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ALMOST PERIODIC SOLUTIONS OF C-WELL-POSED EVOLUTION EQUATIONS

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Abstract. This paper is concerned with the existence and uniqueness of almost periodic mild solutions of evolution equations of the form $\dot{u}(t) = Au(t) + f(t)$ where A is the generator of a holomorphic C-semigroup on a Banach space and f is an almost periodic function. A sufficient condition in terms of spectral properties of A and f is obtained that extends a well known result in this subject.

1. Introduction

In this paper we deal with conditions for the existence and uniqueness of almost periodic mild solutions with specific spectra to evolution equations of the form

$$\frac{du}{dt} = Au + f(t), \tag{1.1}$$

where A is a (unbounded) linear operator which generates a holomorphic C-semigroup of linear operators on a Banach space $\mathbb{X}$ and f is an almost periodic function taking values in $\mathbb{X}$.

This kind of problems has been of great interest to many mathematicians for decades. Actually, it goes back to the far-reaching result of O. Perron on the characterization of exponential dichotomy of linear ordinary differential equations. We refer the reader to [3, 4, 5, 9, 14, 17, 19, 20, 21, 23, 24, 28, 29] and the references therein for more information on recent and related results. Notice that most results (except for those in [4]) were obtained for evolution equations of the form (1.1) that are well-posed. As shown in many recent works (see, for instance, [6, 7, 10, 11, 13, 15, 16, 18, 25, 26, 27, 30] and the references therein) an important class of ill posed evolution equations can be treated in the framework of C-semigroups that was introduced by Da Prato [7], Davies and Pang [6].

As is well-known (see, e.g.,[9, 14, 19, 23, 28]), *if A generates a holomorphic C_0-semigroup and $\sigma(A) \cap i\overline{\sigma_b(f)} = \emptyset$, then Eq. (1.1) has a unique almost periodic mild solution u such that $\sigma_b(u) \subset \overline{\sigma_b(f)}$* (here $\sigma_b(f)$ denotes the

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Bohr spectrum of the given almost periodic function f whose definition is given in the next section). In the simplest case, when A generates a C_0-semigroup and the spectrum of A does not cut the imaginary axis, the above problem is directly concerned with the exponential dichotomy or stability of the semigroup generated by A (see e.g. [22]). Hence, for arbitrary closed operator A the above condition is not sufficient for the existence of an almost periodic mild solution.

In this paper, we aim at extending this result to the case where A is the generator of a holomorphic C-semigroup. Our main result (Theorem 3.8) becomes the above when $C = I$. To this end, we will use the method of sums of commuting operators (see [8, 12]). A modification of a result in [1] on the spectra of sums of commuting operators will be made in the next section that is the key tool to derive the main result of this paper (Theorem 3.8).

2. Preliminaries

2.1. **Notation.** Throughout the paper, $\mathbb{R}$, $\mathbb{C}$, $\mathbb{X}$ stand for the sets of real, complex numbers and a complex Banach space, respectively; $L(\mathbb{X})$, $BC(\mathbb{R},\mathbb{X})$, $BUC(\mathbb{R},\mathbb{X})$, $AP(\mathbb{X})$ denote the spaces of all linear bounded operators on $\mathbb{X}$, all $\mathbb{X}$-valued bounded and continuous functions, all $\mathbb{X}$-valued bounded uniformly continuous and all almost periodic functions in Bohr's sense (see the definition below) with sup-norm, respectively. For a linear operator A, we denote by $D(A)$, $\sigma(A)$ the domain and the spectrum of A.

2.2. **Almost periodic functions.** In this paper by almost periodic functions we mean the almost periodic functions in the sense of Bohr (we refer the reader to [17] for the definition and basic properties of such almost periodic functions). If f is an almost periodic function, the following limit

$$a(\lambda, f) := \lim_{T\to\infty} \frac{1}{2T}\int_{-T}^{T} f(t)e^{-i\lambda t}dt, \quad \text{for all } \lambda \in \mathbb{R} \tag{2.1}$$

exists and is called *the Bohr transform* of f. As is known, there is an at most countable set of reals λ such that the above limit differs from zero. This set will be denoted by $\sigma_b(f)$ and called *Bohr spectrum* of f. The Approximation Theorem (see [17]) says that for every almost periodic function f there exists a sequence of trigonometric polynomials $P_n(t) = \sum_{k=1}^{N_n} a_{k,n}e^{\lambda_{k,n}t}$, where $\lambda_{k,n} \in \sigma_b(f)$ and $a_{k,n} \in \mathbb{X}$ for all k, n, that converges uniformly in $t \in \mathbb{R}$ to f as $n \to \infty$.

