SELF-HOMOTOPY OF THE DOUBLE SUSPENSION OF THE REAL 7-PROJECTIVE SPACE

TOSHIYUKI MIYAUCHI

ABSTRACT. We determine the group structure of the self-homotopy set of the double suspension of the real 7-dimensional projective space.

1. INTRODUCTION

In this paper, all spaces, maps and homotopies are based. We use the same notation as [10] and [5]. Let $\Sigma^n X$ be an $n$-fold suspension of a space $X$ and $P^n$ be the $n$-dimensional real projective space. The purpose of the present paper is to determine the group structure of the homotopy set $[\Sigma^2 P^7, \Sigma^2 P^7]$. We denote by $\gamma_n : S^n \to P^n$ the covering map. According to [9], $\Sigma^2 \gamma_6 = 0$, $\Sigma^2 P^7 = \Sigma^2 P^6 \vee S^9$, and so

$$[\Sigma^2 P^7, \Sigma^2 P^7] \cong [\Sigma^2 P^6, \Sigma^2 P^6] \oplus \pi_9(\Sigma^2 P^6) \oplus \pi_9(S^9).$$

Let $\mathbb{Z}$ be the group of integers and set $\mathbb{Z}_n = \mathbb{Z}/n\mathbb{Z}$. The notation $(\mathbb{Z}_n)^m$ means a direct sum of $m$-copies of $\mathbb{Z}_n$. Our result is stated as follows.

**Theorem 1.1.** $[\Sigma^2 P^7, \Sigma^2 P^7] \cong \mathbb{Z} \oplus (\mathbb{Z}_8)^2 \oplus (\mathbb{Z}_2)^7$.

In this paper we sometimes identify a map with its homotopy class. For $m < n$, let $i_{m,n} : P^m \to P^n$ and $p_{n,m} : P^n \to P^n/P^m$ be the inclusion and collapsing maps, respectively. Especially, we write $M^n = \Sigma^{n-2} P^2$, $i_n = \Sigma^{n-2} i_{1,2} : S^{n-1} \to M^n$ and $p_n = \Sigma^{n-2} p_{2,1} : M^n \to S^n$ for $n \geq 2$. We denote by $[\alpha, \beta]$ the Whitehead product of homotopy classes $\alpha$ and $\beta$. To determine the group structure of $\pi_9(\Sigma^2 P^6)$, we use the following.

**Theorem 1.2.** $[\Sigma^2 \gamma_5, \Sigma^2 i_{1,5}] = 0 \in \pi_9(\Sigma^2 P^5)$.

2. SOME HOMOTOPY GROUPS

We denote by $\iota_X \in [X, X]$ the identity class of a space $X$ and let $\iota_n = \iota_{S^n}$. For the Hopf maps $\eta_2 \in \pi_3(S^2)$ and $\nu_4 \in \pi_7(S^4)$, we set $\eta_n = \Sigma^{n-2} \eta_2$, $\eta_n^2 = \eta_n \eta_{n+1}$, $\eta_n^3 = \eta_n \eta_{n+1} \eta_{n+2}$ for $n \geq 2$ and $\nu_n = \Sigma^{n-4} \nu_4$ for $n \geq 4$. We recall from [7] that there is an element $\tilde{\eta}_2 \in \pi_4(\mathbb{M}^3)$ such that $p_3 \tilde{\eta}_2 = \eta_3$ and $\Sigma \tilde{\eta}_2 = \tilde{\eta}_3$, where $\tilde{\eta}_3$ is a coextension of $\eta_3$. Let $\tilde{\eta}_3 \in [M^5, S^3]$ be an extension of $\eta_3$ and set $\hat{\eta}_n = \Sigma^{n-2} \tilde{\eta}_2$ for $n \geq 2$ and $\hat{\eta}_n = \Sigma^{n-3} \tilde{\eta}_3$ for $n \geq 3$.

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Let $\nu'$ be a generator of the group $\pi_6(S^3) \cong \mathbb{Z}_{12}$ and $\lambda_2$ be the attaching map of the 7-cell of the Stiefel manifold $V_{5,2} = M^4 \cup_2 e^7$. We recall that $\pi_6(M^4) = \mathbb{Z}_4\{\lambda_2\} \oplus \mathbb{Z}_2\{\eta_3\eta_5\}$ [6]. We note $\pi_9(M^4) = (\mathbb{Z}_2)^3$ [12, Theorem 5.8] and, by use of these facts and the homotopy exact sequence of a pair $(V_{5,2}, M^4)$, we determine the generators.

