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SOME HOMOTOPY GROUPS OF HOMOGENEOUS SPACES

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Abstract. The symplectic group is embedded in the rotation group and the quotient set equipped with the identification topology is a homogeneous space. The purpose of this paper is to determine some homotopy groups of the homogeneous space. Exact sequences induced from fibrations are frequently used, and homotopy groups of Lie groups and other homogeneous spaces which are obtained by several authors are referred heavily.

1. Introduction

Let SO_n be the rotation group, U_n the unitary group and Sp_n the symplectic group. The unitary group U_n is embedded in the rotation group SO_{2n} and so the homogeneous space SO_{2n}/U_n is defined. Let $\Gamma_n = SO_{2n}/U_n$. Similarly, the symplectic group Sp_n can be considered as the subgroup of U_{2n} or SO_{4n}. Denote by X_n and Y_n homogeneous spaces U_{2n}/Sp_n and SO_{4n}/Sp_n, respectively. Bott's results of the stable homotopy group of Γ_n and X_n are well known (see [1]) and some nonstable homotopy groups are determined by several authors in [3, 6, 11, 15, 16]. Some homotopy groups of Y_n are studied together with Γ_n and X_n in [2].

The main purpose of this paper is to calculate homotopy groups $\pi_{4n+k}(Y_n)$ for $k \leq 5$ and $n \geq 2$. To state our result, we use the following notations. Let $\mathbb{Z}$ be the group of integers and set $\mathbb{Z}_n = \mathbb{Z}/n\mathbb{Z}$ for a positive integer n. The direct sum $\mathbb{Z}_n \oplus \cdots \oplus \mathbb{Z}_n$ of m copies of $\mathbb{Z}_n$ is denoted by $(\mathbb{Z}_n)^m$. Let (n, m) be the greatest common divisor of natural numbers n and m. Note that, when n is even, the groups $\pi_{4n}(Y_n) \cong \pi_{4n+1}(Y_n) \cong (\mathbb{Z}_2)^3$ are already obtained in [2]. Our result is stated as follows.

Theorem 1.1. *Let $n \geq 2$. If $-1 \leq k \leq 5$, the homotopy group $\pi_{4n+k}(Y_n)$ is isomorphic to the group given by the following table except $\pi_{13}(Y_3)$ and $\pi_{15}(Y_3)$. $\pi_{13}(Y_3)$ is isomorphic to $(\mathbb{Z}_2)^3$ or $\mathbb{Z}_4 \oplus \mathbb{Z}_2$.*

$n\backslash k$	-1	0	1	2	3	4	5
even	$\mathbb{Z} \oplus \mathbb{Z}_4$	$(\mathbb{Z}_2)^3$	$(\mathbb{Z}_2)^3$	$\mathbb{Z}_{8(3,n+1)}$	$\mathbb{Z} \oplus \mathbb{Z}_2$	$\mathbb{Z}_2$	$(\mathbb{Z}_2)^3$
odd	$\mathbb{Z}$	$(\mathbb{Z}_2)^2$	$(\mathbb{Z}_2)^3$	$\mathbb{Z}_{4(3,n+1)} \oplus \mathbb{Z}_2$	$\mathbb{Z} \oplus \mathbb{Z}_4$	$\mathbb{Z}_2$	$(\mathbb{Z}_2)^2$

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Our main method is to use exact sequences induced from fibrations. Calculations need group structures of many homotopy groups of classical groups SO_n, U_n, Sp_n and homogeneous spaces Γ_n, X_n. We rely heavily on results of several authors [1, 3, 6, 8, 10, 11, 13, 15, 16].

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2. Fundamental facts

The unitary group U_n and the symplectic group Sp_n are embedded in the rotation group SO_{2n} and the unitary group U_{2n}, respectively. These inclusions are denote by $r_n : U_n \to SO_{2n}$ and $c_n : Sp_n \to U_{2n}$. The subscript n may be omitted if no confusion occurs.

The fibration $Sp_n \xrightarrow{rc} SO_{4n} \to Y_n$ induces an exact sequence

$$\pi_{4n-1}(Sp_n) \xrightarrow{(rc)_*} \pi_{4n-1}(SO_{4n}) \to \pi_{4n-1}(Y_n) \to \pi_{4n-2}(Sp_n).$$

Since $\pi_{4n-2}(Sp_n) = 0$ (see [1]), the group $\pi_{4n-1}(Y_n)$ is isomorphic to the cokernel of $(rc)_*$. To determine the cokernel, we study maps r_* and c_* by making use of exact sequences induced from fibrations $Sp_n \xrightarrow{c} U_{2n} \to X_n$ and $U_{2n} \xrightarrow{r} SO_{4n} \to \Gamma_{2n}$. Then we have the following.

