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SELF-HOMOTOPY OF THE DOUBLE SUSPENSION OF THE REAL 7-PROJECTIVE SPACE

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Abstract. We determine the group structure of the self-homotopy set of the double suspension of the real 7-dimensionnal projective space.

1. Introduction

In this paper, all spaces, maps and homotopies are based. We use the same notation as [10] and [5]. Let $\Sigma^n X$ be an n-fold suspension of a space X and P^n be the n-dimensional real projective space. The purpose of the present paper is to determine the group structure of the homotopy set $[\Sigma^2\mathrm{P}^7, \Sigma^2\mathrm{P}^7]$. We denote by $\gamma_n : S^n \to \mathrm{P}^n$ the covering map. According to [9], $\Sigma^2\gamma_6 = 0$, $\Sigma^2\mathrm{P}^7 = \Sigma^2\mathrm{P}^6 \vee S^9$, and so

$$[\Sigma^2\mathrm{P}^7, \Sigma^2\mathrm{P}^7] \cong [\Sigma^2\mathrm{P}^6, \Sigma^2\mathrm{P}^6] \oplus \pi_9(\Sigma^2\mathrm{P}^6) \oplus \pi_9(S^9).$$

Let $\mathbf{Z}$ be the group of integers and set $\mathbf{Z}_n = \mathbf{Z}/n\mathbf{Z}$. The notation $(\mathbf{Z}_n)^m$ means a direct sum of m-copies of $\mathbf{Z}_n$. Our result is stated as follows.

Theorem 1.1. $[\Sigma^2\mathrm{P}^7, \Sigma^2\mathrm{P}^7] \cong \mathbf{Z} \oplus (\mathbf{Z}_8)^2 \oplus (\mathbf{Z}_2)^7$.

In this paper we sometimes identify a map with its homotopy class. For $m < n$, let $i_{m,n} : \mathrm{P}^m \to \mathrm{P}^n$ and $p_{n,m} : \mathrm{P}^n \to \mathrm{P}^n/\mathrm{P}^m$ be the inclusion and collapsing maps, respectively. Especially, we write $M^n = \Sigma^{n-2}\mathrm{P}^2$, $i_n = \Sigma^{n-2}i_{1,2} : S^{n-1} \to M^n$ and $p_n = \Sigma^{n-2}p_{2,1} : M^n \to S^n$ for $n \geq 2$. We denote by $[\alpha, \beta]$ the Whitehead product of homotopy classes α and β. To determine the group structure of $\pi_9(\Sigma^2\mathrm{P}^6)$, we use the following.

Theorem 1.2. $[\Sigma^2\gamma_5, \Sigma^2 i_{1,5}] = 0 \in \pi_9(\Sigma^2\mathrm{P}^5)$.

2. Some homotopy groups

We denote by $\iota_X \in [X, X]$ the identity class of a space X and let $\iota_n = \iota_{S^n}$. For the Hopf maps $\eta_2 \in \pi_3(S^2)$ and $\nu_4 \in \pi_7(S^4)$, we set $\eta_n = \Sigma^{n-2}\eta_2$, $\eta_n^2 = \eta_n\eta_{n+1}$, $\eta_n^3 = \eta_n\eta_{n+1}\eta_{n+2}$ for $n \geq 2$ and $\nu_n = \Sigma^{n-4}\nu_4$ for $n \geq 4$. We recall from [7] that there is an element $\widetilde{\eta}_2 \in \pi_4(M^3)$ such that $p_3\widetilde{\eta}_2 = \eta_3$ and $\Sigma\widetilde{\eta}_2 = \widetilde{\eta}_3$, where $\widetilde{\eta}_3$ is a coextension of η_3. Let $\overline{\eta}_3 \in [M^5, S^3]$ be an extension of η_3 and set $\widetilde{\eta}_n = \Sigma^{n-2}\widetilde{\eta}_2$ for $n \geq 2$ and $\overline{\eta}_n = \Sigma^{n-3}\overline{\eta}_3$ for $n \geq 3$.

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Let ν' be a generator of the group $\pi_6(S^3) \cong \mathbf{Z}_{12}$ and λ_2 be the attaching map of the 7-cell of the Stiefel manifold $V_{5,2} = M^4 \cup_{\lambda_2} e^7$. We recall that $\pi_6(M^4) = \mathbf{Z}_4\{\lambda_2\} \oplus \mathbf{Z}_2\{\widetilde{\eta}_3\eta_5\}$ [6]. We note $\pi_9(M^4) = (\mathbf{Z}_2)^3$ [12, Theorem 5.8] and, by use of these facts and the homotopy exact sequence of a pair $(V_{5,2}, M^4)$, we determine the generators.

Lemma 2.1. $\pi_9(M^4) = \mathbf{Z}_2\{\lambda_2\nu_6\} \oplus \mathbf{Z}_2\{[\lambda_2, i_4]\eta_8\} \oplus \mathbf{Z}_2\{\widetilde{\eta}_3\nu_5\eta_8\}$.

Let $s : S^5 \to \Sigma^2\mathrm{P}^3 = M^4 \vee S^5$ be the inclusion to the second factor. Then, we recall

$$\Sigma^2\gamma_3 = 2s \pm (\Sigma^2 i_{2,3})\widetilde{\eta}_3. \tag{2.1}$$

By the Hilton-Milnor theorem, we obtain

$$\pi_i(\Sigma^2\mathrm{P}^3) \cong \pi_i(M^4) \oplus \pi_i(S^5) \oplus \pi_i(M^8) \oplus \pi_i(\Sigma(M^7 \wedge M^3)), \tag{2.2}$$

for $i \leq 9$. By Lemma 2.1 and the facts that $\pi_8(M^4) = \mathbf{Z}_2\{\lambda_2\eta_6^2\} \oplus \mathbf{Z}_2\{[i_4, \lambda_2]\} \oplus \mathbf{Z}_2\{\widetilde{\eta}_3\nu_5\}$ [8, Lemma 2.4], $\pi_8(S^5) = \mathbf{Z}_{24}\{\nu_5\}$, $\pi_8(M^8) = \mathbf{Z}_2\{i_8\eta_7\}$, $\pi_9(S^5) = \mathbf{Z}_2\{\nu_5\eta_8\}$, $\pi_9(M^8) = \mathbf{Z}_4\{\widetilde{\eta}_7\}$ and $\pi_9(\Sigma(M^7 \wedge M^3)) = \mathbf{Z}_2\{\Sigma(i_7 \wedge i_3)\}$, we have the following.

