0)$ ($i = 0, \ldots, k$) in $T_pM$ such that $\exp_p : V_i \to U$ are diffeomorphisms, where $\exp_p : T_pM \to M$ denotes the exponential map at $p$. We may also assume that for any minimal geodesic $\gamma$ parameterized by arclength from $p$ to a point $r \in U$ the tangent vector $d(p, r)\dot{\gamma}(0) \in T_pM$ belongs to one of the corresponding $V_i$'s. Then we set

$$F_i := (\exp_p|V_i)^{-1}$$
that is a diffeomorphism from $U$ onto $V_i$ for each $i = 0, \ldots, k$.

We set $X_i := -\dot{\gamma}_i(l), l = d(p, q), i = 0, \ldots, k$ which are pairwise different unit vectors of $T_qM$. Namely, $\{X_i\}_{0 \leq i \leq k}$ is the set of initial directions of geodesics from $q$ to $p$ parametrized by arclength. Now we define the notion of nondegenerate cut points as follows:

**Definition 2.1.** A cut point $q \in C(p)$ of order $k + 1$ is said to be nondegenerate, if $X_0, \ldots, X_k$ are in general position in $T_qM$.

This means that the dimension of the affine subspace of the tangent space $T_qM$ spanned by $\{X_0, \ldots, X_k\}$ is equal to $k$. This is also equivalent to the condition that $X_0 - X_1, X_0 - X_2, \ldots, X_0 - X_k$ (or equivalently, for fixed $i$, $X_i - X_j$ ($j \neq i$)) are linearly independent. Then either $X_0, \ldots, X_k$ are linearly independent and spans a $k$-dimensional affine subspace that does not contain the origin, or they are linearly dependent and spans a $k$-dimensional vector subspace. For instance, cut points of order 2 or of order 3 are always nondegenerate. Note that if $q$ is a nondegenerate cut point, then its order $k + 1$ is at most $n + 1$.

**Remark 2.2.** (1) Tangent cut loci of a point in 2-dimensional flat tori are in general hexagons, in which case all cut points are nondegenerate. If the tangent cut locus of $p$ is given by a rectangle, then the cut point $q$ which is furthest to $p$ and is given by vertices of the rectangle of the tangent cut locus is degenerate. Indeed, we have four minimal geodesics from $p$ to $q$. However, after slightly deforming the lattice such degenerate cut points disappear in this case. All cut points of $p$ in an $n$-dimensional flat torus are nondegenerate if and only if the cut points, that are local maximum points of $d_p$ and given by the vertices of the tangent cut locus, are of order $n + 1$.

(2) The distance function $d_p$ is a (germ of) min-type function in the sense of [4], namely, in a neighborhood of $q$ we have $d_p(r) := \min\{\|F_i(r)\| \mid 0 \leq i \leq k\}$.

Now suppose that all cut points of $p$ are nondegenerate. We denote by $C_{k+1} \subset C(p)$ the set of cut points of $p$ of order $k + 1$. We assume that $C_{k+1}$ is nonempty. Now for $q \in C_{k+1}$ we denote by $\gamma_0, \ldots, \gamma_k$ the set of minimal geodesics parametrized by arclength joining $p$ to $q$. Recall that for any $r \in U \cap C_{k+1}$ and any minimal geodesic $\gamma$ joining $p$ to $r$, $d(p, r)\dot{\gamma}(0)$
belongs to $V_i$ for some $0 \leq i \leq k$, and that $U \cap C(p)$ consists of cut points of $p$ of order not greater than $k + 1$ (see e.g., [15]).

Now, we consider a smooth map $G : U \rightarrow \mathbb{R}^k$ defined by

$$G(r) := (\|F_0(r)\| - \|F_1(r)\|, \ldots, \|F_0(r)\| - \|F_k(r)\|),$$

where $F_i = (\exp_p |V_i|)^{-1} : U \rightarrow V_i \subset T_r M, i = 0, \ldots, k$, are given by (2.1). Then we easily see that

$$G^{-1}(0) = C_{k+1} \cap U,$$

and that for every $r \in U$ the gradient vector $\nabla G_j(r)$ of the $j$-th coordinate function $G_j(r) := \|F_0(r)\| - \|F_j(r)\|$ of $G$ is given by

$$\nabla G_j(r) = X_j - X_0 \quad (j = 1, \ldots, k)$$

by the first variation formula.

Since $\nabla G_j(r)$ are linearly independent for $r \in G^{-1}(0)$ by the assumption, the differential $DG(r) : T_r U \rightarrow \mathbb{R}^k$ of $G$ at any $r \in G^{-1}(0)$ is of rank $k$. It follows by the implicit function theorem that $C_{k+1}$ is a submanifold of $M$ of codimension $k$. Equivalently, the hypersurfaces $G_j^{-1}(0) (j = 1, \ldots, k)$ intersect transversally at $r \in G^{-1}(0)$. However, note that $C_{k+1}$ is not necessarily connected, and we denote by $C_{k+1,q}$ the connected component of $C_{k+1}$ containing $q \in C_{k+1}$.

Since we have $d_p(r) = \|F_0(r)\| = \|F_j(r)\|, j = 1, \ldots, k$ for $r \in C_{k+1}$ and $F_0$ is a diffeomorphism, we see that $f := d_p|C_{k+1,q}$ is a smooth function for $q \in C_{k+1}$. Summing up we get

**Lemma 2.3.** Suppose that $C_{k+1} (\neq \emptyset)$ consists of nondegenerate cut points. Then $C_{k+1}$ is a submanifold of codimension $k$ of $M$, and $d_p$ is a smooth function when restricted to each connected component $C_{k+1,q}$ of $C_{k+1}$.

It may happen that some $C_{k+1}$ is empty. For instance, the cut locus of the $n$-dimensional real projective space with canonical Riemannian metric of constant curvature $1$ consists of nondegenerate cut points of order $2$, and $C_{k+1} (k \geq 2)$ is empty. In this case the cut locus is an $(n - 1)$-dimensional projective subspace and indeed a smooth submanifold. From the condition (C) we see that $C_{k+1}$ is nonempty for some $k \geq 1$ and then so is $C_{l+1}$ for
1 \leq l \leq k$ from the nondegeneracy condition. For $q \in C_{k+1}$ we have

$$U \cap C(p) = \bigcup_{1 \leq l \leq k} C_{l+1} \cap U \quad \text{and} \quad U \setminus C(p) = \bigcup_{i=0}^{k} D_i,$$

where $D_i := \{ r \in U \mid \| F_i(r) \| < \| F_j(r) \|; j \neq i \}$. Note that the closure $\bar{C}_{k+1,q}$ of $C_{k+1,q}$ is given by $\bigcup_{l \geq k} C_{l+1,r}$, where $r \in \bar{C}_{k+1,q}$ is of order $l+1$. If $C_{k+2} = \emptyset$ then we see that $\bar{C}_{k+1,q}$ is a smooth submanifold. It follows that we have a stratification of the cut locus by submanifolds $C_{k+1,q}$, and it is easy to verify the Whitney’s condition (B) in our case ([8]). Hence we get

**Proposition 2.4.** Suppose a compact Riemannian manifold $(M, g)$ satisfies the condition (C) at $p \in M$ and cut points $q \in C(p)$ are nondegenerate. Then the cut locus $C(p)$ of $p$ has a Whitney stratification given as above.

Next we give a description of the tangent cone of a cut point $q \in C(p)$. Suppose $C_{k+1}$ consists of nondegenerate cut points. Then recall that $C_{k+1}$ is an $(n-k)$-dimensional submanifold of $M$, and $d_p$ is a smooth function when restricted to each connected component $C_{k+1,q}$.

**Theorem 2.5.** Suppose the condition (C) is satisfied at $p$ and all cut points of $p$ are nondegenerate. Let $q \in C_{k+1}$ and let $\gamma_0, \ldots, \gamma_k$ be the minimal geodesics from $p$ to $q$. Set $X_i = -\gamma_i(l) \in U_q M$, $i = 0, \ldots, k$, with $l = d(p, q)$. We denote by $S(q)$ the cut locus of a finite subset $\{X_0, \ldots, X_k\}$ of $U_q M$, which is considered as the unit $(n-1)$-dimensional sphere with the canonical Riemannian metric. Then $C(p) \cap U$ is homeomorphic to the cone over $S(q)$ in $T_q M$ with origin as the vertex, if we take a sufficiently small open neighborhood $U$ of $q$.

First we recall the structure of the cut locus $S(q)$ of the finite set $\{X_0, \ldots, X_k\}$ in the unit sphere $U_q M$ with respect to the canonical metric. Indeed, we have $X \in S(q)$ if and only if there exists at least two minimizing geodesics of $U_q M$ from the set $\{X_0, \ldots, X_k\}$ to $X$. Namely, $S(q)$ consists of the parts of the bisectors of $X_i$ and $X_j$ ($i < j$) in the sphere $U_q M$ that are closer or of equidistance to other $X_i$’s, $l \neq i, j$. In our case, $X_0, \ldots, X_k$ spans a $k$-dimensional affine subspace $V_1$ and they are contained in a $(k-1)$-dimensional great or small sphere in $U_q M$. Note that they are contained in a great sphere $S^{k-1}$ if and only if the affine subspace spanned by them contains
the origin and is a vector subspace. If they are contained in a small hypersphere \( \hat{S}^{k-1} \) in the \( k \)-dimensional great sphere \( S^k := U_q M \cap \langle X_0, \cdots, X_k \rangle \mathbf{R} \), we consider the parallel great sphere \( S^{k-1} \) in \( S^k \) and the corresponding unit vectors \( \tilde{X}_0, \cdots, \tilde{X}_k \) in \( S^{k-1} \) that are projections of \( X_0, \cdots, X_k \) from the north pole respectively. Let \( \tilde{V}_1 = \langle \tilde{X}_0, \cdots, \tilde{X}_k \rangle \mathbf{R} \) denote the \( k \)-dimensional vector subspace determined by \( S^{k-1} \). If \( \{X_0, \cdots, X_k\} \) are contained in a \((k - 1)\)-dimensional great sphere \( S^{k-1} \) of \( V_1 = \langle X_0, \cdots, X_k \rangle \mathbf{R} \), we set \( \tilde{X}_i = X_i(i = 0, \ldots, k) \) and \( \tilde{V}_1 = V_1 \).