**Lemma 2.1.**

$\pi_9(M^4) = \mathbb{Z}_2\{\lambda_2\lambda_6\} \oplus \mathbb{Z}_2\{\lambda_2, i_4|\eta_8\} \oplus \mathbb{Z}_2\{\eta_3\nu_5\eta_8\}$.

Let $s : S^5 \to \Sigma^2P^3 = M^4 \vee S^5$ be the inclusion to the second factor. Then, we recall

\[(2.1) \quad \Sigma^2\gamma_3 = 2s \pm (\Sigma^2i_{2,3})\tilde{\gamma}_3.\]

By the Hilton-Milnor theorem, we obtain

\[(2.2) \quad \pi_i(\Sigma^2P^3) = \pi_i(M^4) \oplus \pi_i(S^5) \oplus \pi_i(M^8) \oplus \pi_i(\Sigma(M^7 \land M^3)),\]

for $i \leq 9$. By Lemma 2.1 and the facts that $\pi_8(M^4) = \mathbb{Z}_2\{\lambda_2\eta_6^2\} \oplus \mathbb{Z}_2\{i_4, \lambda_2\}\oplus \mathbb{Z}_2\{\eta_3\nu_5\}$ [8, Lemma 2.4], $\pi_8(S^5) = \mathbb{Z}_2\{\nu_5\}$, $\pi_8(M^8) = \mathbb{Z}_2\{i_8\eta_7\}$, $\pi_9(S^5) = \mathbb{Z}_2\{\nu_5\eta_8\}$, $\pi_9(M^8) = \mathbb{Z}_4\{\eta_7\}$ and $\pi_9(\Sigma(M^7 \land M^3)) = \mathbb{Z}_2\{i_7 \land i_3\}$, we have the following.

**Lemma 2.2.**

\[
\begin{align*}
(1) \quad \pi_8(\Sigma^2P^3) &= \mathbb{Z}_2\{(\Sigma^2i_{2,3})\lambda_2\eta_6^2\} \oplus \mathbb{Z}_2\{(\Sigma^2i_{2,3})[i_4, \lambda_2]\} \\
&\quad \oplus \mathbb{Z}_2\{(\Sigma^2i_{2,3})\eta_3\nu_5\} \oplus \mathbb{Z}_2\{s\nu_5\} \oplus \mathbb{Z}_2\{[\Sigma^2i_{1,3}, s]\eta_7\}, \\
(2) \quad \pi_9(\Sigma^2P^3) &= \mathbb{Z}_2\{(\Sigma^2i_{2,3})\lambda_2\nu_6\} \oplus \mathbb{Z}_2\{(\Sigma^2i_{2,3})[i_4, \lambda_2]\eta_8\} \\
&\quad \oplus \mathbb{Z}_2\{(\Sigma^2i_{2,3})\eta_3\nu_5\eta_8\} \oplus \mathbb{Z}_2\{s\nu_5\eta_8\} \\
&\quad \oplus \mathbb{Z}_4\{[\Sigma^2i_{1,3}, s]\eta_7\} \oplus \mathbb{Z}_2\{[\Sigma^2i_{1,3}, s], \Sigma^2i_{2,3}\}. 
\end{align*}
\]

Let $X$ be a connected finite CW-complex and $X^* = X \cup_{\theta} e^n$ for $\theta : S^{n-1} \to X$ a complex formed by attaching an $n$-cell. We denote by

\[\omega_n(X^*, X) \in \pi_n(X^*, X)\]

the characteristic map of the $n$-cell $e^n$ of $X^*$. Let $CY$ be a cone of a space $Y$. For an element $\alpha \in \pi_m(Y)$, we denote by $\tilde{\alpha}' \in \pi_{m+1}(CY, Y)$ an element satisfying $\partial'(\tilde{\alpha}') = \alpha$, where $\partial' : \pi_{m+1}(CY, Y) \to \pi_m(Y)$ is the connecting bijection. For $\alpha \in \pi_m(S^{n-1})$, we set

\[\tilde{\alpha} = \omega_n(X^*, X) \circ \tilde{\alpha}' \in \pi_{m+1}(X^*, X).\]

We note the following:

\[\partial(\tilde{\alpha}) = \theta \circ \alpha \quad \text{and} \quad p_*\tilde{\alpha} = \Sigma\alpha,\]

where $\partial : \pi_{m+1}(X^*, X) \to \pi_m(X)$ is the boundary map and $p : (X^*, X) \to (S^n, \ast)$ is the collapsing map. Now we show the following.

