Math. J. Okayama Univ. **48** (2006), 159–172

NEW ULTIMATE BOUNDEDNESS AND PERIODICITY RESULTS FOR CERTAIN THIRD-ORDER NONLINEAR VECTOR DIFFERENTIAL EQUATIONS

Cemİl TUNÇ and Ercan TUNÇ

Abstract. The principle aim of this paper is to present some new results related to the ultimate boundedness and existence of periodic of solutions a certain non-linear ordinary vector differential equation of third order. Our results improve some well-known results in the literature.

1. Introduction

The boundedness and existence of periodic solutions are very important in the theory and applications of differential equations. Till now, many authors have done very excellent works; see, for example, [22] as a survey book and [1], [2], [3], [4], [5], [6], [8], [9], [10], [11], [12], [13], [14], [15], [18], [19], [20], [23], [24], [25], [26], [27] and [28]. However, it should be clarified that the number of results related to the ultimate boundedness and existence of periodic solutions of certain third order nonlinear vector differential equations is very few in comparison to that on the certain scalar nonlinear differential equations of third order. In fact, to our knowledge these results can be presented here, briefly, as follows: Namely, in this way, in 1966, 1983 and 1993, respectively, Ezeilo&Tejumola [8], Afuwape [2] and Meng [20] investigated the ultimately boundedness and existence of periodic solutions of the nonlinear vector differential equation of the form

$$\dddot{X} + A\ddot{X} + B\dot{X} + H(X) = P(t, X, \dot{X}, \ddot{X}).$$

Afterward, in 1985, Afuwape [4] also considered the vector differential equation

$$\dddot{X} + A\ddot{X} + G(\dot{X}) + H(X) = P(t, X, \dot{X}, \ddot{X})$$

and for the above equation the author proved ultimate boundedness results which are generalizations of earlier conclusions of Ezeilo and Tejumola [8]. Along with the above works, in 1985, Abou-El-Ela [1] also established sufficient conditions which ensure that all solutions of real vector differential equations as follows

$$\dddot{X} + F(X, \dot{X})\ddot{X} + G(\dot{X}) + H(X) = P(t, X, \dot{X}, \ddot{X})$$

Mathematics Subject Classification. 34D20, 34D40.

Key words and phrases. Nonlinear differential equations, third order, ultimate boundedness, existence of periodic solutions.

are ultimately bounded. Later, in 1995, Feng [14] demonstrated a result associated with the existence of unique periodic solution of the similar type equation

$$\dddot{X} + A(t)\ddot{X} + B(t)\dot{X} + H(X) = P(t, X, \dot{X}, \ddot{X}).$$

Further, in 1999, Tiryaki [23] obtained some sufficient conditions which make certain that all the solutions of

$$\dddot{X} + A\ddot{X} + G(\dot{X}) + H(X) = P(t, X, \dot{X}, \ddot{X})$$

are ultimately bounded and he also gave some sufficient conditions which guarantee that there exists at least one periodic solution of the equation just mentioned above. In the same year, the author in [25] also proved some theorems on the same topic for nonlinear vector differential equation

$$\dddot{X} + F(X, \dot{X})\ddot{X} + B\dot{X} + H(X) = P(t, X, \dot{X}, \ddot{X}).$$

Recently, that is in 2005, Tunç and Ates [26] investigated, for the cases $P \equiv 0$ and $P \neq 0$, respectively, the asymptotic stability of the zero solution and boundedness of all solutions of the third order non-linear ordinary vector differential equation

$$\dddot{X} + F(X, \dot{X}, \ddot{X})\ddot{X} + B(t)\dot{X} + H(X) = P(t, X, \dot{X}, \ddot{X}).$$

With respect to our observation in the literature, ostensibly, the last work proceeded on ultimate boundedness of solutions of third order non-linear vector differential equation has been made, in 2004, by Afuwape and Omeike [6]. That is to say that, Afuwape and Omeike [6], inspiring from the papers of Ezeilo ([7], [9], [10]) and Tiryaki [23], established two results contain sufficient conditions on the theme for nonlinear vector differential equation

$$\dddot{X} + F(\ddot{X}) + G(\dot{X}) + H(X) = P(t, X, \dot{X}, \ddot{X}).$$

During establishment of the results, Afuwape and Omeike [6] defined the following relations with respect to the vectors F, G and H:

$$F(\ddot{X}_1) = F(\ddot{X}_2) + A_f(\ddot{X}_1, \ddot{X}_2)(\ddot{X}_1 - \ddot{X}_2), \tag{1.1}$$

$$G(\dot{X}_1) = G(\dot{X}_2) + B_g(\dot{X}_1, \dot{X}_2)(\dot{X}_1 - \dot{X}_2) \tag{1.2}$$

and

$$H(X_1) = H(X_2) + A_h(X_1, X_2)(X_1 - X_2) \tag{1.3}$$

where $A_f(\ddot{X}_1, \ddot{X}_2)$, $B_g(\dot{X}_1, \dot{X}_2)$ and $A_h(X_1, X_2)$ are $n \times n$-continuous operators, having real eigenvalues $\lambda_i(A_f(\ddot{X}_1, \ddot{X}_2))$, $\lambda_i(B_g(\dot{X}_1, \dot{X}_2))$ and $\lambda_i(C_h(X_1, X_2))$ such that