Now we give a description of the structure of the cut locus \( S(q) \).

**Lemma 2.6.** (i) The cut locus of \( \{\tilde{X}_0, \cdots, \tilde{X}_k\} \) in the unit sphere \( U_q M \) coincides with \( S(q) \), the cut locus of \( \{X_0, \ldots, X_k\} \) in \( U_q M \).

(ii) The cut locus \( \hat{S}_{k-2}(q) \) of \( \{\tilde{X}_0, \cdots, \tilde{X}_k\} \) in \( S^{k-1} \) is given by the union of the boundaries of the Dirichlet domains (or Voronoi diagrams) \( \{u \in S^{k-1} \mid \angle(u, \tilde{X}_i) < \angle(u, \tilde{X}_j) \text{ for all } j \neq i, 0 \leq j \leq k\} \) determined by \( \tilde{X}_i \)'s \((0 \leq i \leq k)\). These Dirichlet domains are spherical \((k-1)\)-dimensional simplices with totally geodesic boundaries in \( S^{k-1} \) and gives a triangulation of \( S^{k-1} \). Then \( \hat{S}_{k-2}(q) \) is the \((k - 2)\)-skeleton of the triangulation consisting of \( k(k + 1)/2 \) facets, and \((k-l-1)\)-dimensional faces are given by \( \{u \in S^{k-1} \mid \angle(u, \tilde{X}_{i_0}) = \cdots = \angle(u, \tilde{X}_{i_l}) < \angle(u, \tilde{X}_j); 0 \leq j \leq k, j \neq i_0, \ldots, i_l\} \), which are totally geodesic submanifolds of \( S^{k-1} \).

(iii) The whole cut locus \( S(q) \) is given by the spherical join of \( \hat{S}_{k-2}(q) \) and \( S^{n-k-1} \), where \( S^{n-k-1} \) consists of points in \( U_q M \) of spherical distance \( \pi/2 \) (or orthogonal) to the given \( S^{k-1} \).

**Proof of Lemma.** If \( k = 1 \), \( S(q) \) is given by a great sphere \( S^{n-2} \) of \( S^{n-1} \) obtained as the bisector of \( X_0 \) and \( X_1 \), and (ii), (iii) hold setting \( \hat{S}_{k-2}(q) = \emptyset \). So we assume \( k \geq 2 \) in the proof of (ii) and (iii).

(i) Denoting by \( n \) the unit vector in \( V_1 := \langle X_0, \cdots, X_k \rangle \mathbf{R} \cong \mathbf{R}^{k+1} \) representing the north pole of \( S^k \), we have

\[
X_i = \cos \theta \tilde{X}_i + \sin \theta n,
\]

where \( \theta = \angle(X_i, \tilde{X}_i) \) is the spherical distance between \( S^{k-1} \) and \( \hat{S}^{k-1} \). It follows that for \( u \in S^{n-1} \) we have \( \angle(u, X_i) = \angle(u, X_j) \) (resp. \( \angle(u, X_i) < \angle(u, X_j) \)) if and only if \( \angle(u, \tilde{X}_i) = \angle(u, \tilde{X}_j) \) (resp. \( \angle(u, \tilde{X}_i) < \angle(u, \tilde{X}_j) \)) holds, and the cut locus of \( \{\tilde{X}_0, \cdots, \tilde{X}_k\} \) coincides with \( S(q) \).
(ii) By the nondegeneracy condition $\tilde{X}_0, \ldots, \tilde{X}_k \in S^{k-1}$ form vertices of a $k$-simplex in $\tilde{V}_1 \cong R^k$. Then (ii) follows from

$$\tilde{S}_{k-2}(q) = \bigcup_{i<j} \{u \in S^{k-1} \mid \angle(u, \tilde{X}_i) = \angle(u, \tilde{X}_j) \leq \angle(u, \tilde{X}_l)\}$$

where $0 \leq l \leq k, l \neq i, j$.

(iii) Note that the sphere $S^{n-1} = U_qM$ with the canonical Riemannian metric is isometric to the spherical join $S^{k-1} \ast S^{n-k-1}$ of $S^{k-1}$ and $S^{n-k-1}$. Namely any $y \in S^{n-1}$ may be written as $\gamma(t), 0 \leq t \leq \pi/2$, where $\gamma$ is a unit-speed geodesic emanating from $x \in S^{k-1}$ perpendicularly to $S^{k-1}$. Then we have

$$\cos \angle(\gamma(t), \tilde{X}_i) = \cos t \cos \angle(x, \tilde{X}_i)$$

for $0 \leq t \leq \pi/2$. It follows that if $x = \gamma(0) \in \tilde{S}_{k-2}(q)$ then we have $\gamma(t) \in S(q), 0 \leq t < \pi/2$ and vice versa. On the other hand, $S^{n-k-1}$ is contained in $S(q)$, since $S^{n-k-1}$ is the set of points of equidistance $\pi/2$ to $\tilde{X}_i$ ($i = 0, \ldots, k$), namely the set of furthest points to $\{\tilde{X}_i\}$. Note that the $(n-k)$-dimensional vector subspace $V_0$ of $T_qM$ containing $S^{n-k-1}$ is characterized as the set of points in $T_qM$ which are of equidistance from $\tilde{X}_0, \ldots, \tilde{X}_k$ with respect to the Euclidean metric.

In the case where $X_0, \ldots, X_k$ lie on a small sphere in $S^k$, the cut locus $\tilde{S}_{k-1}(q)$ of these unit vectors in $S^k$ is the spherical suspension of the cut locus $\tilde{S}_{k-2}(q)$ of $\tilde{X}_0, \ldots, \tilde{X}_k$ in $S^{k-1}$, and the cut locus $S_{k-2}(q)$ of these unit vectors in the original small $(k-1)$-sphere $\tilde{S}^{k-1}$ is the intersection of $\tilde{S}_{k-1}(q)$ and the $k$-dimensional affine subspace $V_1$ determined by these $X_i, 0 \leq i \leq k$. Then $S_{k-2}(q)$ is indeed homeomorphic to $\tilde{S}_{k-2}(q)$. Further note that the cone over $S(q)$ in $T_qM$ is homeomorphic to the product of the subspace $V_0$ and the cone $T_1$ over $\tilde{S}_{k-2}(q)$ in $\tilde{V}_1$.

Now we turn to the proof of Theorem 2.5, namely study the local structure of the cut locus $C(p)$ around $q \in C_{k+1}$. We set for $r \in U$

$$x_i(r) := \|F_i(r)\|, \quad X_i(r) := -\nabla x_i(r), \quad i = 0, \ldots, k,$$

where $F_i$ is given in (2.1) and recall that we have $X_i = X_i(q) = -\nabla x_i(q)$.

Then from Lemma 2.3 we may regard that we have "local coordinates" $(x_0, \ldots, x_k, x_{k+1}, \ldots, x_n)$, where $(x_{k+1}, \ldots, x_n)$ denotes local coordinates
for $C_{k+1}$ around $q$ guaranteed by Lemma 2.3 on an open neighborhood $U$ of $q$. In the above, we think $(x_0, \cdots, x_k)$ as “local coordinates” around $q$ for a submanifold $N$ complementary to $C_{k+1}$ with tangent space spanned by \{ $X_i - X_0, \ i = 1, \cdots k$\}. More precisely, setting $y_i := x_0 - x_i, \ i = 1, \cdots, k$, we have local coordinates $(y_1, \cdots, y_k, x_{k+1}, \cdots, x_n)$, where $(y_1, \cdots, y_k)$ gives a local coordinates system for $N$ taking $U(\ni q)$ smaller if necessary. However, we also use the above notation. Then around $q$ the cut locus $C(p)$ consists of $k(k+1)/2$ pieces of hypersurfaces $\tilde{C}_{i,j}$ ($0 \leq i < j \leq k$) of $M$ given by
\[
\tilde{C}_{i,j} := \{ r \in C(p) \cap U \mid G_{ij}(r) := x_i(r) - x_j(r) = 0; \ x_l(r) > x_i(r) = x_j(r) \ \text{for} \ \ l \neq i,j \} \]
corresponding to minimal geodesics $\gamma_i$ and $\gamma_j$ from $p$ to $r$. Note that these hypersurfaces are also characterized in terms of $y_i, 1 \leq i \leq k$, namely for $1 \leq i < j \leq k$ we obtain
\[
\tilde{C}_{i,0} := \{ r \in U \mid 0 = y_i(r) \geq y_l(r), \ l \neq 0, i \}\]
\[
\tilde{C}_{i,j} := \{ r \in U \mid 0 \leq y_i(r) = y_j(r) \geq y_l(r), \ l \neq 0, i, j \}. \]
The intersection of these hypersurfaces is nothing but $C_{k+1} \cap U$. Now for $I := \{ 0 \leq i_0 < i_1 < \cdots < i_a \leq k \}$ we set $\tilde{C}_I := \tilde{C}_{i_0,i_1} \cap \tilde{C}_{i_0,i_2} \cap \cdots \cap \tilde{C}_{i_0,i_a}$. Then $C_I := \tilde{C}_I \setminus \bigcup \{ \tilde{C}_J \mid J \ \text{contains} \ I \ \text{with} \ \#J = a + 1 \}$ are submanifolds of codimension $l$ in $M$ that give the stratification of the cut locus $C(p)$ in Proposition 2.4. In terms of the coordinates $y_i$ we have
\[
\tilde{C}_l = \{ r \in U \mid 0 = y_{i_1} = \cdots = y_{i_0} \geq y_l(r), l \not\in I \} \ (0 = i_0 \in I), \]
\[
C_l = \{ r \in U \mid 0 = y_{i_1} = \cdots = y_{i_0} > y_l(r), l \not\in I \} \ (0 = i_0 \in I), \]
\[
(2.8) \]
\[
\tilde{C}_l = \{ r \in U \mid 0 \leq y_{i_0} = \cdots = y_{i_0} \geq y_l(r), l \not\in I \} \ (0 \not\in I), \]
\[
C_l = \{ r \in U \mid 0 < y_{i_0} = \cdots = y_{i_0} > y_l(r), l \not\in I \} \ (0 \not\in I). \]