**Lemma 2.3.**

$2((\Sigma^2i_{3,4})\circ \pi_9(\Sigma^2P^3)) = 0.$
Proof. We consider the homotopy exact sequence of a pair \((\Sigma^2 P^4, \Sigma^2 P^3)\):
\[
\pi_{10}(\Sigma^2 P^4, \Sigma^2 P^3) \xrightarrow{\partial_{10}} \pi_9(\Sigma^2 P^3) \xrightarrow{(\Sigma^2 i_{3,4})_*} \pi_9(\Sigma^2 P^4).
\]
There exists an element \([\omega_6, s] \in \pi_{10}(\Sigma^2 P^4, \Sigma^2 P^3)\) for \(\omega_6 = \omega_6^{(\Sigma^2 P^4, \Sigma^2 P^3)}\). By the relations \([1, (3.5)]\) and \((2.1)\), we have
\[
\partial_{10}([\omega_6, s]) = -[\Sigma^2 \gamma_3, s] = \pm((\Sigma^2 i_{2,3})\tilde{\eta}_3, s) = \pm[\Sigma^2 i_{2,3}, s]\tilde{\eta}_7.
\]
Hence, by Lemma 2.2 (2), we obtain \(2((\Sigma^2 i_{3,4})_*\pi_9(\Sigma^2 P^3)) = 0\). This completes the proof.

Since \(\Sigma^2 \gamma_3 \circ \eta_5^3 = (\Sigma^2 i_{2,3})\tilde{\eta}_3 \circ \nu_5 = 0\) by \((2.1)\), there exists an element \(\tilde{\eta}_3^3 \in \{\Sigma^2 i_{3,4}, \Sigma^2 \gamma_3, \eta_5^3\} \subset \pi_9(\Sigma^2 P^4)\) such that \((\Sigma^2 p_{4,3})\tilde{\eta}_3^3 = \eta_6^3\). For this element, we show the following.

Lemma 2.4. \(\{\tilde{\eta}_3\eta_5^2, \eta_7, 2t_8\} = \tilde{\eta}_3\nu_5\eta_8\) and the order of \(\eta_5^3\) is two.

Proof. By the properties of Toda brackets and by \([3, \text{Lemma 4.1}]\), we have
\[
\{\tilde{\eta}_3\eta_5^2, \eta_7, 2t_8\} \circ \pi_9 = -((\tilde{\eta}_3\eta_5^2 \circ \eta_7, 2t_8, p_8)) = \tilde{\eta}_3\eta_5^2\eta_7 = \tilde{\eta}_3\nu_5\eta_8 p_9.
\]
Since \(p_9^* : \pi_9(M^4) \to [M^9, M^4]\) is a monomorphism by Lemma 2.1, we obtain the first. By \((2.1)\), the relation \((\Sigma^2 i_{2,4})\tilde{\eta}_3\nu_5\eta_8 = \pm2((\Sigma^2 i_{3,4})s) \circ \tilde{\eta}_3\nu_5\eta_8 = 0\) holds. So, by the first and Lemma 2.3, we have
\[
2\tilde{\eta}_3^3 = \{\Sigma^2 i_{3,4}, \Sigma^2 \gamma_3, \eta_5^3\} \circ 2t_9
\]
\[
\quad = -(\Sigma^2 i_{3,4} \circ \{\Sigma^2 \gamma_3, \eta_5^3, 2t_8\})
\]
\[
\quad \supset -(\Sigma^2 i_{2,4} \circ \{\tilde{\eta}_3\eta_5^2, \eta_7, 2t_8\})
\]
\[
\quad = ((\Sigma^2 i_{2,4})\tilde{\eta}_3\nu_5\eta_8 = 0 \mod 2((\Sigma^2 i_{3,4})_*\pi_9(\Sigma^2 P^3)) = 0).
\]
This leads to the second and completes the proof.

Next we compute the homotopy groups of the homotopy fibre of \(\Sigma^2 p_{4,3} : \Sigma^2 P^4 \to S^6\) to determine \(\pi_9(\Sigma^2 P^4)\). Let \(K\) be the homotopy fibre of \(\Sigma^2 p_{4,3}\). By \([2, \text{Corollary 5.8}]\), the 10-skeleton of \(K\) has a cellular decomposition
\[
K^{(10)} = \Sigma^2 P^3 \cup_{[\Sigma^2 P^4, \Sigma^2 \gamma_3]} C\Sigma^6 P^3.
\]
For \(m < n\), we denote by \(i^K_{m,n} : K^{(m)} \to K^{(n)}\) and \(i^K_m : K^{(m)} \to K\) the inclusion maps and \(p^K_{n,m} : K^{(n)} \to K^{(n)}/K^{(m)}\) the collapsing map.

Lemma 2.5.