2.3. **Spectral theory of functions.** In the present paper, for a given $u \in BC(\mathbb{R},\mathbb{X})$, $sp(u)$ stands for the Carleman spectrum of u, which consists of all $\xi \in \mathbb{R}$ such that the Carleman-Fourier transform of u, defined by

$$\hat{u}(\lambda) := \begin{cases} \int_0^\infty e^{-\lambda t} u(t)dt & (Re\lambda > 0) \\ \\ -\int_0^\infty e^{\lambda t} u(-t)dt & (Re\lambda < 0), \end{cases}$$

has no holomorphic extension to any neighborhood of $i\xi$ (see [2, 23]).

Proposition 2.1. Let $u, u_n, v \in BC(\mathbb{R}, \mathbb{X})$ such that $\lim_{n\to\infty} \|u_n - u\| = 0$. Then

(i) $sp(u(\cdot + h)) = sp(u)$ for any $h \in \mathbb{R}$;
(ii) $sp(Bu(\cdot)) \subset sp(u)$ for any $B \in L(\mathbb{X})$;
(iii) $sp(\alpha u(\cdot) \subset sp(u)$ for any $\alpha \in \mathbb{C}$;
(iv) If $sp(u_n) \subset \Lambda$, for all n, then $sp(u) \subset \overline{\Lambda}$, (here Λ is a subset of $\mathbb{R}$);
(v) If for $u \in BUC(\mathbb{R}, \mathbb{X})$, $sp(u)$ is countable, and $\mathbb{X}$ does not contain any subspace which is isomorphic to the space of numerical sequences c_0, then u is almost periodic;
(vi) If u is uniformly continuous and $sp(u)$ is discrete, then u is almost periodic;
(vii) If u is almost periodic, then $sp(u) = \overline{\sigma_b(u)}$.

Proof. We refer the reader to [2], [23, Prop. 0.4, Prop. 0.6, Theorem 0.8, p. 20-25] and [17, Chap. 6] for the proofs. □

If Λ is a closed subset of $\mathbb{R}$, then $\Lambda(\mathbb{X})$ stands for the subspace of $BUC(\mathbb{R}, \mathbb{X})$ consisting of all $g \in BUC(\mathbb{R}, \mathbb{X})$ such that $sp(g) \subset \Lambda$. As a consequence of the above properties, $\Lambda(\mathbb{X})$ *is a closed subspace of* $BUC(\mathbb{R}, \mathbb{X})$ *that is invariant under translations.* In particular, we have (see e.g. [19])

Lemma 2.2. *Let Λ be a closed subset of the real line. Then*

$$\sigma(\mathcal{D}_{\Lambda(\mathbf{X})}) = i\Lambda. \tag{2.2}$$

Let us denote by $AP_\Lambda(\mathbb{X})$ the subspace of $AP(\mathbb{X})$ consisting of all almost periodic functions g such that $sp(g) \subset \Lambda$. Then we can see that if f is almost periodic, then $\Lambda(\mathbb{X})$ is exactly $AP_\Lambda(\mathbb{X})$.

2.4. **C-semigroups.**

Definition 2.3. Let $\mathbb{X}$ be a Banach space and let C be an injective operator in $L(\mathbb{X})$. A family $\{T(t);\ t \geq 0\}$ in $L(\mathbb{X})$ is called an exponentially bounded C-semigroup if the following conditions are satisfied:

(i) $T(0) = C$,
(ii) $T(t + s)C = T(t)T(s)$ for $t,\ s \geq 0$,
(iii) $T(\cdot)x\colon\ [0, \infty) \to \mathbb{X}$ is continuous for any $x \in \mathbb{X}$,
(iv) There are $M \geq 0$ and $a \in \mathbb{R}$ such that $\|T(t)\| \leq Me^{at}$ for $t \geq 0$.

We define an operator A as follows:

$$\begin{aligned} D(A) &= \{x \in \mathbb{X} : \lim_{h \to 0^+} (T(h)x - Cx)/h \in R(C)\} \\ Ax &= C^{-1} \lim_{h \to 0^+} (T(h)x - Cx)/h, \quad \text{for all } x \in D(A). \end{aligned}$$

This operator is called the *generator* of $(T(t))_{t \geq 0}$. It is known that A is closed but not necessarily densely defined. Next we define the operator

$$\begin{aligned} Gx &= \lim_{t \to 0} C^{-1} \frac{T(t)x - Cx}{t} \\ D(G) &:= \{x \in R(C) : \exists \lim_{t \to 0^+} C^{-1} \frac{T(t)x - Cx}{t}\}. \end{aligned}$$

The *complete infinitesimal generator* of $(T(t))_{t \geq 0}$ is defined to be the operator $\overline{G}$ if $R(C)$ is dense in $\mathbb{X}$.