Lemma 2.2.

$$\begin{aligned}
(1)\quad \pi_8(\Sigma^2\mathrm{P}^3) = \;& \mathbf{Z}_2\{(\Sigma^2 i_{2,3})\lambda_2\eta_6^2\} \oplus \mathbf{Z}_2\{(\Sigma^2 i_{2,3})[i_4, \lambda_2]\} \\
& \oplus \mathbf{Z}_2\{(\Sigma^2 i_{2,3})\widetilde{\eta}_3\nu_5\} \oplus \mathbf{Z}_{24}\{s\nu_5\} \oplus \mathbf{Z}_2\{[\Sigma^2 i_{1,3}, s]\eta_7\}, \\
(2)\quad \pi_9(\Sigma^2\mathrm{P}^3) = \;& \mathbf{Z}_2\{(\Sigma^2 i_{2,3})\lambda_2\nu_6\} \oplus \mathbf{Z}_2\{(\Sigma^2 i_{2,3})[i_4, \lambda_2]\eta_8\} \\
& \oplus \mathbf{Z}_2\{(\Sigma^2 i_{2,3})\widetilde{\eta}_3\nu_5\eta_8\} \oplus \mathbf{Z}_2\{s\nu_5\eta_8\} \\
& \oplus \mathbf{Z}_4\{[\Sigma^2 i_{2,3}, s]\widetilde{\eta}_7\} \oplus \mathbf{Z}_2\{[[\Sigma^2 i_{1,3}, s], \Sigma^2 i_{1,3}]\}.
\end{aligned}$$

Let X be a connected finite CW-complex and $X^* = X \cup_\theta e^n$ for $\theta : S^{n-1} \to X$ a complex formed by attaching an n-cell. We denote by

$$\omega_n^{(X^*,X)} \in \pi_n(X^*, X)$$

the characteristic map of the n-cell e^n of X^*. Let CY be a cone of a space Y. For an element $\alpha \in \pi_m(Y)$, we denote by $\widehat{\alpha}' \in \pi_{m+1}(CY, Y)$ an element satisfying $\partial'(\widehat{\alpha}') = \alpha$, where $\partial' : \pi_{m+1}(CY, Y) \to \pi_m(Y)$ is the connecting bijection. For $\alpha \in \pi_m(S^{n-1})$, we set

$$\widehat{\alpha} = \omega_n^{(X^*,X)} \circ \widehat{\alpha}' \in \pi_{m+1}(X^*, X).$$

We note the following:

$$\partial(\widehat{\alpha}) = \theta \circ \alpha \quad \text{and} \quad p_*\widehat{\alpha} = \Sigma\alpha,$$

where $\partial : \pi_{m+1}(X^*, X) \to \pi_m(X)$ is the boundary map and $p : (X^*, X) \to (S^n, *)$ is the collapsing map. Now we show the following.

Lemma 2.3. $2((\Sigma^2 i_{3,4})_*\pi_9(\Sigma^2\mathrm{P}^3)) = 0$.

Proof. We consider the homotopy exact sequence of a pair $(\Sigma^2\mathrm{P}^4, \Sigma^2\mathrm{P}^3)$:

$$\pi_{10}(\Sigma^2\mathrm{P}^4, \Sigma^2\mathrm{P}^3) \xrightarrow{\partial_{10}} \pi_9(\Sigma^2\mathrm{P}^3) \xrightarrow{(\Sigma^2 i_{3,4})_*} \pi_9(\Sigma^2\mathrm{P}^4).$$

There exists an element $[\omega_6, s] \in \pi_{10}(\Sigma^2\mathrm{P}^4, \Sigma^2\mathrm{P}^3)$ for $\omega_6 = \omega_6^{(\Sigma^2\mathrm{P}^4,\Sigma^2\mathrm{P}^3)}$. By the relations [1, (3.5)] and (2.1), we have

$$\partial_{10}([\omega_6, s]) = -[\Sigma^2\gamma_3, s] = \pm[(\Sigma^2 i_{2,3})\widetilde{\eta}_3, s] = \pm[\Sigma^2 i_{2,3}, s]\widetilde{\eta}_7.$$

Hence, by Lemma 2.2 (2), we obtain $2((\Sigma^2 i_{3,4})_*\pi_9(\Sigma^2\mathrm{P}^3)) = 0$. This completes the proof. □

Since $\Sigma^2\gamma_3 \circ \eta_5^3 = (\Sigma^2 i_{2,3})\widetilde{\eta}_3 \circ 4\nu_5 = 0$ by (2.1), there exists an element $\widetilde{\eta_5^3} \in \{\Sigma^2 i_{3,4}, \Sigma^2\gamma_3, \eta_5^3\} \subset \pi_9(\Sigma^2\mathrm{P}^4)$ such that $(\Sigma^2 p_{4,3})\widetilde{\eta_5^3} = \eta_6^3$. For this element, we show the following.

Lemma 2.4. $\{\widetilde{\eta}_3\eta_5^2, \eta_7, 2\iota_8\} = \widetilde{\eta}_3\nu_5\eta_8$ *and the order of* $\widetilde{\eta_5^3}$ *is two.*

Proof. By the properties of Toda brackets and by [3, Lemma 4.1], we have

$$\{\widetilde{\eta}_3\eta_5^2, \eta_7, 2\iota_8\} \circ p_9 = -(\widetilde{\eta}_3\eta_5^2 \circ \{\eta_7, 2\iota_8, p_8\}) = \widetilde{\eta}_3\eta_5^2\overline{\eta}_7 = \widetilde{\eta}_3\nu_5\eta_8 p_9.$$

Since $p_9{}^* : \pi_9(M^4) \to [M^9, M^4]$ is a monomorphism by Lemma 2.1, we obtain the first. By (2.1), the relation $(\Sigma^2 i_{2,4})\widetilde{\eta}_3\nu_5\eta_8 = \pm 2((\Sigma^2 i_{3,4})s) \circ \widetilde{\eta}_3\nu_5\eta_8 = 0$ holds. So, by the first and Lemma 2.3, we have

$$\begin{aligned}
2\widetilde{\eta_5^3} &\in \{\Sigma^2 i_{3,4}, \Sigma^2\gamma_3, \eta_5^3\} \circ 2\iota_9 \\
&= -(\Sigma^2 i_{3,4} \circ \{\Sigma^2\gamma_3, \eta_5^3, 2\iota_8\}) \\
&\supset -(\Sigma^2 i_{2,4} \circ \{\widetilde{\eta}_3\eta_5^2, \eta_7, 2\iota_8\}) \\
&= (\Sigma^2 i_{2,4})\widetilde{\eta}_3\nu_5\eta_8 = 0 \quad \mathrm{mod}\ 2((\Sigma^2 i_{3,4})_*\pi_9(\Sigma^2\mathrm{P}^3)) = 0.
\end{aligned}$$