Proposition 2.1. *If $n \geq 2$, then $\pi_{4n-1}(Y_n) \cong \mathbb{Z} \oplus \mathbb{Z}_4$ when n is even and $\pi_{4n-1}(Y_n) \cong \mathbb{Z}$ when n is odd.*

Similarly, the group structure of $\pi_k(Y_n)$ for $1 \leq k \leq 4n-2$ is obtained easily by the Bott periodicity.

Proposition 2.2. *If $1 \leq k \leq 4n-2$, the group structure of $\pi_k(Y_n)$ is as follows.*

$k =$	$8s$	$8s+1$	$8s+2$	$8s+3$	$8s+4$	$8s+5$	$8s+6$	$8s+7$
	$\mathbb{Z}_2$	$\mathbb{Z}_2$	0	0	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_4$

Let $V_{n,m}$ be the real Stiefel manifold SO_n/SO_{n-m} for $n > m$. There exist natural homeomorphisms $SO_{2n+1}/U_n \approx SO_{2n+2}/U_{n+1}$ and $U_{2n+1}/Sp_n \approx U_{2n+2}/Sp_{n+1}$ (see [3]). Then SO_{4n+3}/Sp_n and SO_{4n+4}/Sp_{n+1} are homeomorphic, and so the fibration

$$SO_{4n}/Sp_n \to SO_{4n+3}/Sp_n \to SO_{4n+3}/SO_{4n}$$

is written as $Y_n \to Y_{n+1} \to V_{4n+3,3}$. In addition, we also use fibrations $X_n \to Y_n \to \Gamma_{2n}$ and $U_n \to SO_{2n+1} \to \Gamma_{n+1}$. In almost all cases, the notation Δ means the connecting homomorphism of the exact sequence induced from a fibration.

Hereafter, we will not distinguish the maps and their homotopy classes. Notations and results of [18] are used. Let $\iota_n \in \pi_n(S^n)$ for $n \geq 1$ and $\eta_n \in \pi_{n+1}(S^n)$ for $n \geq 2$ be generators, and let η_n^k be the composition

$\eta_n \cdots \eta_{n+k-1}$. By abuse of notation, $\nu_n \in \pi_{n+3}(S^n) \cong \mathbb{Z}_{24}$ for $n \geq 5$ is used to denote the generator of $\pi_{n+3}(S^n)$. For a cyclic group G, we denote by $G\{\alpha\}$ the cyclic group isomorphic to G with the generator α.

Let $\mathbb{F}$ be the reals $\mathbb{R}$, the complexes $\mathbb{C}$ or the quaternions $\mathbb{H}$. Denote by $G_n(\mathbb{F})$ the rotation group SO_n, the unitary group U_n or the symplectic group Sp_n. Let $\gamma_n(\mathbb{F}) : S^{d(n+1)-2} \to G_n(\mathbb{F})$ be the characteristic map for the bundle $G_n(\mathbb{F}) \to G_{n+1}(\mathbb{F}) \to S^{d(n+1)-1}$, where $d = \dim_{\mathbb{R}} \mathbb{F}$.

Consider the exact sequence

$$\pi_{4n+3}(S^{4n}) \xrightarrow{\Delta} \pi_{4n+2}(SO_{4n}) \xrightarrow{i_*} \pi_{4n+2}(SO_{4n+1}),$$

where $i : SO_{4n} \to SO_{4n+1}$ is the inclusion. The image of Δ is generated by $\gamma_{4n}(\mathbb{R})\nu_{4n-1}$. From the proof of [15, Theorem 4], the nontriviality of $i_*r_*\gamma_{2n}(\mathbb{C})\eta_{4n}^2 \in \pi_{4n+2}(SO_{4n+1})$ is given. Then

$$r_*\gamma_{2n}(\mathbb{C})\eta_{4n}^2 \neq 0 \text{ and } r_*\gamma_{2n}(\mathbb{C})\eta_{4n}^2 \neq \gamma_{4n}(\mathbb{R})\eta_{4n-1}^3 = 12\gamma_{4n}(\mathbb{R})\nu_{4n-1}. \tag{2.1}$$

It is also obtained in the proof of [15, Theorem 4] that i_* is epimorphic, and

$$\pi_{4n+2}(SO_{4n}) \cong \begin{cases} \mathbb{Z}_8 \oplus \mathbb{Z}_{24} & n \text{ is even and } n \geq 2, \\ \mathbb{Z}_4 \oplus \mathbb{Z}_{24} & n = 3, \\ \mathbb{Z}_4 \oplus \mathbb{Z}_{48} & n \text{ is odd and } n \geq 5, \end{cases} \tag{2.2}$$

which is generated by two elements $r_*c_*\gamma_n(\mathbb{H})$ and $\gamma_{4n}(\mathbb{R})\nu_{4n-1}$. Since $\pi_{4n+2}(SO_{4n+1}) \cong \mathbb{Z}_8$ (see [8]), the image of Δ is isomorphic to $\mathbb{Z}_{12}$ when $n = 3$ and $\mathbb{Z}_{24}$ when $n \neq 3$, that is,

$$12\gamma_{12}(\mathbb{R})\nu_{11} = 0 \text{ and } 12\gamma_{4n}(\mathbb{R})\nu_{4n-1} \neq 0 \text{ when } n \neq 3. \tag{2.3}$$