As in this paper we consider only exponentially bounded C-semigroups which, for the sake of brevity, will be referred to as C-semigroups unless otherwise stated.

Lemma 2.4. *Let C be an injective linear operator and let $(T(t))_{t \geq 0}$ be a C-semigroup with generator A. Then, the following assertions hold true:*

(i) $T(t)T(s) = T(s)T(t)$, *for all* $t, s \geq 0$,

(ii) *If $x \in D(A)$, then $T(t)x \in D(A)$, $AT(t)x = T(t)Ax$ and*

$$\int_0^t T(\xi)Axd\xi = T(t)x - Cx, \quad \text{for all } t \geq 0,$$

(iii) $\int_0^t T(\xi)xd\xi \in D(A)$ *and* $A\int_0^t T(\xi)xd\xi = T(t)x - Cx$ *for every* $x \in \mathbb{X}$ *and* $t \geq 0$,

(iv) *A is closed and satisfies $C^{-1}AC = A$,*

(v) $R(C) \subset \overline{D(A)}$,

(vi) *If $R(C)$ is dense in $\mathbb{X}$, then $\overline{D(G)} = \mathbb{X}$ and $G \subset A$.*

For more information about C-semigroups and the relations between them and the so-called integrated semigroups we refer the reader to [10, 15, 18, 16] and the references therein.

2.5. **Holomorphic C-semigroups.** Let C be an injective operator in $L(\mathbb{X})$ with range $R(C)$ dense in $\mathbb{X}$.

Definition 2.5. A *holomorphic C-semigroup* is a family of operators $\{T(t),\ t \in \mathbb{C}, \arg t < \delta\}$ in $L(\mathbb{X})$ (where $0 < \delta \leq \frac{\pi}{2}$) satisfying

(i) $T(t)T(s) = T(t+s)C$, for $|\arg t| < \delta$; $|\arg s| < \delta$ and $T(0) = C$;

(ii) $T(t)$ is holomorphic on $|\arg t| < \delta$;

(iii) $\lim_{t \to 0, |\arg t| \leq \delta - \epsilon} T(t)x = Cx$ for all $x \in \mathbb{X}$ and $\epsilon \in (0, \delta)$;

(iv) For each $\epsilon \in (0,\delta)$ there exist a real number a and a positive constant M_ϵ such that $\|T(t)\| \le M_\epsilon e^{aRet}$ for $|\arg t| \le \delta - \epsilon$.

The following result is due to Tanaka [25].

Theorem 2.6. *A closed linear operator A is the complete infinitesimal generator of a holomorphic C-semigroup $\{T(t), |\arg t| < \delta\}$ (where $0 < \delta \le \frac{\pi}{2}$) if and only if the following conditions hold*

(A_1) *There is a real number a such that for any $|\arg(\lambda - a)| < \frac{\pi}{2} + \delta$, $\lambda - A$ is injective, $R(\lambda - A) \supset R(C)$ and $(\lambda - A)^{-1}C$ is holomorphic,*

(A_2) *For $\epsilon \in (0,\pi)$ there is a positive constant M_ϵ such that*

$$\|(\lambda - A)^{-1}C\| \le M_\epsilon |\lambda - a|^{-1}$$

for $|\arg(\lambda - a)| \le \frac{\pi}{2} + \delta - \epsilon$.

(A_3) *$(\lambda - A)^{-1}Cx = C(\lambda - A)^{-1}x$ for all $x \in D((\lambda - A)^{-1})$ and $|\arg(\lambda - a)| < \frac{\pi}{2} + \delta$,*

(A_4) *$D(A)$ is dense in $\mathbb{X}$,*

(A_5) *$C(D(A))$ is a core for A.*

Proof. For the proof see [25, Theorem 1]. □

In the sequel, we need the following

Theorem 2.7. *Let C be an injective bounded linear operator with dense range in $\mathbb{X}$, and let A be a closed operator that satisfies the conditions (A_1)–(A_4) in Theorem 2.6. Then the family of operators $T(t)$ defined as*

$$T(t) := \begin{cases} \frac{1}{2\pi i}\int_\gamma e^{\lambda t}(\lambda - A)^{-1}C d\lambda, & \text{for } \ |\arg t| \le \delta - 2\epsilon, \\ C, & \text{for } \ t = 0, \end{cases}$$

where $\epsilon \in (0, \delta/2)$, δ is the constant defined in Theorem 2.6, and γ is a curve running in the sector $\{|\arg\lambda| < \pi/2 + \delta\}$ from $\infty e^{-i\phi}$ to $\infty e^{i\phi}$ with $\phi = \pi/2 + \delta - \epsilon$, forms a C-semigroup (with $t \ge 0$) (whose generator may be different from A).