This leads to the second and completes the proof. □

Next we compute the homotopy groups of the homotopy fibre of $\Sigma^2 p_{4,3} : \Sigma^2\mathrm{P}^4 \to S^6$ to determine $\pi_9(\Sigma^2\mathrm{P}^4)$. Let K be the homotopy fibre of $\Sigma^2 p_{4,3}$. By [2, Corollary 5.8], the 10-skeleton of K has a cellular decomposition

$$K^{(10)} = \Sigma^2\mathrm{P}^3 \cup_{[\iota_{\Sigma^2\mathrm{P}^3}, \Sigma^2\gamma_3]} C\Sigma^6\mathrm{P}^3.$$

For $m < n$, we denote by $i_{m,n}^K : K^{(m)} \to K^{(n)}$ and $i_m^K : K^{(m)} \to K$ the inclusion maps and $p_{n,m}^K : K^{(n)} \to K^{(n)}/K^{(m)}$ the collapsing map.

Lemma 2.5.

(1) $\pi_8(K) = \mathbf{Z}_2\{i_4^K[i_4, \lambda_2]\} \oplus \mathbf{Z}_2\{i_4^K\widetilde{\eta}_3\nu_5\} \oplus \mathbf{Z}_{24}\{i_5^K s\nu_5\} \oplus \mathbf{Z}_2\{i_5^K[\Sigma^2 i_{1,3}, s]\eta_7\}$,

(2) $\pi_9(K) = \mathbf{Z}_2\{i_4^K\lambda_2\nu_6\} \oplus \mathbf{Z}_2\{i_5^K s\nu_5\eta_8\} \oplus \mathbf{Z}_2\{i_5^K[[\Sigma^2 i_{1,3}, s], \Sigma^2 i_{1,3}]\}$.

Proof. We consider the homotopy exact sequence of a pair $(K^{(8)}, \Sigma^2\mathrm{P}^3)$:

$$\pi_{10}(K^{(8)}, \Sigma^2\mathrm{P}^3) \xrightarrow{\partial_{10}} \pi_9(\Sigma^2\mathrm{P}^3) \xrightarrow{i^K_{5,8_*}} \pi_9(K^{(8)}) \xrightarrow{j_*} \pi_9(K^{(8)}, \Sigma^2\mathrm{P}^3)$$

$$\xrightarrow{\partial_9} \pi_8(\Sigma^2\mathrm{P}^3) \xrightarrow{i^K_{5,8_*}} \pi_8(K^{(8)}) \xrightarrow{j_*} \pi_8(K^{(8)}, \Sigma^2\mathrm{P}^3) \xrightarrow{\partial_8} \pi_7(\Sigma^2\mathrm{P}^3).$$

The group structures $\pi_8(K^{(8)}, \Sigma^2\mathrm{P}^3) = \mathbf{Z}\{\omega_8\}$ and $\pi_9(K^{(8)}, \Sigma^2\mathrm{P}^3) = \mathbf{Z}_2\{\widehat{\eta}_7\}$ are obtained by the Blakers-Massey theorem, where $\omega_8 = \omega_8^{(K^{(8)},\Sigma^2\mathrm{P}^3)}$. By (2.1) and the relation $[i_4, \iota_{M^4}] = \lambda_2 p_6$ [8, Lemma 1.5], the attaching map of the 8-cell of $K^{(8)}$ is $[\iota_{\Sigma^2\mathrm{P}^3}, \Sigma^2\gamma_3]\circ\Sigma^6 i_{1,3} = (\Sigma^2 i_{2,3})[i_4, \iota_{M^4}]\widetilde{\eta}_5 = (\Sigma^2 i_{2,3})\lambda_2\eta_6$. So we have $\partial_8(\omega_8) = (\Sigma^2 i_{2,3})\lambda_2\eta_6$ and $\partial_9(\widehat{\eta}_7) = (\Sigma^2 i_{2,3})\lambda_2\eta_6^2$. By (2.2), the order of these elements are two. Therefore, there exists an element $\varphi \in \pi_8(K^{(8)})$ such that $p^K_{8,5}\varphi = 2\iota_8$. Here we note that φ is taken as a representative of the Toda bracket

$$\varphi \in \{i^K_{5,8}[\Sigma^2 i_{2,3}, \Sigma^2\gamma_3], i_8, 2\iota_7\}.$$

So, by Lemma 2.2 (1), we have

$$\begin{aligned}\pi_8(K^{(8)}) =&\mathbf{Z}_2\{i^K_{4,8}[i_4,\lambda_2]\} \oplus \mathbf{Z}_2\{i^K_{4,8}\widetilde{\eta}_3\nu_5\} \oplus \mathbf{Z}_{24}\{i^K_{5,8}s\nu_5\} \\ &\oplus \mathbf{Z}_2\{i^K_{5,8}[\Sigma^2 i_{1,3}, s]\eta_7\} \oplus \mathbf{Z}\{\varphi\}.\end{aligned} \tag{2.3}$$

We have $\pi_{10}(K^{(8)}, \Sigma^2\mathrm{P}^3) = \mathbf{Z}_2\{\widehat{\eta_7^2}\} \oplus \mathbf{Z}_2\{[\omega_8, \Sigma^2 i_{1,3}]\}$ by the James exact sequence [4, Theorem 2.1]. Since $\partial_{10}(\widehat{\eta_7^2}) = (\Sigma^2 i_{2,3})\lambda_2\eta_6^3 = 0$ and

$$\partial_{10}([\omega_8, \Sigma^2 i_{1,3}]) = [(\Sigma^2 i_{2,3})\lambda_2\eta_6, \Sigma^2 i_{1,3}] = (\Sigma^2 i_{2,3})[\lambda_2, i_4]\eta_8,$$

we obtain

$$\begin{aligned}\pi_9(K^{(8)}) =&\mathbf{Z}_2\{i^K_{4,8}\lambda_2\nu_6\} \oplus \mathbf{Z}_2\{i^K_{4,8}\widetilde{\eta}_3\nu_5\eta_8\} \oplus \mathbf{Z}_2\{i^K_{5,8}s\nu_5\eta_8\} \\ &\oplus \mathbf{Z}_4\{i^K_{5,8}[\Sigma^2 i_{2,3}, s]\widetilde{\eta}_7\} \oplus \mathbf{Z}_2\{i^K_{5,8}[[\Sigma^2 i_{1,3}, s], \Sigma^2 i_{1,3}]\}.\end{aligned} \tag{2.4}$$