Now, for any tangent vector $u$ to $\tilde{C}_{i,j}$ at $q$ we see that $u$ is at the same spherical distance to $X_i$ and $X_j$. Indeed, taking a curve $s \mapsto x(s)$ in $\tilde{C}_{i,j}$ tangent to $u$, we have $x_i(x(s)) = ||F_i(x(s))|| = ||F_j(x(s))|| = x_j(x(s))$. Then by the first variation formula, it follows that $\langle u, X_i \rangle = \langle u, X_j \rangle$. By the same argument we have $\langle u, X_i \rangle \leq \langle u, X_j \rangle$ for $l \neq i, j$, namely, we have for the spherical distance
\[
\angle(u, X_i) = \angle(u, X_j) \leq \angle(u, X_l) \ \text{for} \ l \neq i, j. \]
It follows that $T_q C_{I_a} = \{ u \in T_q M \mid \langle u, X_{i_0} \rangle = \langle u, X_{i_1} \rangle = \cdots = \langle u, X_{i_a} \rangle \}$ and $V_0 = T_q C_{k+1} = T_q C_K$ with $K = \{0, 1, \ldots, k\}$.

Now take a section $N(\subset U)$ through $q \in C_{k+1}$ defined by $x_\alpha = \text{const.}$ $(\alpha = k+1, \ldots, n)$ in $U$, that is tangent to $\tilde{V}_1 = \langle X_1(q) - X_0(q), \ldots, X_k(q) - X_0(q) \rangle R = \langle \tilde{X}_1(q) - \tilde{X}_0(q), \ldots, \tilde{X}_k(q) - \tilde{X}_0(q) \rangle R$. Note that $T_q N$ is the orthogonal complement of $V_0 = T_q C_{k+1}$ in $T_q M$. Then we have

$$C_{0j} \cap N = \{ r \in N \mid 0 = y_j(r) > y_l(r), l \neq j \} \quad \text{for} \quad 1 \leq j \leq k,$$

$$C_{ij} \cap N = \{ r \in N \mid 0 < y_i(r) = y_j(r) > y_l(r) \} \quad \text{for} \quad 1 \leq i < j \leq k.$$

and also have the similar expressions for $\tilde{C}_I \cap N$ and $C_I \cap N$ as in (2.8). Then in terms of the local coordinates $(y_1, \ldots, y_k)$, $C(p) \cap N$ is a cone, and the tangent cone to $C(I) \cap N$ (resp. $C_I \cap N$) at $q$ is given by

$$\bigcup_{t \geq 0} t \{ u \in U_q N \mid \angle(X_{i_0}, u) = \cdots = \angle(X_{i_a}, u) \leq \angle(X_{i_l}, u), l \neq i_0, \ldots, i_a \},$$

$$\bigcup_{t \geq 0} t \{ u \in U_q N \mid \angle(X_{i_0}, u) = \cdots = \angle(X_{i_a}, u) < \angle(X_{i_l}, u), l \neq i_0, \ldots, i_a \}$$

respectively. It follows that the tangent cone to $N \cap C(p)$ at $q$ is nothing but the cone over $\tilde{S}_{k-2}(q)$ in $\tilde{V}_1 = T_q N$ as described by the above arguments. Therefore $N \cap C(p)$ is homeomorphic to the cone $\tilde{T}$ over $\tilde{S}_{k-2}(q)$ in $\tilde{V}_1$. Note that the above fact also holds for $r \in U \cap C_{k+1,q}$, and taking $U (\ni q)$ small if necessary, $U \cap C(p)$ is homeomorphic to the product of $\tilde{T}$ and an open $(n-k)$-disk, and the latter is homeomorphic to the cone over $S(q)$ in $T_q M$.

In the case of $k = 1$, $C(p) \cap U$ is a hypersurface of $M$ and is homeomorphic to open $(n-1)$-disk, that is the cone over $S^{n-2}$ in $U_q M$. \hfill \Box

**Remark 2.7.** We show that Theorem 2.5 does not hold without assuming the nondegeneracy condition. Let $(T, g_0)$ be the 3-dimensional flat torus obtained by identifying the opposite faces of the cube $A := [-10, 10] \times [-10, 10] \times [-10, 10]$ in $\mathbb{R}^3$, and we denote by $\phi : A \rightarrow T$ this identifying map. Then note that the tangent cut locus of the origin $o = \phi((0, 0, 0))$ coincides with the boundary $\tilde{C}$ of $A$, and the cut locus $C$ is given by $C = \phi(\tilde{C})$. Now there are exactly four minimal geodesics from the origin to any point in the segment $E = \{ \phi((10, 10, t)) \mid -10 < t < 10 \}$, and the cut locus around the point is given by four half planes gathering along the segment $E$. 


Now for any positive integer \( n \), let \( B_n \) (resp. \( B'_n \)) be the \( \frac{1}{2^{n+1}} \)-ball centered at \((1, 1, \tan \frac{1}{2^n})\) (resp. \((1, -1, \tan \frac{1}{2^n})\)). We denote by \( \frac{1}{2}B_n \) (resp. \( \frac{1}{2}B'_n \)) the \( \frac{1}{4^{n+1}} \)-ball with the same center as \( B_n \) (resp. \( B'_n \)). Now for an \( \epsilon > 0 \) small enough take a smooth function \( \chi_n \) (resp. \( \chi'_n \)) that is equal to \( \epsilon/2^n \) on \( \frac{1}{2}B_n \) (resp. \( \frac{1}{2}B'_n \)), vanishes outside \( B_n \) (resp. \( B'_n \)), and is nonincreasing along radii. Then setting \( g_n := (1 + \chi_n)g_0 \) (resp. \( g'_n := (1 + \chi'_n)g_0 \)), we get a new 3-dimensional almost flat torus. Note that \( s \to \phi(s, s, s\tan \frac{1}{2^n}) \) (resp. \( \phi(s, -s, s\tan \frac{1}{2^n}) \), \( 0 \leq s \leq 10 \), is not a \( g_n \) (resp. \( g'_n \))-minimal geodesic from \( o \) to \( q_n := \phi((10, 10, 10\tan \frac{1}{2^n})) \) (resp. \( \phi((10, -10, 10\tan \frac{1}{2^n})) \)). But still there are exactly three \( g_n \) (resp. \( g'_n \))-minimal geodesics from \( o \) to \( q_n \). Namely, \( q_n \) is a cut point of \( o \) and the cut locus around \( q_n \) with respect to the metric \( g_n \) (resp. \( g'_n \)) locally consists of three half planes \( P_1, P_2, P_3 \) (resp. \( P'_1, P'_2, P'_3 \)) gathering along the segment \( E \), where we set \( P_1 : y = 10, x > 10; P_2 : x = 10, y > 10; P_3 : x = y < 10 \) (resp. \( P'_1 : y = 10, x < 10; P'_2 : x = 10, y < 10; P'_3 : x = y > 10 \)).

Note that \( \{B_{2n}, B_{2n+2}\}_{n=1}^{\infty} \) are pairwise disjoint, and they are disjoint from the segment \( s \mapsto \phi(s, s, 0) \). We take a new 3-dimensional almost flat torus \( (T, \tilde{g}) \) given by
\[
\tilde{g} = (1 + \sum_n (\chi_{2n} + \chi'_{2n+1}))g_0.
\]
Then \( q := \phi(10, 10, 0) \) is again a cut point of \( o \) with respect to the deformed Riemannian metric \( \tilde{g} \) and in fact there are exactly four \( \tilde{g} \)-minimal geodesics from the origin, that are also \( g_0 \)-minimal geodesics. It follows that the cut locus \( S(q) \) in \( U_1T \) consists of four half great circles joining two antipodes, and the cone \( P \) over \( S(q) \) consists of four half planes gathering along a segment. However, the cut locus of the origin with respect to \( \tilde{g} \) is not homeomorphic to \( C \), since both of the sequences \( \{q_{2n}\} \) and \( \{q_{2n+1}\} \) converges to \( q \) and the local structure of the cut locus around \( \{q_{2n}\} \) is different from the one around \( \{q_{2n+1}\} \).

**Remark 2.8.** Even for real analytic metrics, the assumption of nondegeneracy seems necessary in Theorem 2.5, as is suggested by the following example: Take the functions \( f_1 := z + c, f_2 := -z + c, f_3 := y + x^2 + c, f_4 := -y + x^2 + c \) defined on \( \mathbb{R}^3 = \{ (x, y, z) \mid x, y, z \in \mathbb{R} \} \), where \( c \) is a positive constant. Put \( f := \min\{f_1, f_2, f_3, f_4\} \) and consider the set \( C \) of points in \( \mathbb{R}^3 \).
such that there are at least two $1 \leq i < j \leq 4$ with $f = f_i = f_j$. Although $f$ is precisely not a distance function, the set $C$ is similar to the cut locus. Note that the origin $(0,0,0)$ belongs to $C$ with $f = f_i = c(1 \leq i \leq 4)$, and that gradient vectors of $f_i$ at the origin are given by $\pm \partial/\partial z, \pm \partial/\partial y$. In this case $C$ consists of the 4 half parabolic cylinders (given by $y = z - x^2, z \leq 0$; $y = z + x^2, z \geq 0$; $y = -z - x^2, z \geq 0$; $y = -z + x^2, z \leq 0$) and 2 cusp regions $x \geq \sqrt{|y|}, x \leq -\sqrt{|y|}$ in the $xy$-plane. On the other hand, $S((0,0,0))$ consists of 4 great half circles of $S^2$ joining the antipodes. Hence the cone over $S((0,0,0))$ is not homeomorphic to $C$.