(1) \(\pi_8(K) = Z_2\{i^K_{4}[i_4, \lambda_2]\} \oplus Z_2\{i^K_3\tilde{\eta}_3\nu_5\} \oplus Z_2\{i^K_4\nu_5\} \oplus Z_2\{i^K_5[\Sigma^2 i_{1,3}, s]\tilde{\eta}_7\},\)

(2) \(\pi_9(K) = Z_2\{i^K_4\lambda_2\nu_6\} \oplus Z_2\{i^K_5\nu_5\eta_8\} \oplus Z_2\{i^K_5[\Sigma^2 i_{1,3}, s], \Sigma^2 i_{1,3}\}\).
Proof. We consider the homotopy exact sequence of a pair \((K^{(8)}, \Sigma^2P^3)\):

\[
\pi_0(K^{(8)}, \Sigma^2P^3) \xrightarrow{\partial_0} \pi_9(\Sigma^2P^3) \xrightarrow{i_{K,8}^{8}} \pi_9(K^{(8)}) \xrightarrow{j_{\ast}} \pi_9(K^{(8)}, \Sigma^2P^3) \]

\[
\partial_9 \pi_8(\Sigma^2P^3) \xrightarrow{i_{K,8}^{8}} \pi_8(K^{(8)}) \xrightarrow{j_{\ast}} \pi_8(K^{(8)}, \Sigma^2P^3) \xrightarrow{\partial_8} \pi_7(\Sigma^2P^3).
\]

The group structures \(\pi_8(K^{(8)}, \Sigma^2P^3) = \mathbb{Z}\{\omega_8\}\) and \(\pi_9(K^{(8)}, \Sigma^2P^3) = \mathbb{Z}_2\{\tilde{\eta}_7\}\) are obtained by the Blakers-Massey theorem, where \(\omega_8 = \omega_8(K^{(8)}, \Sigma^2P^3)\). By (2.1) and the relation \([i_4, \iota_{M^4}] = \lambda_2p_6\) [8, Lemma 1.5], the attaching map of the 8-cell of \(K^{(8)}\) is \(\iota_{\Sigma^2P^3, \Sigma^2\gamma_3} \circ \iota_{\Sigma_6}^{8} i_{1,3} = (\Sigma^2i_{2,3})[i_4, \iota_{M^4}^{8}] \eta_7 = (\Sigma^2i_{2,3})\lambda_2\eta_6\). So we have \(\partial_8(\omega_8) = (\Sigma^2i_{2,3})\lambda_2\eta_6\) and \(\partial_9(\tilde{\eta}_7) = (\Sigma^2i_{2,3})\lambda_2\eta_6^2\). By (2.2), the order of these elements are two. Therefore, there exists an element \(\varphi \in \pi_8(K^{(8)})\) such that \(p_{8,5}^{K,9}\varphi = 2i_8\). Here we note that \(\varphi\) is taken as a representative of the Toda bracket

\[
\varphi \in \{i_{K,8}^{5}\Sigma^2i_{2,3}, \Sigma^2\gamma_3, i_8, 2i_7\}.
\]

So, by Lemma 2.2 (1), we have

\[
\pi_8(K^{(8)}) = \mathbb{Z}_2\{i_{K,8}^{5}\iota_4, \lambda_2\} \oplus \mathbb{Z}_2\{i_{K,8}^{5}\eta_3\nu_5\} \oplus \mathbb{Z}_2\{i_{5,8}^{5}\nu_5\}
\]

\[
\oplus \mathbb{Z}_2\{i_{5,8}^{5}\Sigma^2i_{1,3}, \lambda_2\eta_6^2\} \oplus \mathbb{Z}\{\varphi\}.
\]

We have \(\pi_10(K^{(8)}, \Sigma^2P^3) = \mathbb{Z}_2\{\tilde{\eta}_7\} \oplus \mathbb{Z}_2\{\omega_8, \Sigma^2i_{1,3}\}\) by the James exact sequence [4, Theorem 2.1]. Since \(\partial_10(\tilde{\eta}_7) = (\Sigma^2i_{2,3})\lambda_2\eta_6^3 = 0\) and

\[
\partial_10(\omega_8, \Sigma^2i_{1,3}) = [(\Sigma^2i_{2,3})\lambda_2\eta_6, \Sigma^2i_{1,3}] = (\Sigma^2i_{2,3})[\lambda_2, i_4]\eta_8,
\]

we obtain

\[
\pi_9(K^{(8)}) = \mathbb{Z}_2\{i_{K,8}^{5}\lambda_2\nu_6\} \oplus \mathbb{Z}_2\{i_{K,8}^{5}\eta_3\nu_5\eta_8\} \oplus \mathbb{Z}_2\{i_{5,8}^{5}\nu_5\eta_8\}
\]

\[
\oplus \mathbb{Z}_4\{i_{5,8}^{5}\Sigma^2i_{1,3}, s\eta_7\} \oplus \mathbb{Z}_2\{i_{5,8}^{5}[\Sigma^2i_{1,3}, s, \Sigma^2i_{1,3}]\}.
\]