Furthermore, the relation of [15, Theorem 4] implies that the homomorphism $(rc)_* : \pi_{4n+2}(Sp_n) \to \pi_{4n+2}(SO_{4n})$ has the image

$$\mathrm{Im}(rc)_* \cong \begin{cases} \mathbb{Z}_{24/(3,n+1)} & n \text{ is even and } n \geq 2, \\ \mathbb{Z}_{24} & n = 3, \\ \mathbb{Z}_{48/(3,n+1)} & n \text{ is odd and } n \geq 5. \end{cases} \tag{2.4}$$

3. Proof of the theorem

The proof of the main theorem relies on the next lemma.

Lemma 3.1. *Let n be odd and $n \geq 3$. Then*

1. $\pi_{4n}(SO_{4n}) = \mathbb{Z}_2\{r_*\gamma_{2n}(\mathbb{C})\} \oplus \mathbb{Z}_2\{\gamma_{4n}(\mathbb{R})\eta_{4n-1}\}$.
2. $\pi_{4n+1}(SO_{4n}) = \mathbb{Z}_2\{r_*\gamma_{2n}(\mathbb{C})\eta_{4n}\} \oplus \mathbb{Z}_2\{\gamma_{4n}(\mathbb{R})\eta_{4n-1}^2\}$.
3. $(\eta_{4n+1})^* : \pi_{4n+1}(SO_{4n}) \to \pi_{4n+2}(SO_{4n})$ *is monomorphic for $n \geq 5$, and the kernel of $(\eta_{13})^*$ is $\mathbb{Z}_2\{\gamma_{12}(\mathbb{R})\eta_{11}^2\}$.*

Proof. Assume that n is odd and $n \geq 3$. Note that $\pi_{4n}(SO_{4n}) \cong (\mathbb{Z}_2)^2$ and $\pi_{4n+1}(SO_{4n}) \cong (\mathbb{Z}_2)^2$ are already obtained in [8]. In the exact sequence

$$\pi_{4n}(U_{2n}) \xrightarrow{r_*} \pi_{4n}(SO_{4n}) \to \pi_{4n}(\Gamma_{2n}) \xrightarrow{\Delta} \pi_{4n-1}(U_{2n}),$$

it is known that $\pi_{4n}(\Gamma_{2n}) \cong \mathbb{Z}_2$ (see [3]) and $\pi_{4n-1}(U_{2n}) \cong \mathbb{Z}$ (see [1]). Then $\Delta = 0$ and, by the group structure $\pi_{4n}(U_{2n}) = \mathbb{Z}_{(2n)!}\{\gamma_{2n}(\mathbb{C})\}$ (see [1]), there exists a direct summand $\mathbb{Z}_2\{r_*\gamma_{2n}(\mathbb{C})\}$ in $\pi_{4n}(SO_{4n})$. Next, consider the exact sequence

$$\pi_{4n+1}(S^{4n}) \to \pi_{4n}(SO_{4n}) \xrightarrow{i_*} \pi_{4n}(SO_{4n+1}) \to \pi_{4n}(S^{4n}),$$

where $i : SO_{4n} \to SO_{4n+1}$ is the inclusion. Similarly, since $\pi_{4n+1}(S^{4n}) = \mathbb{Z}_2\{\eta_{4n}\}$, $\pi_{4n}(SO_{4n+1}) \cong \mathbb{Z}_2$ (see [8]) and $\pi_{4n}(S^{4n}) \cong \mathbb{Z}$, there exists a direct summand $\mathbb{Z}_2\{\gamma_{4n}(\mathbb{R})\eta_{4n-1}\}$ in $\pi_{4n}(SO_{4n})$. By the exact sequence

$$\pi_{4n}(U_{2n}) \xrightarrow{(ir)_*} \pi_{4n}(SO_{4n+1}) \to \pi_{4n}(\Gamma_{2n+1})$$

and the group structure $\pi_{4n}(\Gamma_{2n+1}) = 0$ (see [1]), we have $i_*r_*\gamma_{2n}(\mathbb{C}) \neq 0$. Note that $i_*\gamma_{4n}(\mathbb{R})\eta_{4n-1} = 0$. Hence, $r_*\gamma_{2n}(\mathbb{C})$ is not equal to $\gamma_{4n}(\mathbb{R})\eta_{4n-1}$. This leads us to (1).