Now we set $f := d_p|C_{k+1,q}$ and give the gradient vector $\nabla f$ of $f$ at $x \in C_{k+1,q}$. Indeed, let $u \in T_x C_{k+1,q}$ and $s \mapsto x(s)$ be a curve in $C_{k+1,q}$ with $\dot{x}(0) = u$. Then noting that $f(x(s)) = \|F_i(x(s))\| = x_i(x(s))$ for any $0 \leq i \leq k$, we obtain by the first variation formula (see e.g., [16])

$$\langle \nabla f, u \rangle = \frac{d}{ds} \|F_i(x(s))\|_{s=0} = -\langle X_i, u \rangle,$$

where $\langle X_i, u \rangle$ is independent of $i$ by the definition of $C_{k+1}$. It follows that $\nabla f(x)$ is the orthogonal projection of any $-X_i$ to $T_x C_{k+1,q}$ for $i = 0, \ldots, k$. Therefore, $x$ is a critical points of $f = d_p|C_{k+1,q}$ in usual sense if and only if all of $X_i$ ($i = 0, \ldots, k$) are orthogonal to $C_{k+1,q}$ and are located in a $(k - 1)$-dimensional great sphere of $U_q M$.

Now how about the Hessian $D^2 f(x)$ of $f$ at a critical point $x \in C_{k+1,q}$ of $f$? Let $u$ and $s \mapsto x(s)$ be as before. Then we have $D^2 f(u,u) = \frac{d^2}{ds^2} \|F_i(x(s))\|_{s=0}$ for each $i$ ($0 \leq i \leq k$). Take a variation of $\gamma_i$ given by

$$\alpha(t, s) := \exp_p t \frac{l}{l} F_i(x(s)), \quad (0 \leq t \leq l := d(p, x), -\epsilon \leq s \leq \epsilon).$$

Then the variation vector field is a unique Jacobi field $Y_i(t)$ along $\gamma_i$ with $Y_i(0) = 0$ and $Y_i(l) = u$. Note that $Y_i$ is perpendicular to $\gamma_i$. Then we get by the second variation formula (see e.g., [16])

$$D^2 f(u,u) = \int_0^l \{ \langle \nabla_{\gamma_i} Y_i(t), \nabla_{\gamma_i} Y_i(t) \rangle - \langle R(Y_i(t), \gamma_i(t)) \gamma_i(t), Y_i(t) \rangle \} dt$$

$$- \langle X_i, \nabla_u \dot{x}(s) \rangle = \langle u, \nabla_{\gamma_i} Y_i(l) \rangle - \langle X_i, \nabla_u \dot{x}(s) \rangle,$$

where we set $l = d(p, x)$. 

Now, if a point \( p \) of a compact Riemannian manifold \((M, g)\) admits no conjugate points along all geodesics emanating from \( p \) (e.g., for any point of a nonpositively curved manifold), then the structure of \( C(p) \) may be expressed in terms of the Dirichlet domain of the universal covering space \( \tilde{M} \) of \( M \) with the induced Riemannian metric \( \tilde{g} \). We briefly explain this case (see also [15]). Let \( \pi : \tilde{M} \to M \) be the covering projection, and set \( \pi^{-1}(p) = \{ \tilde{p} = \tilde{p}_0, \tilde{p}_1, \ldots, \tilde{p}_N, \ldots \} \), that may be identified with the deck transformation group \( \Gamma = \{ g_0 = e, g_1, \ldots, g_N, \ldots \} \) via \( \tilde{p}_i = g_i(\tilde{p}_0) \), where \( e \) denotes the identity. Note that \( \tilde{p}_i (i = 0, 1, \ldots) \) are poles, namely \( \exp_{\tilde{p}_i} : T_{\tilde{p}_i}M \to M \) are diffeomorphisms, and for any \( \tilde{q} \in \tilde{M} \) there exists a unique minimal geodesic joining \( \tilde{p}_i \) and \( \tilde{q} \). We fix \( \tilde{p} \) as a base point. Then the cut locus \( C(p) \) may be described as follows: Let \( \Delta_{\tilde{p}} \) be the Dirichlet domain of \( \Gamma \). Namely,
\[
\Delta_{\tilde{p}} = \bigcap_{g(\neq e) \in \Gamma} \{ H_{\tilde{p}, g(\tilde{p})} \mid i = 1, 2, \ldots \}
\]
with \( H_{\tilde{p}, \tilde{q}} = \{ \tilde{r} \in \tilde{M} \mid \tilde{d}(\tilde{p}, \tilde{r}) < \tilde{d}(\tilde{q}, \tilde{r}) \} \). Then we have \( C(p) = \pi(\partial \Delta_{\tilde{p}}) \).

Since \( M \) is compact, it suffices to consider a finite number of \( g \in \Gamma \) such that \( \tilde{d}(\tilde{p}, g\tilde{p}) \leq d(M) \) so that we may write \( \Delta_{\tilde{p}} = \bigcap_{j=1, \ldots, N} H_{\tilde{p}, g_j(\tilde{p})} \).

This also means that the distance function \( d_p \) is a min-type function in the sense of [4], namely we have \( d_p(q) := \min \{ \tilde{d}(q, g_1\tilde{p}), \ldots, \tilde{d}(q, g_N\tilde{p}) \} \), where \( q \in \pi^{-1}(q) \). Let \( q \in C(p) \) be a cut point of order \( k + 1 \) and \( \gamma_0, \ldots, \gamma_k \) be the minimal geodesics from \( p \) to \( q \). As before we set \( X_i = \gamma_i(l) \in U_qM \) (\( i = 0, \ldots, k \)). Take the lift of \( \gamma_0 \) emanating from \( \tilde{p} \) with respect to the universal covering \( \pi \), and we denote by \( \tilde{q} \) the end point of the lift. Then there exist \( g_{i_0} = e, g_{i_1}, \ldots, g_{i_k} \) in \( \Gamma \) such that \( \gamma_j \) is expressed as the projection of a unique minimal geodesic \( \tilde{\gamma}_j \in \tilde{M} \) joining \( \tilde{g}_i \tilde{p} \) and \( \tilde{q} \) (\( j = 0, \ldots, k \)) with \( \tilde{d}(g_{i_j}\tilde{p}, \tilde{q}) = \tilde{d}(\tilde{p}, \tilde{q}) = l \). Now we set for \( I = \{ i_1, \ldots, i_k \} \)
\[
\tilde{C}_I := \{ \tilde{r} \in \tilde{M} \mid \tilde{d}(g_{i_j}\tilde{p}, \tilde{r}) = \tilde{d}(\tilde{p}, \tilde{r}) < \tilde{d}(g_{i_j}\tilde{p}, \tilde{r}) \text{ for any } i \in I, j \notin I \}.
\]
Then \( g_{i_0} = e, g_{i_1}, \ldots, g_{i_k} \) \( g_{i_0} \) are chosen in common in the connected component containing \( q \) of the set of cut points of order \( k+1 \), and we have \( \pi(\tilde{C}_I) \subset C(p) \). If we take the lift of \( \gamma_j \) instead of \( \gamma_0 \) in the above, then we have \( g_{i_j}^{-1} \tilde{q} \) and \( g_{i_j}^{-1} \tilde{C}_I \) instead of \( \tilde{q} \) and \( \tilde{C}_I \), respectively. Now suppose that all cut points of \( p \) are nondegenerate. Then \( \tilde{X}_0, \ldots, \tilde{X}_k \) are in general position in \( T_{\tilde{q}}\tilde{M} \), where
we set $\tilde{X}_i = -\dot{\gamma}_i(l)$. Then if $\tilde{C}_I \neq \emptyset$, $\tilde{C}_I$ (resp. $\pi(\tilde{C}_I)$) is a submanifold of dimension $n - k$ of $\tilde{M}$ (resp. $M$), and the cut locus $C(p)$ is stratified by the strata $\pi(\tilde{C}_I)$ by the same arguments as above.

**Remark 2.9.** (1) We suspect whether Riemannian metrics such that all cut points of $p$ are nondegenerate are open and dense in the set of all Riemannian metrics satisfying the condition (C) at $p$. For two-dimensional case, V. Gershkovich asserts that the above assertion holds ([3], [5]). This also follows applying a recent result of [13] to our situation. We suspect that their approach is helpful for the above problem.

(2) For 3-dimensional case, approximating a Riemannian metric of A. Weinstein ([18]) in §1 by M. Buchner’s cut stable metrics, we get Riemannian metrics satisfying the condition (C) for $p \in M$ such that all cut points of $p$ are nondegenerate ([2], this holds up to dimension 6).

(3) M. van Manen pointed out that Y. Yodomin has considered cones over the $(n - 2)$-skelton of simplices for central sets in $R^n$ in [19] that is related to our cut locus case.