Note that φ is obtained in the follwoing diagram between the cofiber sequences:

$$\begin{tikzcd}
S^7 \arrow[r, "{[\Sigma^2 i_{1,3},\Sigma^2\gamma_3]}"] & \Sigma^2\mathrm{P}^3 \arrow[r, "i^K_{5,8}"] & K^{(8)} \arrow[r, "p^K_{8,5}"] & S^8 \\
S^7 \arrow[r, "i_8"] \arrow[u, "="'] & M^8 \arrow[r, "p_8"] \arrow[u, "{[\Sigma^2 i_{2,3},\Sigma^2\gamma_3]}"'] & S^8 \arrow[r, "2\iota_8"] \arrow[u, "\varphi"'] & S^8. \arrow[u, "="']
\end{tikzcd}$$

Write now the homotopy exact sequence of a pair $(K^{(9)}, K^{(8)})$:

$$\pi_{10}(K^{(9)}, K^{(8)}) \xrightarrow{\partial_{10}} \pi_9(K^{(8)}) \xrightarrow{i^K_{8,9_*}} \pi_9(K^{(9)}) \xrightarrow{j_*} \pi_9(K^{(9)}, K^{(8)})$$

$$\xrightarrow{\partial_9} \pi_8(K^{(8)}) \xrightarrow{i^K_{8,9_*}} \pi_8(K^{(9)}) \to 0.$$

The group structures $\pi_9(K^{(9)}, K^{(8)}) = \mathbf{Z}\{\omega_9\}$ and $\pi_{10}(K^{(9)}, K^{(8)}) = \mathbf{Z}_2\{\widehat{\eta}_8\}$ are obtained by the Blakers-Massey theorem, where $\omega_9 = \omega_9^{(K^{(9)},K^{(8)})}$. By use of the exact sequence of a triple $(K^{(9)}, K^{(8)}, \Sigma^2\mathrm{P}^3)$,

$$\partial' : \pi_9(K^{(9)}, K^{(8)}) \to \pi_8(K^{(8)}, \Sigma^2\mathrm{P}^3)$$

is the map of degree 2. So, by the commutative diagram

$$\begin{tikzcd}
\pi_9(K^{(9)}, K^{(8)}) \arrow[r, "\partial'"] \arrow[d, "\partial_9"] & \pi_8(K^{(8)}, \Sigma^2\mathrm{P}^3) \\
\pi_8(K^{(8)}), \arrow[ur, "j_*"] &
\end{tikzcd}$$

φ is taken as the attaching map of 9-cell of $K^{(9)}$. Hence, by (2.3) and $\pi_8(K^{(9)}) \cong \pi_8(K)$, we obtain (1) and $j_* = 0$. We see that

$$\begin{aligned}
\partial_{10}(\widehat{\eta}_8) &= \varphi \circ \eta_8 \in \{i^K_{5,8}[\Sigma^2 i_{2,3}, \Sigma^2\gamma_3], i_8, 2\iota_7\} \circ \eta_8 \\
&= i^K_{5,8}[\Sigma^2 i_{2,3}, \Sigma^2\gamma_3] \circ \{i_8, 2\iota_7, \eta_7\} \\
&\ni i^K_{5,8}[\Sigma^2 i_{2,3}, \Sigma^2\gamma_3]\widetilde{\eta}_7 \\
&\quad \bmod i^K_{5,8}[\Sigma^2 i_{2,3}, \Sigma^2\gamma_3] \circ (\pi_8(M^8) \circ \eta_8 + i_8 \circ \pi_9(S^7)) = 0.
\end{aligned}$$

Here we used $[\Sigma^2 i_{2,3}, \Sigma^2\gamma_3] i_8 \eta_7^2 = [\Sigma^2 i_{1,3}, \Sigma^2\gamma_3]\eta_7^2 = (\Sigma^2 i_{2,3})\lambda_2\eta_6^3 = 0$. By the fact that $[\Sigma^2 i_{2,3}, \Sigma^2\gamma_3] = 2[\Sigma^2 i_{2,3}, s] + (\Sigma^2 i_{2,3})[\iota_{M^4}, \widetilde{\eta}_3]$ and $[\iota_{M^4}, \widetilde{\eta}_3] = \widetilde{\eta}_3\nu_5 p_8 \pm \lambda_2\overline{\eta}_6$ [5, Lemma 1.2], we obtain

$$\partial_{10}(\widehat{\eta}_8) = 2i^K_{5,8}[\Sigma^2 i_{2,3}, s]\widetilde{\eta}_7 + i^K_{4,8}\widetilde{\eta}_3\nu_5\eta_8,$$

and hence

$$\begin{aligned}
\pi_9(K^{(9)}) =& \mathbf{Z}_2\{i^K_{4,9}\lambda_2\nu_6\} \oplus \mathbf{Z}_2\{i^K_{5,9} s\nu_5\eta_8\} \oplus \mathbf{Z}_4\{i^K_{5,9}[\Sigma^2 i_{2,3}, s]\widetilde{\eta}_7\} \\
&\oplus \mathbf{Z}_2\{i^K_{5,9}[[\Sigma^2 i_{1,3}, s], \Sigma^2 i_{1,3}]\}.
\end{aligned} \tag{2.5}$$

Let $p_M : \Sigma^2\mathrm{P}^3 \to M^4$ be the projection. Then,

$$\iota_{\Sigma^2\mathrm{P}^3} = s\Sigma^2 p_{3,2} + (\Sigma^2 i_{2,3})p_M, \tag{2.6}$$

$$\Sigma^2 p_{3,2} \circ s = \iota_5,\ p_M \circ \Sigma^2 i_{2,3} = \iota_{M^4} \text{ and } p_M \circ s = 0. \tag{2.7}$$