Note that \(\varphi\) is obtained in the following diagram between the cofiber sequences:

\[
\begin{array}{ccc}
S^7 & \xrightarrow{\Sigma^2i_{1,3}, \Sigma^2\gamma_3} & \Sigma^2P^3 & \xrightarrow{i_{K,8}^{5}} & K^{(8)} & \xrightarrow{p_{8,5}^{K,9}} & S^8 \\
\downarrow & & & & & & \\
S^7 & \xrightarrow{i_8} & M^8 & \xrightarrow{p_8} & S^8 & \xrightarrow{2i_8} & S^8.
\end{array}
\]

Write now the homotopy exact sequence of a pair \((K^{(9)}, K^{(8)})\):

\[
\pi_10(K^{(9)}, K^{(8)}) \xrightarrow{\partial_10} \pi_9(K^{(8)}) \xrightarrow{i_{K,9}^{9}} \pi_9(K^{(9)}) \xrightarrow{j_{\ast}} \pi_9(K^{(9)}, K^{(8)})
\]

\[
\partial_9 \pi_8(K^{(8)}) \xrightarrow{i_{K,9}^{9}} \pi_8(K^{(9)}) \rightarrow 0.
\]
The group structures \( \pi_9(K^{(9)}, K^{(8)}) = \mathbb{Z}\omega_9 \) and \( \pi_{10}(K^{(9)}, K^{(8)}) = \mathbb{Z}_2\{\tilde{\eta}_8\} \) are obtained by the Blakers-Massey theorem, where \( \omega_9 = \omega_9^{(9), (8)} \). By use of the exact sequence of a triple \((K^{(9)}, K^{(8)}, \Sigma^2P^3)\),

\[
\partial : \pi_9(K^{(9)}, K^{(8)}) \to \pi_8(K^{(8)}, \Sigma^2P^3)
\]

is the map of degree 2. So, by the commutative diagram

\[
\begin{array}{ccc}
\pi_9(K^{(9)}, K^{(8)}) & \xrightarrow{\partial} & \pi_8(K^{(8)}, \Sigma^2P^3) \\
\downarrow & & \downarrow j_* \\
\pi_8(K^{(8)}) & & 
\end{array}
\]

\( \varphi \) is taken as the attaching map of 9-cell of \( K^{(9)} \). Hence, by (2.3) and \( \pi_8(K^{(9)}) = \pi_8(K) \), we obtain (1) and \( j_* = 0 \). We see that

\[
\partial_1(\tilde{\eta}_8) = \varphi \circ \eta_8 \in \{i_{5,8}^{K}[\Sigma^2i_{2,3}, \Sigma^2\gamma_3], i_8, 2\iota_7\} \circ \eta_8
\]

\[
= i_{5,8}^{K}[\Sigma^2i_{2,3}, \Sigma^2\gamma_3] \circ \{i_8, 2\iota_7, \eta_7\}
\]

\[
\varphi = i_{5,8}^{K}[\Sigma^2i_{2,3}, \Sigma^2\gamma_3][\eta_7]
\]

\[
\mod i_{5,8}^{K}[\Sigma^2i_{2,3}, \Sigma^2\gamma_3] \circ \{\pi_8(M^8) \circ \eta_8 + i_8 \circ \pi_9(S^7)\} = 0.
\]

Here we used \([\Sigma^2i_{2,3}, \Sigma^2\gamma_3]i_8\eta_7^2 = [\Sigma^2i_{1,3}, \Sigma^2\gamma_3]\iota_7^2 = (\Sigma^2i_{2,3})\lambda_2\eta_6^3 = 0. \) By the fact that \([\Sigma^2i_{2,3}, \Sigma^2\gamma_3] = 2[\Sigma^2i_{2,3}, s] + (\Sigma^2i_{2,3})[\iota_{M^4}, \tilde{\eta}_3] \) and \([\iota_{M^4}, \tilde{\eta}_3] = \tilde{\eta}_3\nu_5p_8 + \lambda_2\eta_6 \) [5, Lemma 1.2], we obtain

\[
\partial_1(\tilde{\eta}_8) = 2i_{5,8}^{K}[\Sigma^2i_{2,3}, s][\eta_7] + i_{4,8}\tilde{\eta}_3\nu_5\eta_8,
\]

and hence

\[
\pi_i(K^{(9)}) = \mathbb{Z}_2\{i_{4,9}\lambda_2\nu_6\} \oplus \mathbb{Z}_2\{i_{5,9}s\nu_5\eta_8\} \oplus \mathbb{Z}_4\{i_{5,9}[\Sigma^2i_{2,3}, s][\eta_7]\} \\
\oplus \mathbb{Z}_2\{i_{5,9}[\Sigma^2i_{1,3}, s], \Sigma^2i_{1,3}\}.
\]