We have $\pi_{4n+1}(U_{2n}) = \mathbb{Z}_2\{\gamma_{2n}(\mathbb{C})\eta_{4n}\}$ by making use of the fibration $U_{2n} \to U_{2n+1} \to S^{4n+1}$. From this, and by the argument similar to that of (1), the assertion of (2) is obtained. Properties (2.1) and (2.3) imply (3). $\square$

Let $\theta \in \pi_{4n-1}(Sp_n) \cong \mathbb{Z}$ (see [1]) be a generator. By [13, Theorem 2.1], $\pi_{4n}(Sp_n) = \mathbb{Z}_2\{\theta\eta_{4n-1}\}$ and $\pi_{4n+1}(Sp_n) = \mathbb{Z}_2\{\theta\eta^2_{4n-1}\}$ when n is odd.

Proposition 3.2. *Let n be odd. Then $\pi_{4n}(Y_n) \cong (\mathbb{Z}_2)^2$ for $n \geq 3$ and $\pi_{4n+1}(Y_n) \cong (\mathbb{Z}_2)^3$ for $n \geq 5$.*

Proof. In the calculation of Proposition 2.1, it is obtained that the map $(rc)_* : \pi_{4n-1}(Sp_n) \to \pi_{4n-1}(SO_{4n})$ is monomorphic. Then there exists an exact sequence

$$\pi_{4n}(Sp_n) \xrightarrow{(rc)_*} \pi_{4n}(SO_{4n}) \to \pi_{4n}(Y_n) \to 0.$$

Since the generator $\theta\eta_{4n-1}$ of $\pi_{4n}(Sp_n)$ is of order 2, the element $c_*\theta\eta_{4n-1} \in \pi_{4n}(U_{2n}) = \mathbb{Z}_{(2n)!}\{\gamma_{2n}(\mathbb{C})\}$ is in $((2n)!/2)\pi_{4n}(U_{2n})$, where the integer $(2n)!/2$ is even. Hence, by Lemma 3.1(1), $(rc)_*\theta\eta_{4n-1} = 0$. This implies that $\pi_{4n}(Y_n) \cong \pi_{4n}(SO_{4n}) \cong (\mathbb{Z}_2)^2$ and $(rc)_* : \pi_{4n+1}(Sp_n) \to \pi_{4n+1}(SO_{4n})$ is trivial. Therefore, we obtain the exact sequence

$$0 \to \pi_{4n+1}(SO_{4n}) \xrightarrow{p_*} \pi_{4n+1}(Y_n) \xrightarrow{\Delta} \pi_{4n}(Sp_n) \to 0,$$

where $p : SO_{4n} \to Y_n$ is the projection. Let $\beta \in \pi_{4n+1}(Y_n)$ be an element satisfying $\Delta\beta = \theta\eta_{4n-1}$. Consider the Toda bracket

$$\{rc, \theta\eta_{4n-1}, 2\iota_{4n}\} \subset \pi_{4n+1}(SO_{4n}).$$

If $0 \in \{rc, \theta\eta_{4n-1}, 2\iota_{4n}\}$, then $0 \in rc \circ \{\theta\eta_{4n-1}, 2\iota_{4n}, \eta_{4n}\}$, that is, there exists $\delta \in \{\theta\eta_{4n-1}, 2\iota_{4n}, \eta_{4n}\}$ such that $(rc)_*\delta = 0$. For this δ, by [11, Lemma 2.1], there exists an element $\varepsilon \in \pi_{4n+1}(SO_{4n})$ such that $p_*\varepsilon = 2\beta$ and $0 = (rc)_*\delta = \varepsilon\eta_{4n+1}$. By Lemma 3.1(3), the relation $\varepsilon\eta_{4n+1} = 0$ implies that $\varepsilon = 0$ for $n \geq 5$. Hence, $2\beta = 0$, and the above exact sequence splits for $n \geq 5$. Therefore, we shall prove $0 \in \{rc, \theta\eta_{4n-1}, 2\iota_{4n}\}$. Since $\pi_{4n+2}(Sp_n) \cong \mathbb{Z}_{2\cdot(2n+1)!}$ (see [3]) and

$$\begin{aligned}4\{\theta\eta_{4n-1}, 2\iota_{4n}, \eta_{4n}\} &= -(\theta\eta_{4n-1} \circ \{2\iota_{4n}, \eta_{4n}, 4\iota_{4n+1}\}) \\ &\subset -(\theta\eta_{4n-1} \circ \{2\iota_{4n}, 0, 2\iota_{4n+1}\}) \ni 0 \bmod 0,\end{aligned}$$

the Toda bracket $\{\theta\eta_{4n-1}, 2\iota_{4n}, \eta_{4n}\}$ is the subset of $((2n+1)!/2)\pi_{4n+2}(Sp_n)$. Note that $(2n+1)!/2 \equiv 0 \bmod 24$ for $n \geq 3$ and $(2n+1)!/2 \equiv 0 \bmod 48$ for $n \geq 5$. Then, by (2.2),

$$(\eta_{4n+1})^*\{rc, \theta\eta_{4n-1}, 2\iota_{4n}\} = -((rc)_*\{\theta\eta_{4n-1}, 2\iota_{4n}, \eta_{4n}\}) = 0.$$