By (2.1) and (2.6), we have

$$\begin{aligned}
[\iota_{\Sigma^2\mathrm{P}^3}, \Sigma^2\gamma_3] &= [s\Sigma^2 p_{3,2}, \Sigma^2\gamma_3] + [(\Sigma^2 i_{2,3})p_M, \Sigma^2\gamma_3] \\
&= [s, (\Sigma^2 i_{2,3})\widetilde{\eta}_3] \circ \Sigma^6 p_{3,2} + (\Sigma^2 i_{2,3})[p_M, \widetilde{\eta}_3] + 2[(\Sigma^2 i_{2,3})p_M, s].
\end{aligned}$$

By (2.7), we have $[p_M,\tilde{\eta}_3]\circ\Sigma^6 i_{2,3}=[\iota_{M^4},\tilde{\eta}_3]$. So, by use of the cofiber sequence $M^8\xrightarrow{\Sigma^6 i_{2,3}}\Sigma^6\mathrm{P}^3\xrightarrow{\Sigma^6 p_{3,2}}S^9$,

$$[p_M,\tilde{\eta}_3]\equiv[\iota_{M^4},\tilde{\eta}_3]\circ\Sigma^4 p_M \bmod \pi_9(M^4)\circ\Sigma^6 p_{3,2}.$$

By the same reason,

$$[(\Sigma^2 i_{2,3})p_M,s]\equiv[\Sigma^2 i_{2,3},s]\circ\Sigma^4 p_M \bmod \pi_9(\Sigma^2\mathrm{P}^3)\circ\Sigma^6 p_{3,2}.$$

Hence, by Lemma 2.2 (2) and (2.7), we conclude that

$$[\iota_{\Sigma^2\mathrm{P}^3},\Sigma^2\gamma_3]\circ\Sigma^4 s\equiv\pm[s,(\Sigma^2 i_{2,3})\tilde{\eta}_3] \bmod (\Sigma^2 i_{2,3})\circ\pi_9(M^4)\circ\Sigma^6 p_{3,2}.$$

The attaching map of the 10-cell of $K^{(10)}$ is $i^K_{5,9}[\iota_{\Sigma^2\mathrm{P}^3},\Sigma^2\gamma_3]\Sigma^4 s$. By (2.1), we have

$$i^K_{5,9}[\iota_{\Sigma^2\mathrm{P}^3},\Sigma^2\gamma_3]\Sigma^4 s\equiv\pm i^K_{5,9}[s,\Sigma^2 i_{2,3}]\widetilde{\eta}_7 \bmod i^K_{4,9}\lambda_2\nu_6.$$

So, by the homotopy exact sequence of a pair $(K^{(10)},K^{(9)})$ and (2.5), the group structure of $\pi_9(K)$ is obtained. This completes the proof. $\square$

Lemma 2.6.

$$\begin{aligned}\pi_9(\Sigma^2\mathrm{P}^4)=&\mathbf{Z}_2\{\widetilde{\eta^3_5}\}\oplus\mathbf{Z}_2\{(\Sigma^2 i_{2,4})\lambda_2\nu_6\}\oplus\mathbf{Z}_2\{(\Sigma^2 i_{3,4})s\nu_5\eta_8\}\\&\oplus\mathbf{Z}_2\{(\Sigma^2 i_{3,4})[[s,\Sigma^2 i_{1,3}],\Sigma^2 i_{1,3}]\}.\end{aligned}$$

Proof. We consder the exact sequence induced from the fibration $\Sigma^2 p_{4,3}:\Sigma^2\mathrm{P}^4\to S^6$:

$$\pi_{10}(S^6)=0\to\pi_9(K)\to\pi_9(\Sigma^2\mathrm{P}^4)\to\pi_9(S^6)\xrightarrow{\Delta_9}\pi_8(K)\to\cdots.$$

By [8, Lemma 1.2], we obtain the relations $\Delta_6(\iota_6)=\pm i^K_4\widetilde{\eta}_3+2i^K_5 s$ and

$$\Delta_9(\nu_6)=\Delta_6(\iota_6)\circ\nu_5=\pm i^K_4\widetilde{\eta}_3\nu_5+2i^K_5 s\nu_5.$$

Using the second relation and Lemma 2.5 (1), we obtain Ker $\Delta_9=\mathbf{Z}_2\{\eta^3_6\}$. Therefore, by Lemma 2.4 and 2.5 (2) and by the fact that $i\circ i^K_5=\Sigma^2 i_{3,4}$ ($i:K\to\Sigma^2\mathrm{P}^4$ is the inclusion), we obtain the result. This completes the proof. $\square$

Now we consider the homotopy exact sequence of a pair $(\Sigma^2\mathrm{P}^5,\Sigma^2\mathrm{P}^4)$:

$$\begin{aligned}\pi_{10}(\Sigma^2\mathrm{P}^5,\Sigma^2\mathrm{P}^4)&\xrightarrow{\partial_{10}}\pi_9(\Sigma^2\mathrm{P}^4)\xrightarrow{i_*}\pi_9(\Sigma^2\mathrm{P}^5)\\&\xrightarrow{j_*}\pi_9(\Sigma^2\mathrm{P}^5,\Sigma^2\mathrm{P}^4)\xrightarrow{\partial_9}\pi_8(\Sigma^2\mathrm{P}^4),\end{aligned}$$

where $i=\Sigma^2 i_{4,5}:\Sigma^2\mathrm{P}^4\to\Sigma^2\mathrm{P}^5$. By the James exact sequence, the group structures $\pi_9(\Sigma^2\mathrm{P}^5,\Sigma^2\mathrm{P}^4)=\mathbf{Z}_2\{\widehat{\eta^2_6}\}\oplus\mathbf{Z}_2\{[\omega_7,\Sigma^2 i_{1,4}]\}$ and $\pi_{10}(\Sigma^2\mathrm{P}^5,\Sigma^2\mathrm{P}^4)=\mathbf{Z}_{24}\{\widehat{\nu}_6\}\oplus\mathbf{Z}_2\{[\omega_7,(\Sigma^2 i_{1,4})\eta_3]\}$ are settled, where $\omega_7=\omega_7^{(\Sigma^2\mathrm{P}^5,\Sigma^2\mathrm{P}^4)}$. We recall, from [8, Lemma 1.3], the relation

$$\Sigma^2\gamma_4=(\Sigma^2 i_{3,4})s\eta_5+2(\Sigma^2 i_{2,4})\lambda_2. \tag{2.8}$$