$$0 < \delta_f \leq \lambda_i(A_f(\ddot{X}_1, \ddot{X}_2)) \leq \Delta_f, \tag{1.4}$$

$$0 < \delta_g \le \lambda_i(B_g(\dot{X}_1, \dot{X}_2)) \le \Delta_g, \tag{1.5}$$

$$0 < \delta_h \le \lambda_i(C_h(X_1, X_2)) \le \Delta_h, \quad (i = 1, 2, ..., n), \tag{1.6}$$

with $\delta_f, \delta_g, \delta_h, \Delta_f, \Delta_g$ and Δ_h as fixed constants, and

$$\Delta_h \le k\delta_f\delta_g \tag{1.7}$$

for some positive constant $k(k < 1)$. Their primary reason of defining the above operators is to proceed their results without imposing the differentiability condition on the vector functions $F(\ddot{X}), G(\dot{X})$ and $H(X)$. Through all the papers just pointed out above, the Lyapunov's second (or direct) method [17] is used as a basic tool to achieve the results there. It is reasonable to ask why the Lyapunov's second method has been used as basic tool in all the above works. For instance, in this respect, Iggidr and Sallet [16] states that " The most efficient tool for the study of the stability of given non-linear system is provided by Lyapunov theory. This theory is based on the use positive definite functions that are non-increasing along the solutions of the considered.... But finding an appropriate positive Lyapunov function is in general a difficult, viz." Likewise, the major advantage of this method is that information about stability, boundedness, and existence of periodic solution, viz. can be obtained without any prior knowledge of solutions.

In this paper, we consider nonlinear vector differential equations of the form

$$\dddot{X} + F(X, \dot{X}, \ddot{X})\ddot{X} + G(\dot{X}) + H(X) = P(t, X, \dot{X}, \ddot{X}) \tag{1.8}$$

where $X \in R^n$ and $t \in R$; F is an $n \times n$-symmetric continuous matrix function; $G : R^n \to R^n$, $H : R^n \to R^n$, $H(0) = G(0) = 0$ and $P : R \times R^n \times R^n \times R^n \to R^n$, and G, H and P are continuous.

In what follows it will be convenient to use the equivalent differential system:

$$\begin{aligned} &\dot{X} = Y, \dot{Y} = Z, \\ &\dot{Z} = -F(X, Y, Z)Z - G(Y) - H(X) + P(t, X, Y, Z), \end{aligned} \tag{1.9}$$

which was obtained from (1.8) by setting $\dot{X} = Y, \ddot{X} = Z$.

2. Notations

Corresponding to any pair X, Y in R^n, the symbol $\langle X, Y\rangle$ and the representation $\lambda_i(A)$, $(i = 1, 2, ..., n)$, will denote the usual scalar product $\sum_{i=1}^{n} x_i y_i$ and the eigenvalues of $n\times n$-matrix A, respectively, and, in particular, $\langle X, X\rangle = \|X\|^2$. Next, it is also used, as basic throughout this paper, that δ's and

Δ's with or without suffices will represent positive constants whose magnitudes depend only on the constants associated with the equation under study. The δ's and Δ's with numerical or alphabetical suffices may vary from place to place, but each of them with suffix attached preserves its identity in every place of occurrence.

3. Main results

First the following result is established

Theorem 1. *In addition to the fundamental assumptions imposed on F, G, H and P, we suppose that:*

(i) *There exists an real $n \times n$-symmetric matrix function $F(X, Y, Z)$ and real continuous operators $B_g(Y_1, Y_2)$, $C_h(X_1, X_2)$ for any vectors X, Y, Z, X_1, X_2, Y_1, $Y_2 \in R^n$ such that the functions G, H satisfy* (1.5), (1.6) *and F that*

$$0 \leq \delta_f \leq \lambda_i(F(X, Y, Z)) \leq \Delta_f, \quad (i = 1, 2, ..., n),$$

with δ_f and Δ_f as fixed constants;

(ii) *the operators B_g and C_h are associative and commute pairwise;*

(iii) *the function P satisfies*

$$\begin{aligned} \|P(t, X, Y, Z)\| &\leq p_1(t) + p_2(t) \left\{ \|X(t)\|^2 + \|Y(t)\|^2 + \|Z(t)\|^2 \right\}^{\frac{\rho}{2}} \\ &+ p_3(t) \left\{ \|X(t)\|^2 + \|Y(t)\|^2 + \|Z(t)\|^2 \right\}^{\frac{1}{2}} \end{aligned} \tag{3.1}$$

for any $X, Y, Z \in R^n$ and $t \in R$, where $p_1(t), p_2(t)$ and $p_3(t)$ are continuous function of t and $0 \leq \rho < 1$.