### 3. Morse theory for distance functions

First we recall the notion of a critical point of the distance function $d_p$ in the angle sense (see §1 and [9], [6]). A point $q \in M$ is said to be a critical point of $d_p$, if for any unit tangent vector $v \in T_q M$ there exists a minimal geodesic $\gamma$ from $p$ to $q$ such that $\angle(v, -\dot{\gamma}(l)) \leq \pi/2$ holds with $l = d(p, q)$. Note that any critical point $q$ of $d_p$ is a cut point of $p$. Now we set

$$\Gamma(q) := \{-\dot{\gamma}(l) \in U_q M \mid \gamma : [0, l] \to M; \text{ minimal geodesic from } p \text{ to } q\}$$

and define the set $\tilde{\Gamma}_q \subset T_q M$ as the convex hull of $\Gamma(q)$. Then, the above condition for $q$ to be a critical point of $d_p$ means that $\tilde{\Gamma}_q$ contains the origin 0 of $T_q M$.

**Definition 3.1.** For any critical point $q$ of $d_p$ we define its degree as the dimension of the (vector) subspace spanned by $\Gamma(q)$.

Recall that we set $X_i := -\dot{\gamma}_i(l), l = d(p, q)$ for minimal geodesics $\gamma_i (i = 0, \cdots, k)$ joining $p$ to $q$. If $C(p)$ is nondegenerate and $q \in C_{k+1}$, then $X_0, \cdots, X_k$ span a $k$ or $(k + 1)$-dimensional (vector) subspace $V$. If $q$ is a
critical point, then $V$ is a subspace of dimension $k$, since the convex hull $\hat{\Gamma}_q$ contains 0. Therefore in this nondegenerate case, degree of $q$ is equal to $k$.

Now, suppose there exist no critical points in the annulus $R(r_1, r_2) := d_p^{-1}([r_1, r_2]) = \{x \in M \mid r_1 \leq d_p(x) \leq r_2\}, 0 < r_1 < r_2$. Then the isotopy lemma asserts the following: All the levels $d_p^{-1}(r), r_1 \leq r \leq r_2$ are homeomorphic to each other, and $R(r_1, r_2)$ is homeomorphic to the product $d_p^{-1}(r_1) \times [r_1, r_2]$. This may be proved as in usual Morse theory by considering a gradient-like vector field of $d_p$ (see e.g., [9], [6], [10] for more detail).

Next, suppose that the cut locus $C(p)$ of $p$ consists of nondegenerate cut points. Then $C_{k+1}$ is a submanifold of dimension $n - k$ of $M$, and $d_p$ is a smooth function when restricted to each connected component $C_{k+1};q$. First we will be concerned with the relation between the two kinds of critical points of the distance function, which was also obtained by V. Gershkovich and H. Rubinstein ([4]).

**Lemma 3.2.** Suppose $r \in C_{k+1};q$ is a critical point of $d_p$ in the angle sense. Then $r$ is a critical point of the smooth function $f := d_p \mid C_{k+1};q$ in the usual sense. If $r \in C_{2};q$ is a critical point of the smooth function $f := d_p \mid C_{2};q$, then $r$ is a critical point of $d_p$ in the angle sense.

**Proof.** Suppose $r \in C_{k+1};q$ is a critical point of $d_p$ in the angle sense and let $\gamma_i$ ($i = 0, \ldots, k$) be a minimal geodesic parametrized by arclength joining $p$ to $r$. To see that $r$ is a critical point of $f = d_p \mid C_{k+1};q$ in the usual sense, by (2.9) it suffices to show that $\alpha = \langle u, X_i \rangle$ is equal to 0 for any $u \in T_r C_{k+1};q$, where $X_i = -\gamma_i'(l), l = d(p, q)$. Recall that $\alpha$ is independent of $i$. Then from the assumption we may choose $a_i \geq 0$ ($i = 0, \ldots, k$) with $\sum a_i = 1$ such that $\sum a_i X_i = 0$, and it follows that

$$\alpha = \sum a_i \alpha = \langle u, \sum a_i X_i \rangle = 0.$$ 

Next suppose $r$ is a critical point of $d_p \mid C_{2};q$ in the usual sense. Note that $\dim C_{2};q = n - 1$. We have unit tangent vectors $X_0, X_1$ at $r$ of minimal geodesics $\gamma_0, \gamma_1$ joining $p$ to $r$, respectively. Then $X_0, X_1$ are different unit vectors perpendicular to the hypersubspace $T_r C_{2};q$ of $T_r M$ by (2.6), and therefore should satisfy $X_0 + X_1 = 0$. It follows that $r$ is a critical point of $d_p$ in the angle sense. \hfill $\Box$
In general for $k > 1$, critical points of $d_p | C_{k+1,q}$ are not necessarily critical points of $d_p$ in the angle sense. For instance, if $k = n$ then $\dim C_{n+1,q} = 0$ and every $r \in C_{n+1}$ are critical points in the usual sense. However, $r$ may not assume local maximum of $d_p$ (see Figure 1).

Next we define the notion of nondegeneracy for a distance function as follows:

**Definition 3.3.** Suppose the cut locus $C(p)$ consists of nondegenerate cut points of $p$ under the condition (C). Then we call the distance function $d_p$ nondegenerate, if the following hold:

1. All critical points of $d_p$ in the angle sense are isolated, and $\hat{\Gamma}_q$ contains 0 in its interior for each critical point $q$ of $d_p$. Namely, for a critical point $q \in C_{k+1}$ in the angle sense we may find (unique) $a_i > 0$ $(0 \leq i \leq k)$ with $\sum a_i = 1$ such that $\sum a_i t_i = 0$ holds.

2. For any critical point $q \in C_{k+1}$ of $d_p$ in the angle sense, $q$ is a nondegenerate critical point in the usual sense of $f := d_p | C_{k+1}$, the distance function restricted to the stratum $C_{k+1}$ of $C(p)$.

The last condition in (1) above means that for a critical point $q \in C_{k+1}$, the cone $\bigcup_{t \geq 0} t\hat{\Gamma}_q$ forms a vector subspace of dimension $k = \dim \hat{\Gamma}_q$ that is the orthogonal complement of $T_qC_{k+1}$ in $T_qM$. Note that this property holds for critical points of $d_p$ which are cut points of order 2. For instance,
the distance functions \(d_p\) of generic flat tori are nondegenerate for arbitrary points \(p\).

In the following we want to give a normal form of nondegenerate distance function \(d_p\) in a neighborhood \(U\) of a critical point \(q \in C(p)\) of \(d_p\) in the angle sense by choosing carefully a kind of local coordinates around \(q\) adapted to the local structure of \(C(p)\). For \(q \in C_{k+1}\) recall that \(x_i(r) = \|F_i(r)\|, X_i(r) = -\nabla x_i(r) (r \in U, \ i = 0, \ldots, k)\) were given in (2.7), and we have \(X_i = X_i(q)\). Recall also that a small neighborhood \(U \cap C(p)\) of \(q\) in the cut locus is homeomorphic to the cone over \(S\) in \(T_qM\), where \(S\) is the cut locus of \(\{X_0, \ldots, X_k\}\) in the unit sphere \(U_qM\) (see Theorem 2.5). Then \(U \setminus C(p)\) is divided into \(k+1\) components \(D_0, \ldots, D_k\) where each \(D_i\) contains the direction \(X_i\), namely, contains \(\gamma_i((l-\epsilon,l))\) with \(l = d(p,q)\), where \(\{\gamma_i\}_{0 \leq i \leq k}\) are minimal geodesics from \(p\) to \(q\) and \(\epsilon > 0\) is sufficiently small. \(D_i\) \((i = 0, \ldots, k)\) is indeed given by

\[
D_i = \{r \in U \mid x_i(r) < x_l(r) \text{ for any } l \neq i, 0 \leq l \leq k\},
\]

and note that we have \(d_p(r) = x_i(r)\) for \(r \in D_i\). Then it follows that \(\bar{D}_i \cap \bar{D}_j = \bar{C}_{i,j} \cap U\) for \(i < j\), where \(\bar{D}_i\) denotes the closure of \(D_i\) and \(\bar{C}_{i,j}\) is given just before the formula (2.8).

Now let \(d_p\) be nondegenerate distance function in the sense of Definition 3.3 and \(q\) a critical point of \(d_p\) in the angle sense. To make our approach more understandable we begin with the simplest case where \(q\) is a critical point of order 2, namely \(q \in C_2\). Then we have \(X_0 + X_1 = 0\), and \(S\) is a great hypersphere \(S^{n-2}\) of \(U_qM\). \(U \setminus C_{2,q}\) is divided into domains \(D_0\) and \(D_1\). For every point \(r \in D_i\) \((i = 0, 1)\) take a unique minimal geodesic \(\gamma\) joining \(p\) to \(r\), and denote by \(r_1 = \gamma_x(0) \in C_{2,q}\) the cut point of \(p\) along \(\gamma\).

Note that \(l_r = d(p, r)\) is the cut distance along the geodesic \(\gamma\) and depends smoothly on \(r \in D_i\), since \(\gamma\) intersects \(C_{2,q}\) transversely. Now we set

\[
z := l_r - d(p, r) = i_p(\gamma_r(0)) - d_p(r) > 0.
\]

For \(r \in C_{2,q}\) we set \(r_1 = r\) and \(z = 0\). Then any \(r \in \bar{D}_i, i = 0, 1\), may be uniquely expressed as \(r = (r_1, z) \in C_{2,q} \times \mathbb{R}^+\). Since \(q = (q, 0)\) is a critical point of \(d_p\), \(q\) is a critical point of \(f := d_p | C_{2,q}\) by Lemma 3.2. Since \(d_p\) is nondegenerate, \(q\) is a nondegenerate critical point of \(f\) and taking local
coordinates \((x_2, \ldots, x_n)\) around \(q \in C_{2,q}\) we may write
\[
d_p(r) = d_p(q) + x_2^2 + \cdots + x_n^2 - z.
\]
Therefore in this case, we may define the index of \(d_p\) at \(q\) as the sum of the index of \(f := d_p|_{C_{2,q}}\) at \(q\) and \(1\), where recall that \(1\) is the degree of the critical point \(q\).