By Lemma 2.6 and by the relation $\eta_5\nu_6 = 0$, we obtain

$$\partial_{10}(\widehat{\nu_6}) = (\Sigma^2\gamma_4)\nu_6 = ((\Sigma^2 i_{3,4})s\eta_5 + 2(\Sigma^2 i_{1,4})\lambda_2)\nu_6 = 0. \tag{2.9}$$

The equation $(\Sigma^2 i_{3,4})[\Sigma^2 i_{2,3}, s]\widetilde{\eta}_7 = 0$ is shown in the proof of Lemma 2.6. Then

$$\begin{aligned}\partial_{10}([\omega_7, (\Sigma^2 i_{1,4})\eta_3]) &= [\Sigma^2\gamma_4, (\Sigma^2 i_{1,4})\eta_3] = [(\Sigma^2 i_{3,4})s\eta_5, (\Sigma^2 i_{1,4})\eta_3] \\ &= (\Sigma^2 i_{3,4})[s, \Sigma^2 i_{1,3}]\eta_7^2 = 2(\Sigma^2 i_{3,4})[s, \Sigma^2 i_{2,3}]\widetilde{\eta}_7 = 0.\end{aligned}$$

Therefore $(\Sigma^2 i_{4,5})_* : \pi_9(\Sigma^2 P^4) \to \pi_9(\Sigma^2 P^5)$ is a monomorphism.

By the fact that $\pi_8(\Sigma^2 P^4) = \mathbf{Z}_4\{(\Sigma^2 i_{3,4})s\nu_5\} \oplus \mathbf{Z}_2\{[(\Sigma^2 i_{3,4})s, \Sigma^2 i_{1,4}]\eta_7\} \oplus \mathbf{Z}_2\{(\Sigma^2 i_{2,4})[i_4, \lambda_2]\}$ [8, Lemma 2.5] and by (2.8), we obtain

$$\partial_9(\widehat{\eta_6^2}) = (\Sigma^2\gamma_4)\eta_6^2 = (\Sigma^2 i_{3,4})s\eta_5^3 = 4(\Sigma^2 i_{3,4})s\nu_5 = 0$$

and

$$\partial_9([\omega_7, \Sigma^2 i_{1,4}]) = [\Sigma^2\gamma_4, \Sigma^2 i_{1,4}] = [(\Sigma^2 i_{3,4})s\eta_5, \Sigma^2 i_{1,4}] = [(\Sigma^2 i_{3,4})s, \Sigma^2 i_{1,4}]\eta_7.$$

Then there exists an element $\widetilde{\eta_6^2} \in \{\Sigma^2 i_{4,5}, \Sigma^2\gamma_4, \eta_6^2\} \subset \pi_9(\Sigma^2 P^5)$ such that $(\Sigma^2 p_{5,4})\widetilde{\eta_6^2} = \eta_7^2$. We obtain

$$2\widetilde{\eta_6^2} \in \{\Sigma^2 i_{4,5}, \Sigma^2\gamma_4, \eta_6^2\} \circ 2\iota_9 = -(\Sigma^2 i_{4,5} \circ \{\Sigma^2\gamma_4, \eta_6^2, 2\iota_8\})$$

and

$$\begin{array}{rcl} \{\Sigma^2\gamma_4, \eta_6^2, 2\iota_8\} & \subset & \{\Sigma^2 i_{3,4}, (s\eta_5 + 2(\Sigma^2 i_{2,3})\lambda_2)\eta_6^2, 2\iota_8\} \\ & = & \{\Sigma^2 i_{3,4}, s\eta_5^3, 2\iota_8\} \\ & = & \{\Sigma^2 i_{3,4}, \Sigma^2\gamma_3 \circ 2\nu_5, 2\iota_8\} \\ & \supset & \{\Sigma^2 i_{3,4}, \Sigma^2\gamma_3, \eta_5^3\} \\ & \ni & \widetilde{\eta_5^3} \\ & \bmod & 2\pi_9(\Sigma^2 P^4) + (\Sigma^2 i_{3,4})_*\pi_9(\Sigma^2 P^3) = (\Sigma^2 i_{3,4})_*\pi_9(\Sigma^2 P^3), \end{array}$$

and hence we conclude that $2\widetilde{\eta_6^2} \equiv (\Sigma^2 i_{4,5})\widetilde{\eta_5^3} \mod (\Sigma^2 i_{3,5})_*\pi_9(\Sigma^2 P^3)$. Thus, by Lemma 2.6, we have the following.

Lemma 2.7.

$$\begin{aligned}\pi_9(\Sigma^2 P^5) =& \mathbf{Z}_4\{\widetilde{\eta_6^2}\} \oplus \mathbf{Z}_2\{(\Sigma^2 i_{2,5})\lambda_2\nu_6\} \oplus \mathbf{Z}_2\{(\Sigma^2 i_{3,5})s\nu_5\eta_8\} \\ &\oplus \mathbf{Z}_2\{(\Sigma^2 i_{3,5})[[\Sigma^2 i_{1,3}, s], \Sigma^2 i_{1,3}]\},\end{aligned}$$

where $2\widetilde{\eta_6^2} = (\Sigma^2 i_{4,5})\widetilde{\eta_5^3}$ *for a suitable choice of* $\widetilde{\eta_5^3}$.

3. Proofs of main theorems

First, we show Theorem 1.2.

From the fact that $\Sigma^2\gamma_5 \in \{\Sigma^2 i_{4,5}, \Sigma^2\gamma_4, 2\iota_6\}$, $\Sigma^3\gamma_4 = (\Sigma^3 i_{3,4})(\Sigma s)\eta_6$, we see that

$$\begin{aligned}
[\Sigma^2\gamma_5, \Sigma^2 i_{1,5}]p_9 &= [\iota_{\Sigma^2\mathrm{P}^5}, \Sigma^2 i_{1,5}] \circ \Sigma^4\gamma_5 \circ p_9 \\
&\in [\iota_{\Sigma^2\mathrm{P}^5}, \Sigma^2 i_{1,5}] \circ \{\Sigma^4 i_{4,5}, \Sigma^4\gamma_4, 2\iota_8\} \circ p_9 \\
&\supset [\iota_{\Sigma^2\mathrm{P}^5}, \Sigma^2 i_{1,5}]\{(\Sigma^4 i_{3,5})\Sigma^2 s, \eta_7, 2\iota_8\} \circ p_9 \\
&= -([\Sigma^2 i_{3,5}, \Sigma^2 i_{1,5}]\Sigma^2 s \circ \{\eta_7, 2\iota_8, p_8\}) \\
&\ni [\Sigma^2 i_{3,5}, \Sigma^2 i_{1,5}](\Sigma^2 s)\overline{\eta}_7 \\
\mathrm{mod}\ & [\Sigma^2 i_{4,5}, \Sigma^2 i_{1,5}] \circ \pi_9(\Sigma^4\mathrm{P}^4) \circ p_9.
\end{aligned}$$