By Lemma 3.1(3), this implies that $0 \in \{rc, \theta\eta_{4n-1}, 2\iota_{4n}\}$ for $n \geq 5$. □

Proposition 3.3. *If $n \geq 2$, then $\pi_{4n+2}(Y_n) \cong \mathbb{Z}_{8(3,n+1)}$ when n is even and $\pi_{4n+2}(Y_n) \cong \mathbb{Z}_{4(3,n+1)} \oplus \mathbb{Z}_2$ when n is odd.*

Proof. If n is even, then $\pi_{4n+1}(Sp_n) = 0$ (see [1]). Hence, there exists an exact sequence

$$\pi_{4n+2}(Sp_n) \xrightarrow{(rc)_*} \pi_{4n+2}(SO_{4n}) \to \pi_{4n+2}(Y_n) \to 0.$$

By (2.2) and (2.4), the group $\pi_{4n+2}(Y_{4n})$ is isomorphic to $\mathbb{Z}_{8(3,n+1)}$ when n is even.

If n is odd, then the homomorphism $(rc)_* : \pi_{4n+1}(Sp_n) \to \pi_{4n+1}(SO_{4n})$ is trivial by the proof of Proposition 3.2. So, there is an exact sequence

$$\pi_{4n+2}(Sp_n) \xrightarrow{(rc)_*} \pi_{4n+2}(SO_{4n}) \to \pi_{4n+2}(Y_n) \xrightarrow{\Delta} \pi_{4n+1}(Sp_n) \to 0.$$

Similarly, by (2.2) and (2.4), the cokernel of $(rc)_*$ is isomorphic to $\mathbb{Z}_{4(3,n+1)}$ for $n \geq 3$. By the proof of Proposition 3.2, $\Delta(\beta\eta_{4n+1}) = \theta\eta^2_{4n-1}$. Since $2(\beta\eta_{4n+1}) = 0$, this leads us to the assertion and completes the proof. □

We note that the diagram

(3.1)
$$\begin{tikzcd} \pi_{k+1}(\Gamma_{2n}) \arrow[rr, "\Delta"] \arrow[dr, "\Delta'"'] & & \pi_k(X_n) \\ & \pi_k(U_{2n}) \arrow[ur, "{p'}_*"'] & \end{tikzcd}$$

is commutative, where $p' : U_{2n} \to X_n$ is the projection. We show

Proposition 3.4. *If n is even and $n \geq 2$, then $\pi_{4n+3}(Y_n) \cong \mathbb{Z} \oplus \mathbb{Z}_2$.*

Proof. Consider the exact sequence

$$\pi_{4n+4}(\Gamma_{2n}) \xrightarrow{\Delta} \pi_{4n+3}(X_n) \to \pi_{4n+3}(Y_n) \to \pi_{4n+3}(\Gamma_{2n}) \xrightarrow{\Delta} \pi_{4n+2}(X_n).$$

In the diagram (3.1) for $k = 4n+2$, we know

$$\pi_{4n+2}(U_{2n}) = \mathbb{Z}_{(2n+1)!}\{c_*\gamma_n(\mathbb{H})\} \oplus \mathbb{Z}_2\{\gamma_{2n}(\mathbb{C})\eta^2_{4n}\}$$

(see [15, Theorem 4]). By (2.1), the image of Δ' is in $\mathbb{Z}_{(2n+1)!}\{c_*\gamma_n(\mathbb{H})\}$ and ${p'}_*\Delta' = \Delta : \pi_{4n+3}(\Gamma_{2n}) \to \pi_{4n+2}(X_n)$ is trivial.

Next, consider the diagram (3.1) for $k = 4n+3$. Since $\pi_{4n+4}(SO_{4n}) = 0$ (see [8]), $\pi_{4n+4}(\Gamma_{2n}) \cong \mathbb{Z}_{(12,n)}$ (see [6]) and $\pi_{4n+3}(U_{2n}) \cong \mathbb{Z}_{2(12,n)}$ (see [10]), Δ' is monomorphic and the image is $2\pi_{4n+3}(U_{2n})$. From the group structure of $\pi_{4n+2}(U_{2n})$ as above, $\pi_{4n+2}(Sp_n) = \mathbb{Z}_{(2n+1)!}\{\gamma_n(\mathbb{H})\}$ (see [3]) is naturally embedded in $\pi_{4n+2}(U_{2n})$ and so ${p'}_*$ is epimorphic. Then the cokernel of $\Delta : \pi_{4n+4}(\Gamma_{2n}) \to \pi_{4n+3}(X_n)$ is isomorphic to $\mathbb{Z}_2$ because $\pi_{4n+3}(X_n) \cong \mathbb{Z}_{(24,n)}$ (see [11]) and n is even. Since $\pi_{4n+3}(\Gamma_{2n}) \cong \mathbb{Z}$ (see [3]), the assertion is obtained. □

We show the following to complete the proof of the main theorem.