Now we turn to a critical point \(q \in C_{k+1}\) of \(d_p\) for general \(k \geq 2\). Then \(\{X_0, X_1, \ldots, X_k\}\) span a \(k\)-dimensional subspace of \(T_qM\) orthogonal to the subspace \(T_qC_{k+1,q}\) of dimension \(n - k\), and the convex hull of \(\{X_0, X_1, \ldots, X_k\}\) contains the origin 0 of \(T_qM\) in its interior. Now for any proper subset \(I\) of \(K := \{0, 1, \ldots, k\}\) with \(\#I \geq 2\) we set
\[
C_I = \{r \in C(p) | x_i(r) = x_j(r) < x_l(r) \text{ for any } i, j \in I \text{ and } l \not\in I\},
\]
and \(\bar{C}_I\) denotes the closure of \(C_I\) (see also (2.8)). For \(r \in \bar{C}_I\) we denote by \(T_rC_I := \{u \in T_r(M) | \langle u, X_i(r) \rangle = \langle u, X_j(r) \rangle \text{ for any } i, j \in I\} (= \lim_{r_i \in C_I} T_rC_I)\) the tangent space to \(C_I\) at \(r\). Then \(f_I := d_p|_{C_I}\) is a smooth function and \(\nabla f_I\) denotes the gradient vector of \(f_I\). For \(r \in \bar{C}_I\) we also use the notation \(\nabla f_I(r)(\in T_rC_I)\) that is given as \(\lim_{r_i \in C_I} \nabla f_I(r_i)\).

Then we have the following lemmas.

**Lemma 3.4.** Let \(q \in C_{k+1}\) be a critical point of nondegenerate distance function \(d_p\). Let \(I \subset \{0, 1, \ldots, k\}\) with \(\#I \geq 2\) be a proper subset. Now for any \(i \in I\) and \(r \in \bar{C}_I\) we denote by \(X^\top_I(r)\) the orthogonal projection of \(X_i(r)\) to \(T_rC_I\). Then we have
\[
X^\top_I(r) = -\nabla f_I(r) \text{ for any } i \in I, r \in \bar{C}_I
\]
and we denote the above vector also by \(X^\top_I(r)\). Furthermore, there exist an open neighborhood \(U\) around \(q\) and \(\delta > 0\) such that
\[
\|\nabla f_I(r)\| \geq \delta \text{ on } U \cap C_I.
\]

**Proof.** For any curve \(t \to x(t)\) in \(\bar{C}_I\) emanating from \(r\) we have \(d_p(x(t)) = f_I(x(t)) = x_i(x(t))\) for any \(i \in I\). Differentiating this equation with respect to \(t\) at \(t = 0\), it follows that
\[
\langle \nabla f_I(r), \dot{x}(0) \rangle = \langle \nabla x_i(r), \dot{x}(0) \rangle = -\langle X_i(r), \dot{x}(0) \rangle = -\langle X^\top_I(r), \dot{x}(0) \rangle
\]
from which the first assertion follows. For the second assertion it suffices to show that \(X^\top_I(q) \neq 0\) at the critical point \(q\) of \(d_p\). Indeed, otherwise for a
fixed \( i_0 \in I = \{i_0, \ldots, i_a \} \) we see that \( \{X_i - X_{i_0} \}_{i \in I \setminus \{i_0 \}} \) forms a basis of a subspace \((T_q C_I)^\top \) of dimension \( a \). Since \( q \) is a critical point of \( d_p \), \( \{X_i \}_{i \in I} \) span an \( a \)-dimensional subspace of \( T_q M \). However this contradicts the non-degeneracy condition that the faces of the convex hull of \( \{X_0, X_1, \ldots, X_k \} \) cannot contain the origin 0 of \( T_q M \). \( \square \)

Note that the orthogonal projection of of \( X_i \) to \( T_q C_{k+1} \) is equal to zero since \( q \) is a critical point of \( d_p \), but the lemma asserts that the orthogonal projection \( X_i^\top \) of \( X_i(i \in I) \) to \( T_q C_I \) never vanishes for any proper subset \( I \) of \( \{0, 1, \ldots, k \} \).

**Lemma 3.5.** Under the assumption of the previous lemma there exist positive constants \( a_i \ (i = 0, \ldots, k) \) with \( \sum a_i = 1 \) such that

\[
(3.3) \quad -(\sum_{i \in I} a_i \langle X_i^\top(q), X_i^\top(q) \rangle) = \sum_{j \not\in I} a_j \langle \nabla f_l(q), -X_j(q) \rangle.
\]

Indeed, we have \( \sum_{0 \leq l \leq k} a_l X_l(q) = 0 \) at \( q \) for some \( a_l > 0 \ (l = 0, \ldots, k) \) with \( \sum a_l = 1 \) by Definition 3.3. Then considering the orthogonal projection of \( \sum a_i X_i(q) \) to \( T_q C_I \), we see that

\[
\sum_{i \in I} a_i \langle X_i(q), \nabla f_I(q) \rangle = -\sum_{i \in I} a_i \langle X_i^\top(q), X_i^\top(q) \rangle + \sum_{j \not\in I} a_j \langle \nabla f_l(q), X_j(q) \rangle
\]

vanishes by the previous lemma, and (3.3) follows.

We also note that there exist no critical points of \( d_p \) (and also of \( f_I \)) except \( q \) in \( U \) by the nondegeneracy condition.

Now suppose \( r \in D_{i_0}(0 \leq i_0 \leq k) \). First take a unique minimal geodesic \( \gamma_r \) joining \( p \) to \( r \), and denote by \( r_1 = \gamma_r(l_r) \in C(p) \) the cut point of \( p \) along \( \gamma_r \), where \( l_r = d(p, r_1) = i_p(\gamma_r(0)) \) is the cut distance to \( p \) along \( \gamma_r \). Then \( r_1 \) lies in some \( \tilde{C}_{i_0,i_1} \) that is a subset of the boundary of \( D_{i_0} \). Note that here we do not assume that \( i_0 < i_1 \). For generic \( r \in D_{i_0} \) we have \( r_1 \in C_{i_{0},i_1} := \{r \in U \mid x_{i_0}(r) = x_{i_1}(r) < x_l(r) \text{ for any } l \neq i_0, i_1 \} \) for some \( i_1 \), and \( \gamma_r \) intersects \( C_{i_0,i_1} \) transversely at \( r_1 \). Then we denote the above \( r_1 \) by \( r_{i_1} \), and set \( z_{i_0} := l_r - d(p, r)(> 0) \). By Lemma 3.4 the gradient vector \( \nabla f \) of \( f(= f_{i_0 i_1}) := d_p|C_{i_0,i_1} \) at \( r_{i_1} \) is given by \( -X_{i_0} = -X_{i_1} \) which is the orthogonal projection of \( -X_{i_0} \) (or \( -X_{i_1} \)) to \( T_{r_1} C_{i_0,i_1} \), and does not vanish. Since \( r_{i_1} \) is not a critical point of \( f \), we may move \( r_{i_1} \) along the trajectory of \( \nabla f \) to a point \( r_2 \in \tilde{C}_{i_0,i_1,i_2} \) in general in the following manner. If \( k = 2 \),
then $i_2$ is uniquely determined, and we have $a_{i_0}X_{i_0} + a_{i_1}X_{i_1} + a_{i_2}X_{i_2} = 0$

at $q$ for some $a_{i_j} > 0$. Since we have $f = x_{i_0}(= \|F_{i_0}\|) = x_{i_1}(= \|F_{i_1}\|)$

and $\langle \nabla f(r), -X_{i_0}(r) \rangle = \langle \nabla f(r), -X_{i_1} \rangle = \|\nabla f(r)\|^2 \geq \delta^2 > 0$ by Lemma

3.4, $x_{i_0} = x_{i_1}$ increases along the trajectory of $\nabla f$. On the other hand, $x_{i_2}$

decreases along the trajectory, because from (3.3) we have at $q$

$$\langle -X_{i_2}(q), \nabla f(q) \rangle = -(a_{i_0} + a_{i_1})\|X^\top\|^2(q)/a_{i_2} < 0,$$

and $\langle -X_{i_2}(r), \nabla f(r) \rangle < 0$, $r \in U$ for small $U$.

If $k > 2$, then from Lemma 3.5 there exists at least one index $i_2$ ($0 \leq i_2 \leq k$) different from $i_0, i_1$ such that $\langle X_{i_2}, \nabla f \rangle > 0$. It follows again that $x_{i_0} = x_{i_1}$ increases while $x_{i_2}$ decreases for such an $i_2$ along the trajectory of $\nabla f$.

Therefore, along the trajectory we reach the point $r_2$ such that the values of $x_{i_0}, x_{i_1}, x_{i_2}$ are equal, while the value of other $x_j(= \|F_j\|)$ ($j \neq i_0, i_1, i_2$) is

not less than this value, namely a point of $\tilde{C}_{i_0,i_1,i_2}$ for some $i_2$. Note that for

a starting point $r \in D_{i_0}$ the stratum $C_I$, $I \supset \{i_0, i_1, i_2\}$ of $C(p)$ containing

the above $r_2$ is uniquely determined and the trajectory is transversal to $C_I$

at $r_2$ unless $I = K$. We have $l = 2$ for generic $r$, and then we denote the

above $r_2$ also by $r_{i_2}$. Let $z_{i_2}$ the parameter value of $r_2$ of the trajectory,

namely $z_{i_2} = d(p, r_{i_2}) - d(p, r_{i_1})( > 0)$, which is also uniquely determined

from $r$ and depends smoothly on $r$. Now for generic $r \in D_{i_0}$, repeating

this procedure $k$ times, we may have $r_{i_1}, \ldots, r_{i_k}$ and $z_{i_0}, \ldots, z_{i_{k-1}},$

where $r_{i_k} \in C_{k+1, q}$ and $z_{i_k} = d(p, r_{i_k}) - d(p, r_{i_{k-1}})( > 0)$. Recall that we have

local coordinates $(x_{k+1}, \ldots, x_n)$ of $C_{k+1}$ around $q$ adapted to the smooth

Morse function $f := d_p | C_{k+1, q}$.