Then, there exist constants $\rho_3, \Delta_1, \Delta_2, \Delta_3$ such that if $|p_3(t)| \leq \rho_3$, for all $t \in R$, with ρ_3 chosen small enough, then every solution $X(t)$ of (1.8) *with*

$$X(t_0) = X_0, \dot{X}(t_0) = Y_0, \ddot{X}(t_0) = Z_0,$$

and for any constant r, whatever in the range $\frac{1}{2} \leq r \leq 1$, satisfies

$$\begin{aligned} \left\{ \|X(t)\|^2 + \left\|\dot{X}(t)\right\|^2 + \left\|\ddot{X}(t)\right\|^2 \right\}^r &\leq \Delta_1 \exp\{-\Delta_2(t - t_0)\} \\ &+ \Delta_3 \int_{t_0}^{t} \left\{ p_1^{2r}(\tau) + p_2^{2r/(1-\rho)}(\tau) \right\} \exp\{-\Delta_2(t - \tau)\} \, d\tau \end{aligned} \tag{3.2}$$

for all $t \geq t_0 \geq 0$, where $\Delta_1 \equiv \Delta_1(X_0, Y_0, Z_0)$.

Remark 1. When specialized to the case $n = 1$ with P depending only on t, the above estimate (3.2) reduces to the estimate in Harrow [15].

Remark 2. It should be noted that Theorem 1 mentioned above can be proved here without defining the operator (1.1) and that imposing the differentiability assumption on the matrix function $F(X,Y,Z)$. Hence, in the special case $F(X,Y,Z)=F(Z)$, the above assumptions are less restrictive than those established in Afuwape and Omeike [6; Theorem 1], and our result improves the result proved by them.

Corollary 1. *If* $P\equiv 0$ *and all the conditions of Theorem 1 hold, then every solution* $X(t)$ *of* (1.8) *satisfies*

$$\left\{\|X(t)\|^2+\left\|\dot{X}(t)\right\|^2+\left\|\ddot{X}(t)\right\|^2\right\}\to 0 \quad as \quad t\to\infty,$$

provided that ρ_3 *is small enough. This case can be seen easily when* $\rho_1(t)=\rho_2(t)=0$ *in* (3.2).

Our second result is the following ultimately bounded result, which can be deduced from Theorem 1.

Theorem 2. *Let all the conditions of Theorem 1 be satisfied, and in addition we assume that* $|p_3(t)|\le\rho_3$ *for all* $t\in R$, *with* ρ_3 *chosen small enough, and that the functions* p_1 *and* p_2 *satisfy*

$$|p_1(t)|\le\delta_0 \quad and \quad |p_2(t)|\le\delta_1$$

for all $t\in R$. *Then, there exists a constant* Δ_4 *such that every solution* $X(t)$ *of* (1.8) *ultimately satisfies*

$$\left\{\|X(t)\|^2+\left\|\dot{X}(t)\right\|^2+\left\|\ddot{X}(t)\right\|^2\right\}\le\Delta_4.$$

Remark 3. In the special case $F(X,Y,Z)=F(Z)$, the assumptions of Theorem 2 are less restrictive than those established by Afuwape and Omeike [6, Theorem 2], and our result improves their second result, [6, Theorem 2].

Remark 4. It should become better to say that if P is a bounded function as in Theorem 2, then the constant Δ_4 above can be fixed independent of the initial values X_0, Y_0 and Z_0 as in Theorem 1. This fact is difference between boundedness and ultimately boundedness conceptions.

Finally, we have that

Theorem 3. *In differential system* (1.9), *let* P *satisfies*

$$P(t+\omega,X,Y,Z)=P(t,X,Y,Z)$$

uniformly for all $X,Y,Z\in R^n$. *Assume also that all the conditions of Theorem 2 are satisfied. Then there exists a periodic solution* $X(t)$ *of* (1.9) *with a period* ω.

Remark 5. Theorem 3 yields an additional result to the results of Afuwape and Omeike [6].

4. Preliminaries

In order to reach our main results, we dispose of some well-known algebraic results which will be required in the proofs. The first of these is quite standart one:

Lemma 1 (See [21])**.** *Let D be a real symmetric $n \times n$ matrix. Then for any X in R^n*

$$\delta_d \|X\|^2 \leq \langle DX, X\rangle \leq \Delta_d \|X\|^2$$

where δ_d and Δ_d are, respectively, the least and greatest eigenvalues of the matrix D.

Next, we require the following lemma.

Lemma 2 (See [21])**.** *Let Q, D be any two real $n \times n$ commuting symmetric matrices. Then,*

(i) *The eigenvalues $\lambda_i(QD), (i = 1, 2, ..., n)$, of the product matrix QD are real and satisfy*

$$\max_{1\leq j,k\leq n} \lambda_j(Q)\lambda_k(D) \geq \lambda_i(QD) \geq \min_{1\leq j,k\leq n} \lambda_j(Q)\lambda_k(D).$$

(ii) *The eigenvalues $\lambda_i(Q + D), (i = 1, 2, ..., n)$, of the sum of matrices Q and D are real and satisfy*

$$\left\{\max_{1\leq j\leq n} \lambda_j(Q) + \max_{1\leq k\leq n} \lambda_k(D)\right\} \geq \lambda_i(Q + D) \geq \left\{\min_{1\leq j\leq n} \lambda_j(Q) + \min_{1\leq k\leq n} \lambda_k(D)\right\}$$

where $\lambda_j(Q)$ and $\lambda_k(D)$ are, respectively, the eigenvalues of Q and D.