Let \( p_M : \Sigma^2P^3 \to M^4 \) be the projection. Then,

\[
\iota_{\Sigma^2P^3} = s\Sigma^2p_{3,2} + (\Sigma^2i_{2,3})p_M
\]

\[
(2.6)
\]

\[
\Sigma^2p_{3,2} \circ s = \iota_5, \quad p_M \circ \Sigma^2i_{2,3} = \iota_{M^4} \quad \text{and} \quad p_M \circ s = 0.
\]

By (2.1) and (2.6), we have

\[
[i_{\Sigma^2P^3}, \Sigma^2\gamma_3] = [s\Sigma^2p_{3,2}, \Sigma^2\gamma_3] + [(\Sigma^2i_{2,3})p_M, \Sigma^2\gamma_3]
\]

\[
= [s, (\Sigma^2i_{2,3})\tilde{\eta}_3] \circ \Sigma^6p_{3,2} + (\Sigma^2i_{2,3})[p_M, \tilde{\eta}_3] + 2[(\Sigma^2i_{2,3})p_M, s].
\]
By (2.7), we have \([p_M, \tilde{\eta}_3] \circ \Sigma^6i_{2,3} = [\iota_{M4}, \tilde{\eta}_3]\). So, by use of the cofiber sequence \(M^8 \xrightarrow{\Sigma^6i_{2,3}} \Sigma^6 P^3 \xrightarrow{\Sigma^6p_{3,2}} S^9\),

\([p_M, \tilde{\eta}_3] \equiv [\iota_{M4}, \tilde{\eta}_3] \circ \Sigma^4 p_M \mod \pi_9(M^4) \circ \Sigma^6 p_{3,2}\).

By the same reason,

\([(\Sigma^2 i_{2,3})p_M, s] \equiv [\Sigma^2 i_{2,3}, s] \circ \Sigma^4 p_M \mod \pi_9(\Sigma^2 P^3) \circ \Sigma^6 p_{3,2}\).

Hence, by Lemma 2.2 (2) and (2.7), we conclude that

\([i_{\Sigma^2 P^3}, \Sigma^2 \gamma_3] \circ \Sigma^4 s \equiv \pm [s, (\Sigma^2 i_{2,3})\tilde{\eta}_3] \mod (\Sigma^2 i_{2,3}) \circ \pi_9(M^4) \circ \Sigma^6 p_{3,2}\).

The attaching map of the 10-cell of \(K^{(10)}\) is \(i_{K,9}^K[i_{\Sigma^2 P^3}, \Sigma^2 \gamma_3] \Sigma^4 s\). By (2.1), we have

\(i_{K,9}^K[i_{\Sigma^2 P^3}, \Sigma^2 \gamma_3] \Sigma^4 s \equiv \pm i_{K,9}^K[s, \Sigma^2 i_{2,3}]\tilde{\eta}_7 \mod i_{K,9}^K \lambda_2 \nu_6\).

So, by the homotopy exact sequence of a pair \((K^{(10)}, K^{(9)})\) and (2.5), the group structure of \(\pi_9(K)\) is obtained. This completes the proof.

**Lemma 2.6.**

\[
\pi_9(\Sigma^2 P^4) = Z_2\{\eta_3^5\} \oplus Z_2(\Sigma^2 i_{2,4})\lambda_2 \nu_6 \oplus Z_2(\Sigma^2 i_{3,4}) s \nu_5 \eta_8 \oplus Z_2(\Sigma^2 i_{3,4}) [s, \Sigma^2 i_{1,3}, \Sigma^2 i_{1,3}].
\]

**Proof.** We consider the exact sequence induced from the fibration \(\Sigma^2 P_{4,3} : \Sigma^2 P^4 \rightarrow S^6\):

\[
\pi_{10}(S^6) = 0 \rightarrow \pi_9(K) \rightarrow \pi_9(\Sigma^2 P^4) \rightarrow \pi_9(S^6) \xrightarrow{\Delta_9} \pi_8(K) \rightarrow \cdots.
\]

By [8, Lemma 1.2], we obtain the relations \(\Delta_9(\nu_6) = \pm i_{K,9}^K \tilde{\eta}_3 + 2i_{5}^{K} \nu_5\) and

\(\Delta_9(\nu_6) = \Delta_6(\nu_6) \circ \nu_5 = \pm i_{K,9}^K \tilde{\eta}_3 \nu_5 + 2i_{5}^{K} s \nu_5\).