On the other hand, if $r \in D_{i_0}$ moves along $\gamma_r$ to $r_1 \in C_I, I = \{i_0, \ldots, i_a\}$

($a < k$), then $\gamma_r$ intersects $C_I$ transversely. In this case, we set $r_{i_1} = \cdots = r_{i_a} := r_1$, and $z_{i_0} = d(p, r_1) - d(p, r) = l_r - d(p, r) > 0$, $z_{i_1} = \cdots = z_{i_{a-1}} = 0$.

We make a similar arrangement for the case where the starting point $r$

(resp. $r_j$) belongs to $C_I$ ($a < k$) (resp. $C_I$ ($j < a$)), and we get $r_{i_1}, \ldots, r_{i_k}$

and $z_{i_0}, \ldots, z_{i_{k-1}}(\geq 0)$ for every $r \in D_{i_0}$, the closure of $D_{i_0}$. For instance,

for $r \in C_I$ we set $r_{i_1} = \cdots = r_{i_a} := r$ and $z_{i_0} = \cdots = z_{i_{a-1}} = 0$, $z_{i_a}$

does not change along the trajectory.

Then setting $D_{i_0, I} (I = \{i_0, i_1, \ldots, i_a\} \subset K = \{0, 1, \ldots, k\})$ as the set of points $r \in D_{i_0}$ such that $r$ reaches the point $r_1$ of $C_I$ along the geodesic $\gamma_r$ in

the above manner, $D_{i_0}$ is stratified by $D_{i_0, I}$'s. Indeed, $D_{i_0, I}$ is a submanifold
of codimension $a - 1$ and is independent of the order of $\{i_1, \ldots, i_a\}$, in fact we have $D_{i_0, I} = (\bar{D}_{i_0, i_1} \cap \cdots \cap \bar{D}_{i_0, i_l}) \setminus C(p)$. It follows that

$$U = \{C(p) \cap U\} \cup \{\cup_{0 \leq i_0 \leq k} (\cup I \ni i_0 D_{i_0, I})\},$$

namely, $U$ is stratified into submanifolds given by $\{D_{i_0, I}\}$ and the stratification $\{C_I\}$ of $C(p)$ given in $\S 2$. The boundary $\partial D_{i_0, I}$ of $D_{i_0, I}$ consists of $C_J \subset C(p)$ with $J \supset I$ and $D_{i_0, I}$ with $J \supset I$.

Note that on $D_{i_0}$ we have the smooth unit vector field $-X_{i_0} = \nabla d_p$ which is transversal to the boundary, and on a stratum $C_I \subset C(p)$ where $I$ is a proper subset of $\{0, 1, \ldots, k\}$ with $\# I \geq 2$ we have the nonvanishing smooth vector field $-X_I^T = \nabla f_I$ transversal to the boundary. For $I$ with $\# I \geq 2$ we also write $D_I = C_I$ in the above notation. Then $r_{i_1}, \ldots, r_{i_k}, z_{i_0}, \ldots, z_{i_{k-1}}$ mentioned above are obtained by the successive trajectories of these vector fields on $D_I$. Indeed, for $\emptyset \neq I_0 \subsetneq I_1 \subsetneq \cdots \subsetneq I_l \subsetneq K := \{0, 1, \ldots, k\}$ we denote by $D_{I_0, I_1, \ldots, I_l}$ the set of points $r \in D_{I_0}$ that reach the points $q$ to the Morse function $f := d_p | C_{k+1}$, and we set $x_{k+j}(r) := x_{k+j}(\pi(r))$. Then $D_{I_0, I_1, \ldots, I_l}$ is a submanifold of codimension $k - l - 1$ and its closure consists of $D_{I_0, \ldots, I_{l'}} (l' \leq l)$ with $I_0 \subset I_0 \subset I_1 \subset \cdots \subset I_{l'} \subset I_l$. Once $D_{I_0, I_1, \ldots, I_l}$ is fixed, $(x_{k+1}, \ldots, x_n, z_{i_0}, \ldots, z_{i_{k-1}})$ are uniquely determined for $r \in D_{I_0, I_1, \ldots, I_l}$ and smooth where precisely $(l + 1)$ of $z_i$’s are positive. Note that

$$D_{I_0, I_1, \ldots, I_l} \ni r \mapsto (x_{k+1}(r), \ldots, x_n(r), z_{i_0}(r), \ldots, z_{i_{k-1}}(r))$$

is an embedding into $\mathbb{R}^n$, since $r \mapsto (r_{i_1}, z_{i_0}) \mapsto (r_{i_2}, z_{i_0}, z_{i_1}) \mapsto \cdots \mapsto (r_{i_k}, z_{i_0}, \ldots, z_{i_{k-1}})$ define an embedding at each stage. $U$ is stratified into submanifolds $D_{I_0, I_1, \ldots, I_l}$ and $D_K := C_{k+1} \cap U$ (corresponding to $l = -1$), and above embeddings are piecewise smoothly extended to the whole $U$. Thus we have “local coordinates” $(x_{k+1}, \ldots, x_n, z_1, \ldots, z_k)$ around $q$, where $(z_1, \ldots, z_k)$ is given by $(z_{i_0}, \ldots, z_{i_{k-1}})$ when restricted to $D_{I_0, I_1, \ldots, I_l}$ and Lipschitz on $U$.

Now recalling the local structure around $q$ of $M$ given by the cone structure of the cut locus mentioned above, we see that for a fixed point $r \in C_{k+1} \cap U$ with $z_1 = \ldots = z_k = 0$, the set defined by the equation $z_1 + \ldots + z_k \leq \delta$
dimension of procedure of defining \((z_1, z_2, \ldots, z_k)\). First for a fixed \(r \in C_{k+1} \cap U\), note that the equation \(z_1 = \cdots = z_{k-1} = 0, z_k = \delta\) represents \(k\) points (i.e., 0-disks) \(r_{ik} \in C_{K\{ik\}}\) with \(d(r, r_{ik}) = \delta\) such that \(r_{ik}\) moves to \(r\) along the trajectory of \(\nabla f_{K\{ik\}}\) in \(C_{K\{ik\}}\). Next the intersection of the set \(z_1 = \cdots = z_{k-2} = 0, z_{k-1} + z_k = \delta\) with \(C_{K\{i_{k-1}, ik\}}\) is a curve (i.e., 1-disk) joining \(r_{ik-1}\) and \(r_{ik}\). Then \(z_1 = \cdots = z_{k-2} = 0, z_{k-1} + z_k = \delta\) defines a family of curves corresponding to the 1-skeleton of the triangulation of \(\tilde{S}_{k-2}(q)\) given in Lemma 3.6. Repeating the procedure we see that the intersection of the set \(z_1 + \cdots + z_k = \delta\) with each \(\tilde{D}_{l_0}\) is a \((k-1)\)-disk, and that the set given by \(z_1 + \cdots + z_k = \delta\) is Lipschitz homeomorphic to a \((k-1)\)-sphere.

Now suppose \(k < n\). Since \(q\) is a nondegenerate critical point of \(f := d_p|C_{k+1, q}\) by Lemma 3.2, we may take local coordinates \(\{x_{k+1}, \ldots, x_n\}\) around \(q\) in \(C_{k+1,q}\) so that we may write

\[
d(\pi(r), p) = d(r_{ik}, p) = d(p, q) \pm x_{k+1}^2 \pm \cdots \pm x_n^2.
\]

It follows that we have for \(r\) in an open neighborhood \(U\) of \(q\)

\[
d_p(r) = d(p, q) \pm x_{k+1}^2(r) \pm \cdots \pm x_n^2(r) - z_1(r) - \cdots - z_k(r),
\]

where \(x_{k+j}\) are smooth and \(z_j\) are nonnegative Lipschitz functions. Therefore, we may consider that the index of \(d_p\) at the critical point \(q \in C_{k+1}\) is given by the sum of the index of \(f := d_p|C_{k+1, q}\) at \(q\) and \(k\), where \(k\) is also equal to the degree of \(q\) (see Definition 3.1).

If \(k = n\), then \(\dim C_{n+1} = 0\) and \(C_{n+1}\) consists of vertices, i.e., strata of dimension 0 of \(C(p)\). If such a vertex \(q\) is a critical point of \(d_p\) in the angle sense, then above procedure implies that we may write around \(q\)

\[
d_p(r) = d(p, q) - z_1(r) - \cdots - z_n(r)
\]

with \(z_j \geq 0\) and we may consider that \(q\) is a critical point of index \(n\).

Summing up we have

**Lemma 3.6.** Let \(q \in C_{k+1}\) be a critical point in the angle sense of nondegenerate distance function \(d_p\). Then we have a stratification of a neighborhood \(U\) of \(q\) by submanifolds \(D_{l_0, l_1, \ldots, l_t}'s\) and embeddings

\[
\phi : D_{l_0, l_1, \ldots, l_t} \ni r \mapsto (z_1(r), \ldots, z_k(r), x_{k+1}(r), \ldots, x_n(r)) \in \mathbb{R}^n
\]
with \( z_i(r) \geq 0 \) such that
\[
d_p(r) = d(p, q) - z_1(r) - \cdots - z_k(r) - x_{k+1}(r)^2 - \cdots - x_{k+j}(r)^2 \\
+ x_{k+j+1}(r)^2 + \cdots + x_n(r)^2
\]
where \( j \) is the index of \( f := d_p|C_{k+1} \) at \( q \) in the usual sense. \( i := k + j \) is called the index of \( d_p \) at the critical point \( q \). \( \phi \)'s are piecewise smoothly extended to the whole \( U \) and \( x_{k+i} \)'s are smooth and \( z_i \)'s are Lipschitz functions on \( U \). Moreover for a fixed point \( r \in C_{k+1} \cap U \) with \( z_1 = \ldots = z_k = 0 \), the set defined by the equation \( z_1 + \cdots + z_k \leq \delta \) with \( x_{k+1} = x_{k+1}(r), \ldots, x_n = x_n(r) \) is homeomorphic to \( k \)-disk.