5. The Lyapunov function V

We use the Lapunov function used in Afuwape and Omeike [6] in the proof of the main results. That is, the function $V = V(X, Y, Z)$ defined by

$$\begin{aligned} 2V &= \beta(1-\beta)\delta_g^2 \langle X, X\rangle + \beta\delta_g \langle Y, Y\rangle + \alpha\delta_g\delta_f^{-1} \langle Y, Y\rangle + \alpha\delta_f^{-1} \langle Z, Z\rangle \\ &\quad + \langle Z + \delta_f Y + (1-\beta)\delta_g X, Z + \delta_f Y + (1-\beta)\delta_g X\rangle, \end{aligned} \tag{5.1}$$

where $0 < \beta < 1$ and $\alpha > 0$.

The function and its time derivative, (in the light of Lyapunov's second or direct method), must satisfy some fundamental inequalities.

Now, the first property of the function $V = V(X, Y, Z)$ is summarized with Lemma 3.

Lemma 3. *Assume that all the conditions on F, G and H in Theorem 1 are satisfied. Then, there are positive constants δ_2 and δ_3 such that*

$$\delta_2\left(\|X\|^2+\|Y\|^2+\|Z\|^2\right)\le V(X,Y,Z)\le\delta_3\left(\|X\|^2+\|Y\|^2+\|Z\|^2\right)$$

is valid for every solution of (1.9).

Proof. Since the function V in (5.1) is the same as the function V defined in [6], if one follows the lines indicated as the same as in [6], it can be easily obtained that

$$\delta_2\left(\|X\|^2+\|Y\|^2+\|Z\|^2\right)\le V(X,Y,Z)\le\delta_3\left(\|X\|^2+\|Y\|^2+\|Z\|^2\right),\tag{5.2}$$

where

$$\delta_2=\min\left\{\beta(1-\beta)\delta_g^2,\delta_g(\beta+\alpha\delta_f^{-1}),\alpha\delta_f^{-1}\right\}$$

and

$$\delta_3=\max\Big\{\delta_g(1-\beta)(1+\delta_f+\delta_g),\delta_g(\beta+\alpha\delta_f^{-1})+\delta_f[1+\delta_g(1-\beta)+\delta_f],$$
$$1+\alpha\delta_f^{-1}+\delta_f+\delta_g(1-\beta)\Big\}.$$

This completes the proof of the lemma. □

Now, let $(X,Y,Z)=(X(t),Y(t),Z(t))$ be an arbitrary solution of (1.9). Differentiating the function $V=(X(t),Y(t),Z(t))$ in (5.1) along the system (1.9) we obtain

$$\dot V=\frac{d}{dt}V(X(t),Y(t),Z(t))=-V_1-V_2-V_3-V_4-V_5-V_6-V_7-V_8,\tag{5.3}$$

where

$$\begin{aligned}
V_1=\{&\gamma_1\delta_g(1-\beta)\langle X,H(X)\rangle+\eta_1\delta_f\langle Y,G(Y)-\delta_g(1-\beta)Y\rangle\\
&+\xi_1\alpha\delta_f^{-1}\langle Z,F(X,Y,Z)Z\rangle+\langle Z,F(X,Y,Z)Z-\delta_fZ\rangle\Big\},\\
V_2=\Big\{&\gamma_2\delta_g(1-\beta)\langle X,H(X)\rangle+\xi_2\alpha\delta_f^{-1}\langle Z,F(X,Y,Z)Z\rangle\\
&+(1+\alpha\delta_f^{-1})\langle Z,H(X)\rangle\Big\},\\
V_3=\{&\gamma_3\delta_g(1-\beta)\langle X,H(X)\rangle+\eta_2\delta_f\langle Y,G(Y)-\delta_g(1-\beta)Y\rangle\\
&+\delta_f\langle Y,H(X)\rangle\},\\
V_4=\Big\{&\gamma_4\delta_g(1-\beta)\langle X,H(X)\rangle+\xi_3\alpha\delta_f^{-1}\langle Z,F(X,Y,Z)Z\rangle\\
&+\delta_g(1-\beta)\langle X,F(X,Y,Z)Z-\delta_fZ\rangle\},
\end{aligned}$$

$$V_5 = \{\gamma_5\delta_g(1-\beta)\langle X, H(X)\rangle + \eta_3\delta_f\langle Y, G(Y) - \delta_g(1-\beta)Y\rangle + \delta_g(1-\beta)\langle X, G(Y) - \delta_g Y\rangle\},$$

$$V_6 = \left\{\xi_4\alpha\delta_f^{-1}\langle Z, F(X,Y,Z)Z\rangle + \eta_4\delta_f\langle Y, G(Y) - \delta_g(1-\beta)Y\rangle + (1+\alpha\delta_f^{-1})\langle Z, G(Y) - \delta_g Y\rangle\right\},$$

$$V_7 = \left\{\xi_5\alpha\delta_f^{-1}\langle Z, F(X,Y,Z)Z\rangle + \eta_5\delta_f\langle Y, G(Y) - \delta_g(1-\beta)Y\rangle + \delta_f\langle Y, F(X,Y,Z)Z - \delta_f Z\rangle\right\},$$

$$V_8 = \left\{\left\langle(1-\beta)\delta_g X + \delta_f Y + (1+\alpha\delta_f^{-1})Z, P(t,X,Y,Z)\right\rangle\right\},$$

with $\xi_i, \eta_i, \gamma_i; (i = 1,2,3,4,5)$, are strictly positive constants such that

$$\sum_{i=1}^{5}\xi_i = 1, \quad \sum_{i=1}^{5}\eta_i = 1, \quad \sum_{i=1}^{5}\gamma_i = 1.$$

The next property related to the time derivative of the function $V = V(X,Y,Z)$ is clarified with Lemma 4.