It is easily seen that $\pi_9(\Sigma^4\mathrm{P}^4) = \mathbf{Z}_2\{(\Sigma^4 i_{1,4})\nu_5\eta_8\} \oplus \mathbf{Z}_2\{(\Sigma^4 i_{3,4})(\Sigma^2 s)\eta_7^2\}$. Since $[\iota_3, \iota_3] = 0$ and $(\Sigma^2 i_{3,4})[s, \Sigma^2 i_{2,3}]\widetilde{\eta}_7 = 0$, we obtain

$$[\Sigma^2 i_{4,5}, \Sigma^2 i_{1,5}] \circ (\Sigma^4 i_{1,4})\nu_5\eta_8 = (\Sigma^2 i_{1,5})[\iota_3, \iota_3]\nu_5\eta_8 = 0$$

and

$$[\Sigma^2 i_{4,5}, \Sigma^2 i_{1,5}] \circ (\Sigma^4 i_{3,4})(\Sigma^2 s)\eta_7^2 = [\Sigma^2 i_{4,5}, \Sigma^2 i_{1,5}] \circ (\Sigma^4\gamma_4)\eta_8 = 0.$$

Then $[\Sigma^2 i_{4,5}, \Sigma^2 i_{1,5}] \circ \pi_9(\Sigma^4\mathrm{P}^4) = 0$. By (2.6), the element $[\Sigma^2 i_{3,5}, \Sigma^2 i_{1,5}]\Sigma^2 s$ is changed as follows.

$$\begin{aligned}
[\Sigma^2 i_{3,5}, \Sigma^2 i_{1,5}]\Sigma^2 s &= (\Sigma^2 i_{3,5})[\iota_{\Sigma^2\mathrm{P}^3}, \Sigma^2 i_{1,3}]\Sigma^2 s \\
&= (\Sigma^2 i_{3,5})[s, \Sigma^2 i_{1,3}] + (\Sigma^2 i_{2,5})[p_M, i_4]\Sigma^2 s.
\end{aligned}$$

By the fact that $[p_M, i_4] \in [\Sigma^4\mathrm{P}^3, M^4] = (\Sigma^4 p_{3,2})^*\pi_7(M^4) \oplus (\Sigma^2 p_M)^*[M^6, M^4]$, we obtain

$$\begin{aligned}
(\Sigma^2 i_{2,5})[p_M, i_4]\Sigma^2 s &\in \Sigma^2 i_{2,5} \circ (\pi_7(M^4) \circ \Sigma^4 p_{3,2} + [M^6, M^4] \circ \Sigma^2 p_M) \circ \Sigma^2 s \\
&= \Sigma^2 i_{2,5} \circ \pi_7(M^4).
\end{aligned}$$

We recall from [7, Lemma 2.2] that $\pi_7(M^4) = \mathbf{Z}_2\{\lambda_2\eta_6\} \oplus \mathbf{Z}_2\{\widetilde{\eta}_3\eta_5^2\}$. Since $(\Sigma^2 i_{2,4})\lambda_2\eta_6 = 0$ [8, the proof of Lemma 2.2] and by (2.1), the group $\Sigma^2 i_{2,5} \circ \pi_7(M^4)$ is 0. Then,

$$[\Sigma^2\gamma_5, \Sigma^2 i_{1,5}]p_9 = [\Sigma^2 i_{3,5}, \Sigma^2 i_{1,5}](\Sigma^2 s)\overline{\eta}_7 = (\Sigma^2 i_{3,5})[s, \Sigma^2 i_{1,3}]\overline{\eta}_7.$$

Here we consider an element $[\Sigma^2\gamma_4, \Sigma^2 i_{2,4}] \in [M^9, \Sigma^2\mathrm{P}^4]$. Since $2\iota_{M^4} = i_4\eta_3 p_4$ [11], $(\Sigma^2 i_{2,4})\lambda_2\eta_6 = 0$, $(\Sigma^2 i_{3,4})[s, \Sigma^2 i_{2,3}]\widetilde{\eta}_7 = 0$ and $\eta_2 \wedge \iota_{M^2} = i_4\overline{\eta}_3 + \widetilde{\eta}_3 p_5$, we obtain

$$\begin{aligned}
[\Sigma^2\gamma_4, \Sigma^2 i_{2,4}] &= (\Sigma^2 i_{3,4})[s\eta_5, \Sigma^2 i_{2,3}] + (\Sigma^2 i_{2,4})[2\lambda_2, \iota_{M^4}] \\
&= (\Sigma^2 i_{3,4})[s, \Sigma^2 i_{2,3}] \circ \Sigma(\eta_4 \wedge \iota_{M^3}) + (\Sigma^2 i_{2,4})[\lambda_2, 2\iota_{M^4}]
\end{aligned}$$

$$
\begin{aligned}
&= (\Sigma^2 i_{3,4})[s, \Sigma^2 i_{2,3}] \circ \Sigma(i_7\overline{\eta}_6 + \widetilde{\eta}_6 p_8) + (\Sigma^2 i_{2,4})[\lambda_2, i_4\eta_3 p_4] \\
&= (\Sigma^2 i_{3,4})[s, \Sigma^2 i_{1,3}]\overline{\eta}_7 + (\Sigma^2 i_{2,4})[\lambda_2\eta_6, i_4]p_9 \\
&= (\Sigma^2 i_{3,4})[s, \Sigma^2 i_{1,3}]\overline{\eta}_7.
\end{aligned}
$$

Thus, we get that

$$[\Sigma^2\gamma_5, \Sigma^2 i_{1,5}]p_9 = (\Sigma^2 i_{3,5})[s, \Sigma^2 i_{1,3}]\overline{\eta}_7 = (\Sigma^2 i_{4,5})[\Sigma^2\gamma_4, \Sigma^2 i_{2,4}] = 0.$$

By use of the cofibre sequence $S^8 \xrightarrow{i_9} M^9 \xrightarrow{p_9} S^9 \xrightarrow{2\iota_9} S^9$, we have

$$[\Sigma^2\gamma_5, \Sigma^2 i_{1,5}] \in 2\pi_9(\Sigma^2\mathrm{P}^5) = \mathbf{Z}_2\{2\widetilde{\eta_6^2}\}.$$