Using the second relation and Lemma 2.5 (1), we obtain \(\ker \Delta_9 = Z_2\{\eta_3^5\}\). Therefore, by Lemma 2.4 and 2.5 (2) and by the fact that \(i \circ i_{K}^K = \Sigma^2 i_{3,4}\) (\(i : K \rightarrow \Sigma^2 P^4\) is the inclusion), we obtain the result. This completes the proof.

Now we consider the homotopy exact sequence of a pair \((\Sigma^2 P^5, \Sigma^2 P^4)\):

\[
\pi_{10}(\Sigma^2 P^5, \Sigma^2 P^4) \xrightarrow{\partial_9} \pi_9(\Sigma^2 P^4) \xrightarrow{i} \pi_9(\Sigma^2 P^5),
\]

where \(i = \Sigma^2 i_{4,5} : \Sigma^2 P^4 \rightarrow \Sigma^2 P^5\). By the James exact sequence, the group structures \(\pi_9(\Sigma^2 P^5, \Sigma^2 P^4) = Z_2\{\eta_6^7\} \oplus Z_2(\omega_7, \Sigma^2 i_{1,4})\) and \(\pi_{10}(\Sigma^2 P^5, \Sigma^2 P^4) = Z_24\{\tilde{\eta}_6\} \oplus Z_2(\omega_7, (\Sigma^2 i_{1,4})\eta_3)\) are settled, where \(\omega_7 = \omega_7(\Sigma^2 P^5, \Sigma^2 P^4)\). We recall, from [8, Lemma 1.3], the relation

\[(2.8) \quad \Sigma^2 \gamma_4 = (\Sigma^2 i_{3,4}) s \eta_5 + 2(\Sigma^2 i_{2,4}) \lambda_2.\]
By Lemma 2.6 and by the relation $\eta_5 \nu_6 = 0$, we obtain

$$
\partial_{10}(\tilde{\nu}_6) = (\Sigma^2 \gamma_4) \nu_6 = ((\Sigma^2 i_{3,4}) s \eta_5 + 2(\Sigma^2 i_{1,4}) \lambda_2) \nu_6 = 0.
$$

The equation $(\Sigma^2 i_{3,4})[\Sigma^2 i_{2,3}, s] \tilde{\eta}_7 = 0$ is shown in the proof of Lemma 2.6. Then

$$
\partial_{10}([\omega_7, (\Sigma^2 i_{1,4}) \eta_3]) = [\Sigma^2 \gamma_4, (\Sigma^2 i_{1,4}) \eta_3] = [(\Sigma^2 i_{3,4}) s \eta_5, (\Sigma^2 i_{1,4}) \eta_3] = (\Sigma^2 i_{3,4}) [s, \Sigma^2 i_{3,4}] \eta_7^2 = 2(\Sigma^2 i_{3,4}) [s, \Sigma^2 i_{2,3}] \eta_7 = 0.
$$

Therefore $(\Sigma^2 i_{4,5})_* : \pi_0(\Sigma^2 P^4) \to \pi_0(\Sigma^2 P^5)$ is a monomorphism.

By the fact that $\pi_8(\Sigma^2 P^4) = Z_4\{(\Sigma^2 i_{3,4}) s \nu_5\} \oplus Z_2\{[(\Sigma^2 i_{3,4}) s, (\Sigma^2 i_{1,4}) \eta_7]\} \oplus Z_2\{[(\Sigma^2 i_{2,4}) i_4, \lambda_2]\}$ [8, Lemma 2.5] and by (2.8), we obtain

$$
\partial_9(\tilde{\eta}_6^2) = (\Sigma^2 \gamma_4) \eta_6^2 = (\Sigma^2 i_{3,4}) s \eta_6^2 = 4(\Sigma^2 i_{3,4}) s \nu_5 = 0
$$

and

$$
\partial_9([\omega_7, \Sigma^2 i_{1,4}]) = [\Sigma^2 \gamma_4, \Sigma^2 i_{1,4}] = [(\Sigma^2 i_{3,4}) s \eta_5, \Sigma^2 i_{1,4}] = [(\Sigma^2 i_{3,4}) s, \Sigma^2 i_{1,4}] \eta_7.
$$