Lemma 3.5. *If $n \geq 2$, then*

(1) $\pi_{4n+3}(SO_{4n}) \cong \mathbb{Z} \oplus \mathbb{Z}_2$ *has a direct summand* $\mathbb{Z}_2\{r_*\gamma_{2n}(\mathbb{C})\nu_{4n}\}$.
(2) $r_*c_*\gamma_n(\mathbb{H})\eta_{4n+2} = n(r_*\gamma_{2n}(\mathbb{C})\nu_{4n})$.

Proof. By making use of the fibration $U_{2n} \to U_{2n+1} \to S^{4n+1}$, we have $\pi_{4n+3}(U_{2n}) = \mathbb{Z}_{2(12,n)}\{\gamma_{2n}(\mathbb{C})\nu_{4n}\}$. In the proof of Proposition 3.4, it is shown that the connecting homomorphism $\pi_{4n+4}(\Gamma_{2n}) \to \pi_{4n+3}(U_{2n})$ has the image $2\pi_{4n+3}(U_{2n})$ when n is even. Then (1) is proved when n is even.

Assume that n is odd. Consider the commutative diagram

$$\begin{tikzcd}
\pi_{4n+1}(V_{4n+6,6}) \arrow[r, "\Delta_1"] \arrow[d, "{\nu_{4n+1}}^*"'] & \pi_{4n}(SO_{4n}) \arrow[d, "{\nu_{4n}}^*"] \\
\pi_{4n+4}(V_{4n+6,6}) \arrow[r, "\Delta_2"'] & \pi_{4n+3}(SO_{4n}).
\end{tikzcd}$$

Here, Δ_1 is isomorphic and Δ_2 is monomorphic (see [8]). We use the group structure of $\pi_{4n+l}(V_{4n+k,k})$ (cf. [17]). Let $\alpha \in \pi_{4n+1}(V_{4n+2,2}) \cong \mathbb{Z} \oplus \mathbb{Z}_2$ be a generator of the direct summand $\mathbb{Z}$. The exact sequence induced from the fibration $V_{4n+2,2} \to V_{4n+3,3} \to S^{4n+2}$ leads to the group structure $\pi_{4n+4}(V_{4n+2,2}) = \mathbb{Z}_{24}\{\alpha\nu_{4n+1}\}$. Similarly, by use of the fibration $V_{4n+k,k} \to V_{4n+k+1,k+1} \to S^{4n+k}$ for $2 \leq k \leq 5$, we see that $\pi_{4n+1}(V_{4n+6,6}) \cong (\mathbb{Z}_2)^2$ has the direct summand $\mathbb{Z}_2\{j_*\alpha\}$, where $j : V_{4n+2,2} \to V_{4n+6,6}$ is the map induced from the inclusion $SO_{4n+2} \to SO_{4n+6}$. By the exact sequence

$$\pi_{4n+4}(V_{4n+2,2}) \xrightarrow{j_*} \pi_{4n+4}(V_{4n+6,6}) \to \pi_{4n+4}(V_{4n+6,4})$$

and $\pi_{4n+4}(V_{4n+6,4}) = 0$, we have $\pi_{4n+4}(V_{4n+6,6}) = \mathbb{Z}_2\{j_*\alpha\nu_{4n+1}\}$. Hence, in the above diagram, $\operatorname{Im}\nu_{4n}{}^* = \operatorname{Im}(\nu_{4n}{}^*\Delta_1) = \operatorname{Im}(\Delta_2\nu_{4n+1}{}^*) \cong \operatorname{Im}\nu_{4n+1}{}^* \cong \mathbb{Z}_2$. By Lemma 3.1(1) and the relation $\eta_{4n-1}\nu_{4n} = 0$, we have (1).

In [14, Lemma 2.1], the relation $c_*\gamma_n(\mathbb{H})\eta_{4n+2} = n(\gamma_{2n}(\mathbb{C})\nu_{4n})$ is obtained. This leads to (2) and completes the proof. □

Proposition 3.6. *If n is odd and $n \geq 5$, then $\pi_{4n+3}(Y_n) \cong \mathbb{Z} \oplus \mathbb{Z}_4$.*