Proof. For the proof and related remarks see [7] and [25]. □

2.6. **Sums of commuting operators and holomorphic C-semigroups.** In this section we will extend a result by Arendt-Rabiger-Sourour [1, Theorem 7.3] to the case of holomorphic C-semigroups.

Definition 2.8. Let A and B be operators on a Banach space G with nonempty resolvent set. We say that A *and* B *commute* if one of the following equivalent conditions hold:

(i) $R(\lambda, A)R(\mu, B) = R(\mu, B)R(\lambda, A)$ for some (all) $\lambda \in \rho(A), \mu \in \rho(B)$,

(ii) $x \in D(A)$ implies $R(\mu, B)x \in D(A)$ and $AR(\mu, B)x = R(\mu, B)Ax$ for some (all) $\mu \in \rho(B)$.

For any $\theta \in (0, \pi)$ and $R > 0$ we let $\Sigma(\theta, R) := \{z \in \mathbb{C} : |z| \geq R, |\arg z| \leq \theta\}$.

Theorem 2.9. *Let A and B be commuting operators on a Banach space $\mathbb{X}$ that both commute with the given injective bounded linear operator C, and let $D(A)$ be dense in $\mathbb{X}$. Assume further that there exist $R > 0$ and $\theta \in (0, \pi)$ such that*

(i) *The operator A is the generator of a holomorphic C-semigroup;*
(ii) *There exist $R > 0$ and $\theta_1 \in (0, \theta)$ such that*

$$\Sigma(\frac{\pi}{2} - \theta_1, R) \subset \rho(B)$$

and

$$\sup_{\lambda \in \Sigma(\frac{\pi}{2} - \theta_1, R)} \|\lambda R(\lambda, B)\| < \infty.$$

(iii) $\sigma(A) \cap \sigma(-B) = \oslash$.

Then, $A+B$ is closable and there exists an injective bounded linear operator Q commuting with A, B, C such that

$$Q\overline{A+B} \subset \overline{A+B}Q = C. \tag{2.3}$$

Proof. The proof is suggested by [1, Theorem 7.3]. Choose a rectifiable path Γ_0 lying in $\{z \in \rho(A) \cap \rho(B) : |z| \leq R\}$ starting from the point $Re^{-i(\pi/2+\theta)}$ and ending at $Re^{i(\pi/2+\theta)}$. Then, consider the oriented contour Γ_0 consisting of

$$\{re^{-i(\pi/2+\theta):r\geq R}\}, \quad \Gamma_0, \quad \{re^{i(\pi/2+\theta)} : r \geq R\}$$

Then there exists a partition $\mathbb{C} = \Omega_- \cup \Gamma_0 \cup \Omega_+$, where Ω_-, Ω_+ are open such that

$$int(\Sigma(\theta + \pi/2, R)) \subset \Omega_+, \quad \{re^{i\alpha} : r \geq R, \alpha \in (\theta + \pi/2, 3\pi/2 - \theta)\} \subset \Omega_-.$$

So, we have

$$\{\lambda \in \sigma(A) : |\lambda| \geq R\} \subset \Omega_-, \quad \{\lambda \in \sigma(-B) : |\lambda| \geq R\} \subset \Omega_+.$$

There exist compact subsets K_+, K_- with oriented (piecewise C^1) boundary Γ_+ and Γ_-, respectively such that

$$\Omega_- \cap \sigma(-B) \subset intK_- \subset K_- \subset \Omega_- \backslash \sigma(A)$$

and

$$\Omega_+ \cap \sigma(A) \subset intK_+ \subset K_+ \subset \Omega_+ \backslash \sigma(-B)$$