Now if \( q \) is not a critical point of \( d_p \), we may make use of the isotopy lemma. Next suppose \( q \in C(p) \) is a critical point of index \( i \) of \( d_p \) in the angle sense that is isolated by the nondegeneracy assumption. Then we may apply the usual procedure of Morse theory ([14]). Indeed, around \( q \) we have
\[
d_p(r) = d(p, q) - z_1 - \cdots - z_k - x_{k+1}^2 - \cdots - x_{k+j}^2 + x_{k+j+1}^2 + \cdots + x_n^2,
\]
where \( i = k + j \) is equal to the index of \( q \). Then the subset defined by
\[
-z_1 - \cdots - z_k - x_{k+1}^2 - \cdots - x_{k+j}^2 + x_{k+j+1}^2 + \cdots + x_n^2 = c \\
(\text{resp. } -z_1 - \cdots - z_k - x_{k+1}^2 - \cdots - x_{k+j}^2 + x_{k+j+1}^2 + \cdots + x_n^2 = -c)
\]
for sufficiently small \( c > 0 \) is homeomorphic to \( S^{n-k-j-1} \times I^{k+j} \) (resp. \( (S^{j-1} \times S^{k-1}) \times I^{n-k-j} \)), where \( S^{j-1} \times S^{k-1} \) denotes the spherical join and homeomorphic to \( S^{k+j-1} \).

Then for sufficiently small \( \epsilon > 0 \), levels \( d_p^{-1}(t) \cap U \) are homeomorphic to \( S^{n-i-1} \times I^i \) (resp. \( S^{i-1} \times I^{n-i} \)) for \( t \) with \( 0 < t - l \leq \epsilon \) (resp. \( -\epsilon \leq t - l < 0 \)), and \( \{ r \in U \mid d(p, r) \leq l + \epsilon \} \) has the homotopy type of \( \{ r \in U \mid d(p, r) \leq l - \epsilon \} \) with an \( i \)-cell attached. In the above, we set \( S^{n-i-1} \times I^i = \emptyset \) for \( i = n \). Since we may construct a nowhere vanishing gradient-like vector field of \( d_p \) outside of a small neighborhood of the set of critical points of \( d_p \), we apply the usual procedure of Morse theory to get the following main result in this section.

**Theorem 3.7.** Let \((M, g)\) satisfy the condition \((C)\) at \( p \). Suppose all cut points of \( p \) are nondegenerate cut points and \( d_p \) is a nondegenerate distance function. Let \( q \) be a critical point of \( d_p \) in the angle sense. Note that \( q \) is
a cut point of \( p \) and we denote by \( k \) the degree of \( q \), namely \( q \in C_{k+1}(p) \). Then \( q \) is a critical point of a smooth function \( f := d_p|C_{k+1} \) on \( C_{k+1,q} \) and we define the index of \( d_p \) at \( q \) as the sum of \( k \) and the index of \( f \) at \( q \) in the usual sense. Then \( M \) has the homotopy type of a CW-complex, with one cell of dimension \( i \) for each critical point in the angle sense of index \( i \).

Remark 3.8. Among all Riemannian metrics on \( M \) satisfying the condition (C) at \( p \), is the set of Riemannian metrics on \( M \) such that all cut points of \( p \) are nondegenerate cut points and \( d_p \) is a nondegenerate distance function dense in \( C^\infty \) topology? For two-dimensional case, V. Gershkovich asserts that the assertion holds ([3], [5]).

Now we consider the case where the Riemannian metric \( g \) satisfies the following condition (F):

**Definition 3.9.** A compact Riemannian manifold \((M, g)\) satisfies the condition (F) at \( p \in M \), if for any unit speed geodesic \( \gamma \) emanating from \( p \) and any Jacobi field \( J \) along \( \gamma \) satisfying the initial condition \( Y(0) = 0, \nabla_\gamma Y(0) \neq 0 \), we have \( \langle Y(t), \nabla_\gamma Y(t) \rangle > 0 \) for parameter values \( t > 0 \) up to the cut distance to \( p \) along \( \gamma \).

If \((M, g)\) satisfies the condition (F) at \( p \), then minimal geodesics are conjugate points-free, namely the condition (C) holds. However, considering the real projective spaces of positive constant curvature for which the cut locus of any point is disjoint from the conjugate locus, we see that the converse does not necessarily holds. On the other hand, if \((M, g)\) is of nonpositive sectional curvature, then it satisfies the condition (F) at every point \( p \in M \). Note that the condition (F) means that cut points are not focal points, and is also essentially given in [5].

**Lemma 3.10.** Suppose \((M, g)\) satisfies the condition (F) at \( p \). Then for any critical point \( q \in C_{k+1} \) (\( k < n \)) of \( d_p \) in the angle sense, that is also a critical point of the smooth function \( f = d_p|C_{k+1,q} \), its Hessian \( D^2 f(q) \) is positive definite. Namely, \( f \) assumes a local minimum at \( r \).

**Proof.** To see the assertion recall the second variation formula (2.10). As before choose \( a_i \geq 0 \) (\( 0 \leq i \leq k \)) with \( \sum a_i = 1 \) so that \( \sum a_i X_i = 0 \) holds. Recall that \( \dim C_{k+1,q} > 0 \). Then for any \( u \in T_q C_{k+1}(u \neq 0) \) take a unique
Jacobi field \( Y_i \) along \( \gamma_i \) with \( Y_i(0) = 0, Y_i(l) = u, l = d(p, q) \). By (2.10) we obtain

\[
D^2 f(u, u) = \sum a_i D^2 f(u, u)
\]

\[
= \sum a_i (Y_i(l), \nabla_{\gamma_i} Y_i(l)) - \langle \sum a_i X_i, \nabla_u \dot{x}(s) \rangle
\]

\[
= \sum a_i (Y_i(l), \nabla_{\gamma_i} Y_i(l)) > 0,
\]

and this completes the proof of the lemma. Note that here we do not need to assume that the critical point \( q \) is nondegenerate in the sense of Definition 3.3.

Therefore, we may abbreviate the condition (2) in Definition 3.3 of non-degeneracy for a distance function \( d_p \) if the condition (F) is satisfied. Now suppose a compact Riemannian manifold \((M, g)\) of dimension \( n \) satisfies the condition (F) at \( p \) and the cut locus \( C(p) \) consists of nondegenerate cut points. Let \( q \in C_{k+1} \) be a critical point of a nondegenerate \( d_p \). We want to describe the behavior of the distance function \( d_p \) in a neighborhood \( U \) of \( q \), and follow the argument as before. For instance, if \( k = 1 \) then by (3.2) and Lemma 3.10 we have

\[
d_p(x) = d_p(q) + y_1^2 + \cdots + y_{n-1}^2 - z,
\]

with \( z = d(p, x_1) - d(p, x) \) where \( x_1 \) denotes the cut point of \( p \) along the minimal geodesic joining \( p \) to \( x \). Therefore, we may consider that the index of \( d_p \) at the critical point \( q \) is given by 1, which is also equal to the degree of \( q \) (see Definition 3.1). Next, suppose \( k < n \). Since \( q \) is a local minimum point of \( f := d_p|_{C_{k+1}, q} \) by Lemma 3.10, we may take local coordinates \( \{x_{k+1}, \ldots, x_n\} \) around \( q \) in \( C_{k+1, q} \) so that we may write for \( r \in D_{i_0} \)

\[
d(r_{i_k}, p) = d(p, q) + x_{k+1}^2 + \cdots + x_n^2,
\]

where \( r_{i_k} \in C_{k+1, q} \) is uniquely determined from \( r \) by the procedure given before the statement of Theorem 3.7. It follows that we have for \( r \in U \)

\[
d_p(r) = d(p, q) + x_{k+1}^2 + \cdots + x_n^2 - z_1 - \cdots - z_k,
\]

where \( z_j \) are nonnegative and given by \((z_{i_0}, z_{i_1}, \ldots, z_{i_k})\) as described before. Therefore, we may regard that the index of \( d_p \) at the critical point \( q \in C_{k+1} \) is given by \( k \), which is also equal to the degree of \( q \) (see Definition 3.1). If \( k = n \), then the situation is the same as before.
Note that if \((M, g)\) satisfies the condition (F) at \(p\) then \(C_{n+1} \neq \emptyset\) for \(C(p)\), since points which are furthest from \(p\) are of index \(n\) and therefore of order \(n + 1\).

**Theorem 3.11.** Suppose \((M, g)\) satisfies the condition (F) at \(p\). If all cut points of \(p\) are nondegenerate and \(d_p\) is a nondegenerate distance function, then we may perform the \(k\)-cell attaching at each critical point in \(C_{k+1} \) of \(d_p\) in the angle sense.

Finally we give an easy application of Theorem 3.11.

**Corollary 3.12.** Let \((M, g)\) be a Riemannian manifold satisfying the condition (F) at \(p \in M\). If all cut points of \(p\) are nondegenerate cut points and \(d_p\) is a nondegenerate distance function, then the number of connected components of \(C_{k+1}\) is greater than or equal to the \(k\)-th Betti number of \(M\).

**Proof.** The critical points of the distance function are local minimal points on \(C_{k+1}\). Then the index of each critical points on \(C_k\) is equal to \(k\) and we get the corollary by the Morse inequality.

**Remark 3.13.** In [11] some kind of Morse theory was discussed by using distance function. In particular, in section 3 of [11] one of the authors constructed a metric by using handle attaching in Morse theory.

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