Lemma 4. *Let us assume that all the conditions of Theorem 1 hold. Then, subject to a conveniently chosen of values of constants k_i, $(i = 1,2,3,4,5,6)$, in (1.7), the components of the time derivative of the function V, $V_i = V_i(X,Y,Z)$, $(i = 2,3,...,7)$, and $\dot{V}$ satisfy*

$$V_i(X,Y,Z) \geq 0 \quad \textit{for all } X, Y, Z \in R^n$$

and

$$\dot{V} \leq -\delta_{10}\psi^2 + \delta_{14}\left\{p_1^{2r}(t) + p_2^{2r/(1-\rho)}(t)\right\}\psi^{2(1-r)}$$

for any r in the large $\frac{1}{2} \leq r \leq 1$.

Proof. The function V_3, V_5 and V_8 here are the same as the functions W_3, W_5 and W_8 defined in [6]. The estimates for V_3, V_5 and V_8 in [6] yield that

$$V_3 \geq 0, V_5 \geq 0, |V_8| \leq \sqrt{3}\delta_9\left\{p_3(t)\psi^2 + p_2(t)\psi^{(1+\rho)} + p_1(t)\psi\right\}$$

where δ_9 is a certain positive constant as fixed in [6].

Now, by noting assumptions of (i), (ii) of Theorem 1 and Lemma 1, it follows that

$$\begin{aligned} V_1 &= \{\gamma_1\delta_g(1-\beta)\langle X, H(X)\rangle + \eta_1\delta_f\langle Y, G(Y) - \delta_g(1-\beta)Y\rangle \\ &\quad + \xi_1\alpha\delta_f^{-1}\langle Z, F(X,Y,Z)Z\rangle + \langle Z, [F(X,Y,Z) - \delta_f I]Z\rangle\Big\} \\ &\geq \{\gamma_1\delta_g(1-\beta)\langle X, C_h(X,0)X\rangle + \eta_1\delta_f\langle Y, [B_g(Y,0) \\ &\quad - \delta_g(1-\beta)]Y\rangle + \xi_1\alpha\|Z\|^2 \end{aligned}$$

$$\geq \gamma_1\delta_g\delta_f(1-\beta)\left\|X\right\|^2+\eta_1\delta_g\delta_f\beta\left\|Y\right\|^2+\xi_1\alpha\left\|Z\right\|^2$$
$$\geq \delta_8(\|X\|^2+\|Y\|^2+\|Z\|^2),$$

where

$$\delta_8=\min\left\{\gamma_1\delta_g\delta_f(1-\beta),\eta_1\delta_f\delta_g\beta,\xi_1\alpha\right\}.$$

Next, consider the expression

$$V_2=\Big\{\gamma_2\delta_g(1-\beta)\left\langle X,H(X)\right\rangle+\xi_2\alpha\delta_f^{-1}\left\langle Z,F(X,Y,Z)Z\right\rangle + (1+\alpha\delta_f^{-1})\left\langle Z,H(X)\right\rangle\Big\}.$$

Again, in view of (1.6), assumption (i) of Theorem 1 and Lemma 1, easily, we obtain that

$$\begin{aligned}V_2 &\geq \xi_2\alpha\left\|Z\right\|^2+\left\|k_1\sqrt{1+\alpha\delta_f^{-1}}Z+\frac{1}{2k_1}\sqrt{1+\alpha\delta_f^{-1}}H(X)\right\|^2\\
&\quad -k_1^2(1+\alpha\delta_f^{-1})\left\|Z\right\|^2-\frac{1}{4k_1^2}(1+\alpha\delta_f^{-1})\left\langle H(X),H(X)\right\rangle\\
&\quad +\gamma_2\delta_g(1-\beta)\left\langle X,C_h(X,0)X\right\rangle\\
&\geq \xi_2\alpha\left\|Z\right\|^2-k_1^2(1+\alpha\delta_f^{-1})\left\|Z\right\|^2\\
&\quad -\frac{1}{4k_1^2}(1+\alpha\delta_f^{-1})\left\langle C_h(X,0)X,C_h(X,0)X\right\rangle\\
&\quad +\gamma_2\delta_g(1-\beta)\left\langle X,C_h(X,0)X\right\rangle\\
&\geq [\xi_2\alpha-k_1^2(1+\alpha\delta_f^{-1})]\left\|Z\right\|^2+[\gamma_2\delta_g(1-\beta)\\
&\quad -\frac{1}{4k_1^2}(1+\alpha\delta_f^{-1})\delta_h\Delta_h]\left\|X\right\|^2.\end{aligned}$$