Let $\Gamma = \Gamma_0 \cup (-\Gamma_-) \cup \Gamma_+$. Since $\sup_{\lambda\in\Gamma} \lambda|^2|\|R(\lambda, A)CR(\lambda, -B)\| < \infty$,

$$\begin{aligned} Q &:= \frac{1}{2\pi i}\int_\Gamma R(\lambda, A)CR(\lambda, -B)d\lambda \\ Q_t &:= \frac{1}{2\pi i}\int_\Gamma e^{\lambda t}R(\lambda, A)CR(\lambda, -B)d\lambda, \quad (t > 0) \end{aligned}$$

define bounded operators on $\mathbb{X}$ that satisfy $\lim_{t\to 0^+} Q_t x = Qx$, for all $x \in \mathbb{X}$. By Theorem 2.7, the following family of operators

$$T(t) := \begin{cases} \frac{1}{2\pi i}\int_\Gamma e^{\lambda t}R(\lambda, A)Cd\lambda, & \text{for } \ |\arg t| \le \delta - 2\epsilon, \\ C, \quad \text{for } \ t = 0, \end{cases}$$

forms a C-semigroup (with $t \ge 0$). In particular, we have $\lim_{t\to 0} T(t)x = Cx$ for all $x \in \mathbb{X}$. Since A, B, C commute with each other,

$$(A + B)R(\lambda, A)CR(\lambda, -B) = (R(\lambda, A) - R(\lambda, -B))C.$$

This yields in particular that $Q_t x \in D(A + B)$, for all $x \in \mathbb{X}$. Next, we have

$$(A + B)Q_t x = \frac{1}{2\pi i}\int_\Gamma e^{\lambda t}R(\lambda, A)Cxd\lambda - \frac{1}{2\pi i}\int_\Gamma e^{\lambda t}R(\lambda, -B)Cxd\lambda.$$

Since $\int_\Gamma e^{\lambda t}R(\lambda, -B)d\lambda = 0$ by Cauchy's theorem, we have

$$(A + B)Q_t x = T(t)x, \qquad \text{for all } x \in \mathbb{X}. \tag{2.4}$$

Moreover, for $x \in D(A + B)$, since A is closed and A, B, C are commuting,

$$\begin{aligned} Q_t Ax &= \frac{1}{2\pi i}\int_\Gamma e^{\lambda t}R(\lambda, A)CR(\lambda, -B)Axd\lambda \\ &= A\frac{1}{2\pi i}\int_\Gamma e^{\lambda t}R(\lambda, A)CR(\lambda, -B)xd\lambda = AQ_t x \end{aligned}$$

Similarly, we can show that $Q_t Bx = BQ_t x$ for all $x \in D(A+B)$. This yields that for $x \in D(A + B)$ we have $Q_t(A + B)x = (A + B)Q_t x$. By the same argument, we can show that for all $x \in D(A+B)$, $Q(A+B)x = (A+B)Qx$. Next, for $x \in D(A + B)$ by (2.4) we have

$$Q_t(A + B)x = (A + B)Q_t x = T(t)x.$$

Thus, letting $t \downarrow 0$, since $T(t)x \to Cx$, we obtain

$$Q(A + B)x = Cx, \qquad \text{for all } x \in D(A + B)$$

Therefore, for all $x \in D(A + B)$, we have $(A + B)Qx = Q(A + B)x = Cx$. From the commutativeness of A, B, C for any $\mu \in \rho(A) \cap \rho(B)$ it follows that

$$\begin{aligned} R(\mu, A)R(\mu, B)Cx &= R(\mu, A)R(\mu, B)(A + B)Qx \\ &= (R(\mu, A) - R(\mu, B))Qx. \end{aligned}$$

Hence, if $Qx = 0$, then $R(\mu, A)R(\mu, B)Cx = 0$. The injectiveness of $C, R(\mu, A), R(\mu, B)$ yields that $x = 0$, that is the injectiveness of Q. Next, to show the closability of $A + B$ we suppose that $x_n \in D(A + B)$ such that $x_n \to 0$ and $(A + B)x_n \to y \in \mathbb{X}$ as $n \to \infty$. Then

$$Qy = \lim_{n\to\infty} Q(A + B)x_n = \lim_{n\to\infty} Cx_n = 0.$$

Since Q is injectiveness, $y = 0$, that is, the closability of the operator $A+B$. Next, let $x \in \mathbb{X}$. Then, $\lim_{t\downarrow 0} Q_t x = Qx$ and

$$\lim_{t\downarrow 0}(A + B)Q_t x = \lim_{t\downarrow 0} T(t)x = Cx.$$

Therefore, $Qx \in D(\overline{A + B})$ and

$$\overline{A + B}Qx = Cx, \quad (\text{ for all } x \in \mathbb{X}) \tag{2.5}$$

Conversely, let $x \in D(\overline{A + B})$ and let $x_n \in D(A+B)$ such that $x_n \to x$ and $(A + B)x_n \to \overline{A + B}x$. Then

$$\begin{aligned} Q\overline{A + B}x &= \lim_{n\to\infty} Q(A + B)x_n \\ &= \lim_{n\to\infty} Cx_n = Cx. \end{aligned}$$