Proof. By the same argument in the proof of Proposition 3.4, the connecting homomorphism $\pi_{4n+3}(\Gamma_{2n}) \to \pi_{4n+2}(X_n)$ is trivial. Next, we examine the cokernel of $\pi_{4n+4}(\Gamma_{2n}) \to \pi_{4n+3}(X_n)$ by use of the diagram (3.1). Consider the exact sequence

$$\pi_{4n+3}(Sp_n) \xrightarrow{c_*} \pi_{4n+3}(U_{2n}) \xrightarrow{p'_*} \pi_{4n+3}(X_n)$$

and groups $\pi_{4n+3}(Sp_n) = \mathbb{Z}_2\{\gamma_n(\mathbb{H})\eta_{4n+2}\}$ (see [13]), $\pi_{4n+3}(U_{2n}) \cong \mathbb{Z}_{2(12,n)}$. Since n is odd, $\pi_{4n+3}(U_{2n}) \cong \mathbb{Z}_2 \oplus \mathbb{Z}_{(3,n)}$ and, by Lemma 3.5, $c_*\gamma_n(\mathbb{H})\eta_{4n+2} \neq 0$. Then the image of p'_* is isomorphic to $\mathbb{Z}_{(3,n)}$. In the exact sequence

$$\pi_{4n+4}(SO_{4n}) \to \pi_{4n+4}(\Gamma_{2n}) \xrightarrow{\Delta'} \pi_{4n+3}(U_{2n}),$$

$\pi_{4n+4}(SO_{4n}) \cong \mathbb{Z}_2$ (see [8]) and $\pi_{4n+4}(\Gamma_{2n}) \cong \mathbb{Z}_{2(12,n)} \cong \mathbb{Z}_2 \oplus \mathbb{Z}_{(3,n)}$ (see [6]). So, Δ' maps the odd component isomorphically. Hence, by the diagram (3.1) and the group $\pi_{4n+3}(X_n) \cong \mathbb{Z}_2 \oplus \mathbb{Z}_{(3,n)}$ (see [11]), the cokernel of $\Delta : \pi_{4n+4}(\Gamma_{2n}) \to \pi_{4n+3}(X_n)$ is isomorphic to $\mathbb{Z}_2$. Therefore, there exists an exact sequence

$$0 \to \mathbb{Z}_2 \to \pi_{4n+3}(Y_n) \to \pi_{4n+3}(\Gamma_{2n}) \to 0.$$

By (2.2), $\pi_{4n+3}(\Gamma_{2n}) \cong \mathbb{Z} \oplus \mathbb{Z}_2$ for $n \geq 5$ (see [6]) and so $\pi_{4n+3}(Y_n)$ is isomorphic to $\mathbb{Z} \oplus (\mathbb{Z}_2)^2$ or $\mathbb{Z} \oplus \mathbb{Z}_4$. In the exact sequence

$$\pi_{4n+3}(Sp_n) \xrightarrow{(rc)_*} \pi_{4n+3}(SO_{4n}) \to \pi_{4n+3}(Y_n) \to \pi_{4n+2}(Sp_n),$$

the cokernel of $(rc)_*$ is isomorphic to $\mathbb{Z}$ by Lemma 3.5, and $\pi_{4n+2}(Sp_n)$ is a cyclic group. Therefore, $\pi_{4n+3}(Y_n) \cong \mathbb{Z} \oplus \mathbb{Z}_4$. □

Proposition 3.7. $\pi_{4n+4}(Y_n) \cong \mathbb{Z}_2$ *for $n \geq 2$.*

Proof. Consider the exact sequence

$$\pi_{4n+4}(Sp_n) \to \pi_{4n+4}(SO_{4n}) \to \pi_{4n+4}(Y_n) \to \pi_{4n+3}(Sp_n) \to \pi_{4n+3}(SO_{4n}).$$

By Lemma 3.5 and $\pi_{4n+3}(Sp_n) = \mathbb{Z}_2\{\gamma_n(\mathbb{H})\eta_{4n+2}\}$, the kernel of $(rc)_* : \pi_{4n+3}(Sp_n) \to \pi_{4n+3}(SO_{4n})$ is $\pi_{4n+3}(Sp_n)$ when n is even and 0 when n is odd. If n is even, $\pi_{4n+4}(SO_{4n}) = 0$ leads to $\pi_{4n+4}(Y_n) \cong \mathbb{Z}_2$. If n is odd,

$\pi_{4n+4}(Sp_n) = \mathbb{Z}_2\{\gamma_n(\mathbb{H})\eta^2_{4n+2}\}$ (see [13]). By Lemma 3.5 and the relation $\nu_{4n}\eta_{4n+3} = 0$, the image of $(rc)_* : \pi_{4n+4}(Sp_n) \to \pi_{4n+4}(SO_{4n})$ is generated by $r_*\gamma_{2n}(\mathbb{C})\nu_{4n}\eta_{4n+3} = 0$. Then $\pi_{4n+4}(Y_n) \cong \pi_{4n+4}(SO_{4n}) \cong \mathbb{Z}_2$. □

Proposition 3.8. *If $n \geq 2$, then $\pi_{4n+5}(Y_n) \cong (\mathbb{Z}_2)^3$ when n is even and $\pi_{4n+5}(Y_n) \cong (\mathbb{Z}_2)^2$ when n is odd.*