If we choose

$$k_1^2\leq\frac{\xi_2\alpha\delta_f}{\alpha+\delta_f}\ \text{ and }\Delta_h\leq\frac{4\gamma_2\xi_2\alpha(1-\beta)\delta_f^2\delta_g}{(\alpha+\delta_f)^2},$$

then, clearly,

$$V_2(X,Y,Z)\geq 0\ \text{ for all }X,Y,Z\in R^n.$$

For the terms

$$\begin{aligned}V_4=&\Big\{\gamma_4\delta_g(1-\beta)\left\langle X,H(X)\right\rangle+\xi_3\alpha\delta_f^{-1}\left\langle Z,F(X,Y,Z)Z\right\rangle\Big\}\\
&+\left\{\delta_g(1-\beta)\left\langle X,F(X,Y,Z)Z-\delta_fZ\right\rangle\right\},\end{aligned}$$

similarly, if we take into consideration (1.6), assumption (i) of Theorem 1 and Lemma 1, we have that

$$\begin{aligned}
V_4 &\geq \gamma_4\delta_g\delta_h(1-\beta)\left\|X\right\|^2 + \xi_3\alpha\left\|Z\right\|^2 + \delta_g(1-\beta)\left\langle X, F(X,Y,Z)Z - \delta_f Z\right\rangle \\
&= \left\| \frac{1}{2k_3}\sqrt{\delta_g(1-\beta)}\sqrt{[F(X,Y,Z)-\delta_f I]}X \right. \\
&\qquad\qquad \left. + k_3\sqrt{\delta_g(1-\beta)}\sqrt{[F(X,Y,Z)-\delta_f I]}Z \right\|^2 \\
&\qquad + \gamma_4\delta_g\delta_h(1-\beta)\left\|X\right\|^2 + \xi_3\alpha\left\|Z\right\|^2 \\
&\qquad - \frac{1}{4k_3^2}\left\langle \delta_g(1-\beta)[F(X,Y,Z)-\delta_f I]X, X\right\rangle \\
&\qquad - k_3^2\left\langle \delta_g(1-\beta)[F(X,Y,Z)-\delta_f I]Z, Z\right\rangle \\
&\geq \gamma_4\delta_g\delta_h(1-\beta)\left\|X\right\|^2 + \xi_3\alpha\left\|Z\right\|^2 \\
&\qquad - \frac{1}{4k_3^2}\left\langle \delta_g(1-\beta)[F(X,Y,Z)-\delta_f I]X, X\right\rangle \\
&\qquad - k_3^2\left\langle \delta_g(1-\beta)[F(X,Y,Z)-\delta_f I]Z, Z\right\rangle \\
&\geq \gamma_4\delta_g\delta_h(1-\beta)\left\|X\right\|^2 + \xi_3\alpha\left\|Z\right\|^2 \\
&\qquad - \frac{1}{4k_3^2}\delta_g(1-\beta)(\Delta_f-\delta_f)\left\|X\right\|^2 \\
&\qquad - \delta_g(1-\beta)(\Delta_f-\delta_f)k_3^2\left\|Z\right\|^2 .
\end{aligned}$$

Let us choose

$$\frac{\Delta_f-\delta_f}{4\gamma_4\delta_h} \leq k_3^2 \leq \frac{\xi_3\alpha}{(1-\beta)\delta_g(\Delta_f-\delta_f)}.$$

Hence

$$V_4(X,Y,Z) \geq 0 \ \text{ for all } \ X,Y,Z \in R^n.$$

Similarly, subject to the assumptions of Theorem 1, we easily obtain

$$V_6(X,Y,Z) \geq 0 \ \text{ for all } \ X,Y,Z \in R^n.$$

Lastly, we consider

$$\begin{aligned}
V_7 = \Big\{ &\xi_5\alpha\delta_f^{-1}\left\langle Z, F(X,Y,Z)Z\right\rangle + \eta_5\delta_f\left\langle Y, G(Y) - \delta_g(1-\beta)Y\right\rangle \\
&+ \delta_f\left\langle Y, F(X,Y,Z)Z - \delta_f Z\right\rangle \Big\} .
\end{aligned}$$

By using (1.5), assumptions (i), (ii) of Theorem 1 and Lemma 1, it is clear that

$$\begin{aligned}
V_7 &= \Big\{\xi_5\alpha \|Z\|^2 + \eta_5\delta_f \langle Y, [B_g(Y,0) - \delta_g(1-\beta)I]Y\rangle \\
&\qquad\qquad + \delta_f \langle Y, F(X,Y,Z)Z - \delta_f Z\rangle\} \\
&\ge \xi_5\alpha \|Z\|^2 + \beta\eta_5\delta_f\delta_g \|Y\|^2 + \delta_f \langle Y, F(X,Y,Z)Z - \delta_f Z\rangle \\
&= \left\| \frac{1}{2k_6}\sqrt{\delta_f}\sqrt{[F(X,Y,Z) - \delta_f I]}Y + k_6\sqrt{\delta_f}\sqrt{[F(X,Y,Z) - \delta_f I]}Z \right\|^2 \\
&\qquad + \xi_5\alpha \|Z\|^2 + \beta\eta_5\delta_f\delta_g \|Y\|^2 - \frac{1}{4k_6^2}\langle \delta_f[F(X,Y,Z) - \delta_f I]Y, Y\rangle \\
&\qquad - k_6^2 \langle \delta_f[F(X,Y,Z) - \delta_f I]Z, Z\rangle \\
&\ge \xi_5\alpha \|Z\|^2 + \beta\eta_5\delta_f\delta_g \|Y\|^2 - \frac{1}{4k_6^2}\langle \delta_f[F(X,Y,Z) - \delta_f I]Y, Y\rangle \\
&\qquad - k_6^2 \langle \delta_f[F(X,Y,Z) - \delta_f I]Z, Z\rangle \\
&\ge \xi_5\alpha \|Z\|^2 + \beta\eta_5\delta_f\delta_g \|Y\|^2 - \frac{1}{4k_6^2}\delta_f(\Delta_f - \delta_f)\|Y\|^2 \\
&\qquad - \delta_f(\Delta_f - \delta_f)k_6^2 \|Z\|^2 .
\end{aligned}$$