This and (2.5) prove the theorem. □

3. Almost periodic mild solutions

In this section we consider the equation

$$\frac{dx}{dt} = Ax + f(t), \quad x \in \mathbb{X}, t \in \mathbb{R}, \tag{3.1}$$

where A generates an exponentially bounded C-semigroup of bounded linear operators on $\mathbb{X}$ and f is a bounded and uniformly continuous function. We always assume in this section that C is an injection with $R(C)$ dense in $\mathbb{X}$ unless otherwise stated. The main purpose of this section is to find conditions for the existence and uniqueness of almost periodic mild solutions to Eq. (3.1) when f is almost periodic. To this end, we first recall the following definitions:

Definition 3.1. (i) An $\mathbb{X}$-valued function u on $\mathbb{R}$ is said to be a *solution on* $\mathbb{R}$ to Eq.(3.1) for given linear operator A and $f \in BC(\mathbb{R}, \mathbb{X})$ (or sometimes, *classical solution*) if $u \in BC^1(\mathbb{R}, \mathbb{X}), u(t) \in D(A)$, for all t and u satisfies Eq.(3.1) for all $t \in \mathbb{R}$.

(ii) An $\mathbb{X}$-valued continuous function u on $\mathbb{R}$ is said to be a *mild solution on* $\mathbb{R}$ to Eq.(3.1) for a given $f \in BC(\mathbb{R}, \mathbb{X})$ if u satisfies

$$Cu(t) = T(t - s)u(s) + \int_s^t T(t - \xi)f(\xi)d\xi, \quad \text{for all } t \geq s. \tag{3.2}$$

Lemma 3.2. *Let $(T(t))_{t\geq 0}$ be a C-semigroup with generator A and let f be a continuous function on the real line. Then, u is a solution of Eq. (3.2) if and only if both of the following assertions hold:*

$$\int_0^t Cu(\xi)d\xi \in D(A), \qquad \text{for all } t \in \mathbb{R} \tag{3.3}$$

$$Cu(t) - Cu(0) = A\int_0^t Cu(s)ds + \int_0^t Cf(s)ds, \qquad \text{for all } t \in \mathbb{R}. \tag{3.4}$$

Proof. For the proof see [4, Lemma 2.9]. □

From this lemma, by the injectivity of the operator C, it follows in particular that every classical solution is a mild one.

Definition 3.3. Let Λ be a closed subset of $\mathbb{R}$ and let $(T(t))_{t\geq 0}$ be a C-semigroup on $\mathbb{X}$. Then the following family of bounded linear operators $(T^h)_{h\geq 0}$ on $AP_\Lambda(\mathbb{X})$ is called the evolution semigroup associated with $(T(t))_{t\geq 0}$ on $AP_\Lambda(\mathbb{X})$:

$$(T^h v)(t) := T(h)v(t-h), \quad v \in AP_\Lambda(\mathbb{X}), h \geq 0. \tag{3.5}$$

Let us denote by $\tilde{C}$ the operator of multiplication by C on $AP_\Lambda(\mathbb{X})$.

Lemma 3.4. *Under the above notation the evolution semigroup $(T^h)_{h\geq 0}$ is a $\tilde{C}$-semigroup with generator $-\mathcal{L}_\mathcal{M}$, where $\mathcal{M} := AP(\mathbb{X})$.*

Proof. For the proof see [4, Lemma 2.6]. □

We introduce the following operator $L_\mathcal{M}$.

Definition 3.5. Let $\mathcal{M}$ be a closed subspace of $BUC(\mathbb{R}, \mathbb{X})$. We define the operator $L_\mathcal{M}$ on $\mathcal{M}$ as follows: $u \in D(L_\mathcal{M})$ if and only if $u \in \mathcal{M}$ and there is $f \in \mathcal{M}$ such that

$$Cu(t) = T(t-s)u(s) + \int_s^t T(t-r)f(r)dr, \qquad \text{for all } t \geq s \tag{3.6}$$

and in this case $L_\mathcal{M} u := f$.