Proof. In the exact sequence

$$\pi_{4n+6}(V_{4n+3,3}) \to \pi_{4n+5}(Y_n) \to \pi_{4n+5}(Y_{n+1}) \to \pi_{4n+5}(V_{4n+3,3}),$$

$\pi_{4n+5}(V_{4n+3,3}) \cong \mathbb{Z}_2$ (see [17]). Let P^n be the n-dimensional real projective space and set $\mathrm{P}^n_m = \mathrm{P}^n/\mathrm{P}^{m-1}$ for $n \geq m$. Since the pair $(V_{n,m}, \mathrm{P}^{n-1}_{n-m})$ is $(2n-2m)$-connected (see [5]) and P^{4n+2}_{4n} is of the same homotopy type as $\mathrm{P}^{4n+2}_{4n+1} \vee S^{4n}$, we have $\pi_{4n+6}(V_{4n+3,3}) \cong \mathbb{Z}_2$. Then the above sequence is

$$\mathbb{Z}_2 \to \pi_{4n+5}(Y_n) \to (\mathbb{Z}_2)^3 \to \mathbb{Z}_2.$$

Hence, $\pi_{4n+5}(Y_n)$ is isomorphic to $(\mathbb{Z}_2)^2$, $(\mathbb{Z}_2)^3$, $(\mathbb{Z}_2)^4$, $\mathbb{Z}_4 \oplus \mathbb{Z}_2$ or $\mathbb{Z}_4 \oplus (\mathbb{Z}_2)^2$. Furthermore, when n is even, by the continuation of the above exact sequence and the group structure $\pi_{4n+4}(V_{4n+3,3}) \cong (\mathbb{Z}_2)^2$ (see [17]), there is an exact sequence

$$\mathbb{Z}_2 \to \pi_{4n+5}(Y_n) \to (\mathbb{Z}_2)^3 \to \mathbb{Z}_2 \to \mathbb{Z}_2 \to (\mathbb{Z}_2)^3 \to (\mathbb{Z}_2)^2.$$

This implies that the image of $\pi_{4n+5}(Y_n) \to \pi_{4n+5}(Y_{n+1})$ is isomorphic to $(\mathbb{Z}_2)^2$ and so $\pi_{4n+5}(Y_n)$ is isomorphic to $(\mathbb{Z}_2)^2$, $(\mathbb{Z}_2)^3$ or $\mathbb{Z}_4 \oplus \mathbb{Z}_2$ when n is even.

On the other hand, consider the exact sequence

$$\pi_{4n+5}(SO_{4n}) \xrightarrow{p_*} \pi_{4n+5}(Y_n) \to \pi_{4n+4}(Sp_n) \to \pi_{4n+4}(SO_{4n}),$$

where $p : SO_{4n} \to Y_n$ is the projection. By making use of the isomorphism $\pi_{n+k}(SO_n) \cong \pi_{n+k}(SO_{n+m}) \oplus \pi_{n+k+1}(V_{n+m,m})$ for $m > k+2$, $n > 13$, $k < n-2$ (see [9]), and by [1, 4],

$$\begin{aligned}\pi_{4n+5}(SO_{4n}) &\cong \pi_{4n+5}(SO_{4n+8}) \oplus \pi_{4n+6}(V_{4n+8,8}) \\ &\cong \begin{cases} (\mathbb{Z}_2)^2 & n \text{ is even and } n \geq 4 \\ \mathbb{Z}_2 & n \text{ is odd and } n \geq 5. \end{cases}\end{aligned}$$

By [12], $\pi_{13}(SO_8) \cong (\mathbb{Z}_2)^2$ and, by [7], the free part and the 2-primary component of $\pi_{17}(SO_{12})$ is isomorphic to $\mathbb{Z}_2$. Since $\pi_{4n+4}(Sp_n)$ is isomorphic to $(\mathbb{Z}_2)^2$ when n is even and $\mathbb{Z}_2$ when n is odd (see [13]), the assertion of this proposition is clearly obtained when n is odd.

Assume that n is even. Since $\pi_{4n+4}(SO_{4n}) = 0$, the above sequence is

$$(\mathbb{Z}_2)^2 \xrightarrow{p_*} \pi_{4n+5}(Y_n) \to (\mathbb{Z}_2)^2 \to 0.$$

By the proof of [15, Theorem 2i)], the element $p_*\gamma_{4n}(\mathbb{R})\nu^2_{4n-1}$ is nontrivial and not divisible by two. Then $\pi_{4n+5}(Y_n)$ is isomorphic to $(\mathbb{Z}_2)^3$, $(\mathbb{Z}_2)^4$ or $\mathbb{Z}_4 \oplus (\mathbb{Z}_2)^2$. Therefore, $\pi_{4n+5}(Y_n) \cong (\mathbb{Z}_2)^3$ when n is even. □

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