Taking

$$\frac{\Delta_f - \delta_f}{4\eta_5\beta\delta_g} \le k_6^2 \le \frac{\xi_5\alpha}{\delta_f(\Delta_f - \delta_f)},$$

we get

$$V_7 \ge 0.$$

Bringing together the estimates just obtained for $V_1, V_2, V_3, V_4, V_5, V_6, V_7$ and V_8 in (5.3), using the fact $|p_3(t)| \le \rho_3$ (for all $t \in R$) and follows the line indicated in [6] we get

$$\dot V \le -\left(\delta_8 - \sqrt{3}\delta_9\rho_3\right)\psi^2 + \sqrt{3}\delta_9\left\{p_2(t)\psi^{(1+\rho)} + p_1(t)\psi\right\}$$

and hence

$$\dot V \le -\delta_{10}\psi^2 + \delta_{14}\left\{p_1^{2r}(t) + p_2^{2r/(1-\rho)}(t)\right\}\psi^{2(1-r)}.$$

This completes the proof of Lemma 4. □

6. Proof of Theorem 1

Let $(X,Y,Z) = (X(t), Y(t), Z(t))$ be an arbitrary solution of (1.9). To complete the proof of Theorem 1, it is sufficient to proceed that, subject to the conditions of Theorem 1, the Lyapunov function V defined in (5.1), satisfies for any solution $(X(t), Y(t), Z(t))$ of (1.9) and for any r in the range $\frac{1}{2} \leq r \leq 1$ the inequality as follows

$$\dot{V} \leq -\delta_4 \psi^2 + \delta_5 \left\{ p_1^{2r}(t) + p_2^{2r/(1-\rho)}(t) \right\} \psi^{2(1-r)}$$

for some constants δ_4, δ_5, where $\psi^2(t) = \left\{ \|X(t)\|^2 + \|Y(t)\|^2 + \|Z(t)\|^2 \right\}$. The rest of the proof can be verified proceeding exactly along the lines just indicated [6, Theorem 1]. Hence we omit the detailed proof.

7. Proof of Theorem 2

Consider the function V defined by (5.1). To perfect the proof of Theorem 2, it is enough to show under the assumptions of Theorem 2 that

$$V(X, Y, Z) \to \infty \quad \text{as} \quad \|X\|^2 + \|Y\|^2 + \|Z\|^2 \to \infty \tag{7.1}$$

and

$$\dot{V} \leq -1 \quad \text{provided that} \quad \|X\|^2 + \|Y\|^2 + \|Z\|^2 \geq \delta_{16}. \tag{7.2}$$

If we take into consideration the result of Lemma 3, then the accuracy of (7.1) is clear. Next, since $\dot{V}$ satisfies the inequality

$$\dot{V} \leq -\delta_{10} \psi^2 + \delta_{14} \left\{ p_1^{2r}(t) + p_2^{2r/(1-\rho)}(t) \right\} \psi^{2(1-r)}, \frac{1}{2} \leq r \leq 1,$$

in view of the boundedness of the functions $p_2(t)$ and $p_3(t)$ for all $t \in R$, it follows that there exists a positive constant δ_{15} such that

$$\dot{V} \leq -\delta_{10} \psi^2 + \delta_{15} \psi^{2(1-r)} \leq -1 \text{ provided } \psi \geq \delta_{16} > \left(\delta_{10}^{-1} \delta_{15} \right)^{1/2r}.$$

The proof of Theorem 2 is now complete.

8. Proof of Theorem 3

By an similar argument to that in the proof of the boundedness result of Tejumola [24], one can complete the proof of this theorem. Therefore, we omit the detailed proof for the theorem.

References

[1] Abou-El-Ela, A. M. A., Boundedness of the solutions of certain third-order vector differential equations, *Annals of Differential Equations* 1 (1985), no.2, 127-139.

[2] Afuwape, A.U. Ultimate boundedness results for a certain system of third-order nonlinear differential equations. *J. Math. Anal. App.* 97 (1983), no.1, 140-150.

[3] Afuwape, A. U., Uniform dissipative solutions for a third order nonlinear differential equation. Differential equations (Birmingham, Ala., 1983), 1-6, North-Holland math. Stud., 92, North-Holland, Amsterdam, 1984.

[4] Afuwape, A.U. Further ultimate boundedness results for a third-order nonlinear system of differential equations. *Unione Mathematica Italiana Bollettino C.* Serie VI. 4 (1985), no.1, 347-361.

[5] Afuwape, A. U.; Ukpera, A. S., Existence of solutions of periodic boundary value problems for some vector third order differential equations, *J. Nigerian math. Soc.* 20 (2001), 1-17.