Let us denote the differential operator d/dt in $AP_\Lambda(\mathbb{X})$ by $\mathcal{D}$ and the function space $AP_\Lambda(\mathbb{X})$ by $\mathcal{M}$. It is obvious that $\mathcal{D}, \tilde{A}$ and $\tilde{C}$ commute with each other. Moreover, we have

Lemma 3.6. *Let A be the generator of a C-semigroup. Then, under the above notations, the operator $L_\mathcal{M}$ is well-defined single valued operator. Moreover,*

$$L_\mathcal{M} = \overline{\mathcal{D} - \tilde{A}} \tag{3.7}$$

Proof. The fact that $L_{\mathcal{M}}$ is a single valued operator was established in [4]. By Lemma 2.6 of [4], $-L_{\mathcal{M}}$ is the generator of the evolution $\tilde{C}$-semigroup $(T^h)_{h\geq 0}$. On the other hand, by [16, Theorem 2.1], the generator of this evolution semigroup is nothing but $\overline{-\mathcal{D}+\tilde{A}}$. So, $L_{\mathcal{M}} = \overline{\mathcal{D}-\tilde{A}}$. □

We are now in a position to apply the method of sums of commuting operators to derive conditions for the existence of almost periodic mild solutions to Eq. (1.1). To this end, assume that the operator A in Eq. (1.1) generates a holomorphic C-semigroup $T(t)$ on the Banach space $\mathbb{X}$. Let us denote by Λ the closure of the Bohr spectrum of the almost periodic function f. Consider the operators $\tilde{T}(t)$ of multiplication by $T(t)$ and the operator $\tilde{A}$ of multiplication by A on $\mathcal{M} := AP_\Lambda(\mathbb{X})$. Recall that, by definition, $D(\tilde{A})$ consists of all $g \in AP_\Lambda(\mathbb{X})$ such that $g(t) \in D(A)$, for all $t \in \mathbb{R}$ and $Ag(\cdot) \in AP_\Lambda(\mathbb{X})$.

Lemma 3.7. *Under the above assumptions and notations, the family of operators of multiplication $\tilde{T}(t)$ is a holomorphic $\tilde{C}$-semigroup with generator $\tilde{A}$ on $AP_\Lambda(\mathbb{X})$;*

Proof. First, we show that the range of $\tilde{C}$ is dense in $AP_\Lambda(\mathbb{X})$. In fact, by the assumption, every trigonometric monomial $e^{i\lambda t}a$, $\lambda \in \Lambda, a \in \mathbb{X}$ can be approximated by a sequence of trigonometric monomials $e^{i\lambda t}a_k$ with $a_k \in R(C)$, $k = 1, 2, \cdots$. So, every trigonometric polynomial $P_n(t) := \sum_{k=1}^n a_{k,n}e^{i\lambda_{k,n}t}$, with $\lambda_{k,n} \in \Lambda$ and $a_{k,n} \in \mathbb{X}$, can be approximated by a trigonometric polynomial $Q_n(t) := \sum_{k=1}^n b_{k,n}e^{i\lambda_{k,n}t}$ with $b_{k,n} \in R(C)$. Note that $Q_n \in R(\tilde{C})$. On the other hand, by the Approximation Theorem of Almost Periodic Functions, every function in $AP_\Lambda(\mathbb{X})$ can be approximated by a sequence of trigonometric polynomials in $\mathbb{X}$ with exponents in Λ. Therefore, it can be approximated by a sequence of trigonometric polynomials with coefficients in $R(C)$ and exponents in Λ (that are in $R(\tilde{C})$). So, the range of $\tilde{C}$ is dense in $AP_\Lambda(\mathbb{X})$.