Let $l_1 : \mathrm{P}^4/\mathrm{P}^3 = S^4 \to \mathrm{P}^5/\mathrm{P}^3 = S^4 \vee S^5$ be the canonical inclusion map. By Lemma 2.7 and by the relations $p_{5,3} \circ i_{4,5} = l_1 \circ p_{4,3}$ and $p_{5,3} \circ i_{1,5} = 0$, we obtain

$$
\begin{aligned}
\Sigma^2 p_{5,3} \circ 2\widetilde{\eta_6^2} &= \Sigma^2 p_{5,3} \circ \Sigma^2 i_{4,5} \circ \widetilde{\eta_5^3} \\
&= \Sigma^2 l_1 \circ \Sigma^2 p_{4,3} \circ \widetilde{\eta_5^3} \\
&= (\Sigma^2 l_1)\eta_6^3 \neq 0 \in \pi_9(S^6 \vee S^7) \cong \mathbf{Z}_{24} \oplus \mathbf{Z}_2
\end{aligned}
$$

and

$$\Sigma^2 p_{5,3} \circ [\Sigma^2\gamma_5, \Sigma^2 i_{1,5}] = 0.$$

Therefore we have $[\Sigma^2\gamma_5, \Sigma^2 i_{1,5}] = 0$ and the proof of Theorem 1.2 is complete.

By [8, the proof of Lemma 2.5], we have a relation $(\Sigma^2\gamma_5)\eta_7 = 0$ and we can define a coextension $\widetilde{\eta}'_7 \in \pi_9(\Sigma^2\mathrm{P}^6)$ of η_7 as follows:

$$\widetilde{\eta}'_7 \in \{\Sigma^2 i_{5,6}, \Sigma^2\gamma_5, \eta_7\}.$$

Since $2\widetilde{\eta}'_7 \in \{\Sigma^2 i_{5,6}, \Sigma^2\gamma_5, \eta_7\} \circ 2\iota_9 = -(\Sigma^2 i_{5,6} \circ \{\Sigma^2\gamma_5, \eta_7, 2\iota_8\})$ and

$$\Sigma^2 p_{5,4} \circ \{\Sigma^2\gamma_5, \eta_7, 2\iota_8\} \subset \{2\iota_7, \eta_7, 2\iota_8\} = \eta_7^2,$$

we obtain $2\widetilde{\eta}'_7 \equiv (\Sigma^2 i_{5,6})\widetilde{\eta_6^2} \mod \Sigma^2 i_{5,6} \circ 2\pi_9(\Sigma^2\mathrm{P}^5) + \Sigma^2 i_{4,6} \circ \pi_9(\Sigma^2\mathrm{P}^4) = \Sigma^2 i_{4,6} \circ \pi_9(\Sigma^2\mathrm{P}^4)$. From the exact sequence of a pair $(\Sigma^2\mathrm{P}^6, \Sigma^2\mathrm{P}^5)$ and by Theorem 1.2, we see that $(\Sigma^2 i_{5,6})_* : \pi_9(\Sigma^2\mathrm{P}^5) \to \pi_9(\Sigma^2\mathrm{P}^6)$ is a monomorphism. Thus, $\widetilde{\eta}'_7$ is of order 8 and the group structure of $\pi_9(\Sigma^2\mathrm{P}^6)$ is given as follows.

Lemma 3.1.

$$
\begin{aligned}
\pi_9(\Sigma^2\mathrm{P}^6) =& \mathbf{Z}_8\{\widetilde{\eta}'_7\} \oplus \mathbf{Z}_2\{(\Sigma^2 i_{2,6})\lambda_2\nu_6\} \oplus \mathbf{Z}_2\{(\Sigma^2 i_{3,6})s\nu_5\eta_8\} \\
&\oplus \mathbf{Z}_2\{(\Sigma^2 i_{3,6})[[\Sigma^2 i_{1,3}, s], \Sigma^2 i_{1,3}]\},
\end{aligned}
$$

where $2\widetilde{\eta}'_7 = (\Sigma^2 i_{5,6})\widetilde{\eta_6^2}$ *for a suitable choice of* $\widetilde{\eta_6^2}$.

We denote by $s_1 : S^9 \to \Sigma^2 \mathrm{P}^7 = \Sigma^2 \mathrm{P}^6 \vee S^9$ the inclusion map to the second factor and by $q_1 : \Sigma^2 \mathrm{P}^7 = \Sigma^2 \mathrm{P}^6 \vee S^9 \to \Sigma^2 \mathrm{P}^6$ the map collapsing S^9 to one point. Finally we obtain the following.

Theorem 3.2.

$$
\begin{aligned}
[\Sigma^2\mathrm{P}^7, \Sigma^2\mathrm{P}^7] = & \mathbf{Z}\{s_1\Sigma^2 p_{7,6}\} \oplus \mathbf{Z}_8\{(\Sigma^2 i_{6,7})q_1\} \oplus \mathbf{Z}_8\{(\Sigma^2 i_{6,7})\widetilde{\eta}'_7\Sigma^2 p_{7,6}\} \\
& \oplus \mathbf{Z}_2\{(\Sigma^2 i_{3,7})[s, \Sigma^2 i_{2,3}](\Sigma^2 p_{6,4})q_1\} \\
& \oplus \mathbf{Z}_2\{(\Sigma^2 i_{2,7})\lambda_2(\Sigma^2 \bar{\bar{p}}_{4,3})q_1\} \oplus \mathbf{Z}_2\{(\Sigma^2 i_{2,7})[\lambda_2, i_4](\Sigma^2 p_{6,5})q_1\} \\
& \oplus \mathbf{Z}_2\{(\Sigma^2 i_{3,7})s\nu_5(\Sigma^2 p_{6,5})q_1\} \oplus \mathbf{Z}_2\{(\Sigma^2 i_{2,7})\lambda_2\nu_6\Sigma^2 p_{7,6}\} \\
& \oplus \mathbf{Z}_2\{(\Sigma^2 i_{3,7})s\nu_5\eta_8\Sigma^2 p_{7,6}\} \\
& \oplus \mathbf{Z}_2\{(\Sigma^2 i_{3,7})[[\Sigma^2 i_{1,3}, s], \Sigma^2 i_{1,3}]\Sigma^2 p_{7,6}\}.
\end{aligned}
$$

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