[6] Afuwape, A. U.; Omeike, M. O., Further ultimate boundedness of solutions of some system of third order nonlinear ordinary differential equations, *Acta Univ. Palack. Olomuc. Fac. Rerum Natur. Math.* 43 (2004), 7-20.

[7] Ezeilo, J.O.C. Stability results for the solutions of some third and fourth order differential equations. *Ann. Mat. Pure. Appl.* (4)66, (1964), 233-249

[8] Ezeilo, J.O.C.; Tejumola, H.O., Boundedness and periodicty of solutions of a certain sysstem of third-order non-linear differential equations. *Annali di Matematica Pura ed Applicata.* (4) 74 (1966), 283-316.

[9] Ezeilo, J.O.C., n-dimensional extensions of boundedness and stability theorems for some third order differential equations. *J. Math. Anal. Appl.* 18 (1967), 395-416.

[10] Ezeilo, J.O.C., New properties of the equation $x'''+ax''+bx'+h(x) = p(t,x,x',x'')$ for certain special values of the incrementary ratio $y^{-1}\{h(x+y) - h(x)\}$. *In: Equations Differentielles et Functionelles Non-lineares* (P. Janssens, J. Mawhin and N. Rouche, eds.), Herman, Paris, 1973,447-462.

[11] Ezeilo, J. O. C.; Tejumola, H. O., Further results for a system of third order differential equations, *AttiAccad. Naz. Lincei Rend. Cl. Sci. Fis. Mat. Natur.* (8) 58 (1975), no.2, 143-151.

[12] Feng, C. H., On the existence of almost periodic solutions of nonlinear third-order differential equations. *Annals of Differential Equations* 9 (1993), no. 4, 420-424.

[13] Feng, C., The existence of periodic solutions for a third -order nonlinear differential equation, *(Chinese) Gongcheng Shuxue Xuebao,* 11 (1994), no. 2, 113-117.

[14] Feng, C., On the existence of periodic solutions for a certain system of third order nonlinear differential equations, *Annals of Differential Equations,* 11 (1995), no. 3, 264-269.

[15] Harrow, M., Further results for the solutions of certain third and fourth order differential equations, *Proc. Cambridge Philos. Soc.* 63, (1967), 147-153.

[16] Iggidr, A.; Sallet, G., On the stability of nonautonomous system, *Automatica J. IFAC* 39 (2003), no. 1, 167-171.

[17] Lyapunopv, A. M., stability of Motion, Academic Press, London, 1966, p. 203.

[18] Mehri, B., Periodic solution for certain nonlinear third order differential equations, *Indian Journal of Pure and Applied Mathematics* 21 (1990), no. 3, 203-210.

[19] Mehri, B. and Shadman, D., Periodic solutions of certain third order nonlinear differential equations, *Studia Scientiarum Mathematicarum Hungarica* 33 (1997), no. 4, 345-350.

[20] Meng, F. W., Ultimate boundedness results for a certain system of third order nonlinear differential equations, *J. Math. Anal. Appl.* 177 (1993), no. 2, 496-509.

[21] Mirsky, L. An introduction to linear algebra. Dover Publications, Inc., New York, 1990.

[22] Reissig, R., sansone, G., and Conti, R., Nonlinear Differential Equations of Higher Order, Noordhoff, Groningen, 1974.

[23] Tiryaki, A. Boundedness and periodicty results for a certain system of third order non-linear differential equations. *Indian J. Pure Appl. Math.* 30, 4(1999), 361-372.

[24] Tejumola, H. O., A note on the boundedness and the stability of solutions of certain third-order differential equations, *Ann. Mat. Pura Appl.* (4) 92 (1972), 65-75.
[25] Tunç, C., On the boundedness and periodicity of the solutions of a certain vector differential equation of third-order, Chinese translation in *Appl. Math. Mech.* 20 (1999), no. 2, 153-160. *Appl. Math. Mech.* (English Ed.) 20 (1999), no. 2, 163-170.
[26] Tunç, C.; Ateş, M., Stability and boundedness results for solutions of certain third order nonlinear vector differential equations, *Nonlinear Dynamics. An International Journal of Nonlinear Dynamics and Chaos in Engineering Systems*, Volume 45, Numbers 3-4, (2006), 273 - 281.
[27] Tunç, C.; Ateş, M., On the Periodicity Results for Solutions of Some Certain Third Order Nonlinear Differential Equations, *Advances in Mathematical Sciences and Applications*, Volume 16, Number 1, (2006), 1-14.
[28] Tunç, C., On the boundedness of solutions of certain nonlinear vector differential equations of third order, *Bulletin Mathématique de la Société des Sciences Mathématiques de Roumanie*, Tome 49 (97), (2006), no. 3, 290-299.

Cemİl Tunç
Department of Mathematics,
Faculty of Arts and Sciences,
Yüzüncü Yil University,
65080 ,Van, TURKEY.
e-mail address: cemtunc@yahoo.com

Ercan Tunç
Department of Mathematics,
Faculty of Arts and Sciences,
Gaziosman Paşa University,
60080 ,Tokat, TURKEY.
e-mail address: ercantunc72@yahoo.com

(*Received August 23, 2005*)