The precompactness of the range of each function in $AP_\Lambda(\mathbb{X})$ yields the convergence in $AP_\Lambda(\mathbb{X})$ of the limit $\lim_{t\to 0, |\arg t|\leq \delta-\epsilon} \tilde{T}(t)g = \tilde{C}g$ for all $g \in AP_\Lambda(\mathbb{X})$ and $\epsilon \in (0,\delta)$. Therefore, $\tilde{T}(t)$ is a holomorphic $\tilde{C}$-semigroup. It remains to show that its generator is exactly $\tilde{A}$. First of all, suppose that $g \in D(\tilde{A})$. Then, for each $t \in \mathbb{R}$ there exists $\lim_{h\downarrow 0}(T(h)g(t) - Cg(t))/h \in R(C)$. From the precompactness of the range of g and the strong continuity of $(T(h))_{h\geq 0}$ it follows that this limit is uniform in $t \in \mathbb{R}$. Hence, $\lim_{h\downarrow 0}(\tilde{T}(h)g - Cg)/h$ exists as an element w of $AP_\Lambda(\mathbb{X})$ with range in $R(C)$. Since $C^{-1}w(\cdot) = Ag(\cdot) = \tilde{A}g \in AP_\Lambda(\mathbb{X})$, we have that $g = Cw \in R(\tilde{C})$. Hence, $g \in D(\mathcal{A})$, where $\mathcal{A}$ denotes the generator of the C-semigroup $\tilde{T}(t)$. Obviously, $\tilde{A}g = \mathcal{A}g$. This shows that $\tilde{A} \subset \mathcal{A}$. Conversely, let $g \in D(\mathcal{A})$.

Then, by definition, we can easily see that for each $t \in \mathbb{R}$, $g(t) \in D(A)$ and $Ag(\cdot) \in AP_\Lambda(\mathbb{X})$, that is, $\mathcal{A} \subset \tilde{A}$. Finally, we obtain that $\tilde{A}$ is exactly the generator of the C-semigroup $\tilde{T}(t)$. □

We now state the main result of this paper.

Theorem 3.8. *Let A be the generator of a holomorphic C-semigroup with $R(C)$ dense in $\mathbb{X}$, and let the following conditions be satisfied:*

(i) *There exists an almost periodic function g such that $f(t) = Cg(t)$, for all $t \in \mathbb{R}$;*
(ii) *$\sigma(A) \cap isp(g) = \oslash$.*

Then, the following assertions hold:

(i) *There exists an almost periodic mild solution u to Eq. (3.1) such that $sp(u) \subset sp(g)$;*
(ii) *If there exists another bounded mild solution v to Eq. (3.1) such that $sp(v) \subset sp(g)$, then $u(t) = v(t)$, for all $t \in \mathbb{R}$.*

Proof. (i) Let us denote $\Lambda := sp(g)$. We can easily show $\sigma(\tilde{A}) \subset \sigma(A)$. And by Lemma 2.2 we obtain that $\sigma(\mathcal{D}) = i\Lambda = i\sigma(g)$. By Theorem 2.9, there exists an injective bounded linear operator Q on $AP_\Lambda(\mathbb{X})$ such that $\overline{\mathcal{D} - \tilde{A}}Q = \tilde{C}$. Setting $u = Qg$, we have $\overline{\mathcal{D} - \tilde{A}}Qg = \tilde{C}g = f$. Therefore, $u = Qg$ is an almost periodic mild solution of Eq. (3.1) such that $sp(u) \subset \Lambda = sp(g)$.

(ii) The uniqueness of the solution u is an immediate consequence of the following estimate: For any bounded mild solution w of Eq. (3.1) we have

$$sp(Cw) \quad \subset \quad \sigma_i(A) \cup sp(g) \ , \tag{3.8}$$

where $\sigma_i(A) := \{\xi \in \mathbb{R} : i\xi \in \sigma(A)\}$. Assume that there is $\xi_0 \in \mathbb{R}$ such that $i\xi_0 \in \rho(A)$. Taking the Laplace transforms of both sides of (3.4) we can easily show that (for $Re\lambda \neq 0$ and λ close to $i\xi_0$),

$$\widehat{Cw}(\lambda) = R(\lambda, A)Cw(0) + R(\lambda, A)\widehat{Cf}(\lambda). \tag{3.9}$$

Therefore, if $\xi_0 \notin sp(g)$, (so $\xi_0 \notin sp(f)$ by Proposition 2.1 (ii)), the function $\widehat{Cw}(\lambda)$ has an analytic extension around $i\xi$. Thus, $sp(Cw(\cdot)) \subset \sigma_i(A) \cup sp(Cf(\cdot))$. And hence we have (3.8).

Obviously $[Cu(\cdot) - Cv(\cdot)]$ is an almost periodic mild solution of Eq. (3.1) with f replaced by 0. In view of (3.8), $isp(Cu(\cdot) - Cv(\cdot)) \subset \sigma_i(A)$. On the other hand, since both u and v are in $\Lambda(\mathbb{X})$ we have $isp(Cu(\cdot) - Cv(\cdot)) \subset \sigma_i(A) \cap i\Lambda = \sigma_i(A) \cap isp(g) = \oslash$. This yields that $[Cu(\cdot) - Cv(\cdot)] = 0$.

□

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