Then there exists an element $\sim \eta_6^2 \in \{\Sigma^2 i_{4,5}, \Sigma^2 \gamma_4, \eta_6^2\} \subset \pi_9(\Sigma^2 P^5)$ such that $(\Sigma^2 P^5)_* \tilde{\eta}_6^2 = \eta_7^2$. We obtain

$$
2\tilde{\eta}_6^2 \in \{\Sigma^2 i_{4,5}, \Sigma^2 \gamma_4, \eta_6^2\} \circ 2\nu_9 = -(\Sigma^2 i_{4,5} \circ \{\Sigma^2 \gamma_4, \eta_6^2, 2\nu_8\})
$$

and

$$
\{\Sigma^2 \gamma_4, \eta_6^2, 2\nu_8\} \subset \{\Sigma^2 i_{3,4}, (s \eta_5 + 2(\Sigma^2 i_{2,3}) \lambda_2) \eta_6^2, 2\nu_8\}
$$

$$
= \{\Sigma^2 i_{3,4}, s \eta_6^2, 2\nu_8\}
$$

$$
= \{\Sigma^2 i_{3,4}, \Sigma^2 \gamma_3 \circ 2\nu_5, 2\nu_8\}
$$

$$
\ni \sim \eta_5^3
$$

mod $2\pi_9(\Sigma^2 P^4) + (\Sigma^2 i_{3,4})_* \pi_9(\Sigma^2 P^3) = (\Sigma^2 i_{3,4})_* \pi_9(\Sigma^2 P^3)$, and hence we conclude that $2\tilde{\eta}_6^2 \equiv (\Sigma^2 i_{4,5}) \sim \eta_5^3$ mod $(\Sigma^2 i_{3,5})_* \pi_9(\Sigma^2 P^3)$. Thus, by Lemma 2.6, we have the following.

**Lemma 2.7.**

$$
\pi_9(\Sigma^2 P^5) = Z_4\{\tilde{\eta}_6^2\} \oplus Z_2\{(\Sigma^2 i_{2,5}) \lambda_2 \nu_6\} \oplus Z_2\{(\Sigma^2 i_{3,5}) s \nu_5 \eta_8\}
$$

$$
\oplus Z_2\{(\Sigma^2 i_{3,5})[[\Sigma^2 i_{1,3}, s], \Sigma^2 i_{1,3}]\},
$$

where $2\tilde{\eta}_6^2 = (\Sigma^2 i_{4,5}) \sim \eta_5^3$ for a suitable choice of $\sim \eta_5^3$. 
3. Proofs of main theorems

First, we show Theorem 1.2.

From the fact that $\Sigma^2 \gamma_5 \in \{\Sigma^2 i_{4,5}, \Sigma^2 \gamma_4, 2\iota_6\}$, $\Sigma^3 \gamma_4 = (\Sigma^3 i_{3,4})(\Sigma s)\eta_6$, we see that

$$[\Sigma^2 \gamma_5, \Sigma^2 i_{1,5}]p_9 = [\iota \Sigma^2 \pi^5, \Sigma^2 i_{1,5}] \circ (\Sigma^4 \gamma_5 \circ p_9)$$

$$\in [\iota \Sigma^2 \pi^5, \Sigma^2 i_{1,5}] \circ (\{\Sigma^4 i_{4,5}, \Sigma^4 \gamma_4, 2\iota_8\} \circ p_9)$$

$$\Rightarrow [\iota \Sigma^2 \pi^5, \Sigma^2 i_{1,5}] \circ (\{\Sigma^4 i_{3,5}\} \Sigma^2 s, \eta_7, 2\iota_8 \circ p_9)$$

$$= -([\Sigma^2 i_{3,5}, \Sigma^2 i_{1,5}] \Sigma^2 s \circ \{\eta_7, 2\iota_8, p_8\})$$

$$\Rightarrow [\Sigma^2 i_{3,5}, \Sigma^2 i_{1,5}] \circ (\Sigma^2 s) \eta_7$$

mod $[\Sigma^2 i_{4,5}, \Sigma^2 i_{1,5}] \circ \pi_9 (\Sigma^4 \pi^4) \circ p_9$.

It is easily seen that $\pi_9 (\Sigma^4 \pi^4) = \mathbf{Z}_2 \{\{\Sigma^4 i_{1,4}\} \eta_5 \eta_8\} \oplus \mathbf{Z}_2 \{\{\Sigma^4 i_{3,4}\} (\Sigma^2 s) \eta_7^2\}$.

Since $[\iota_3, \iota_3] = 0$ and $([\Sigma^2 i_{3,4}, \Sigma^2 i_{1,5}] \Sigma^2 s$ is changed as follows.

$$[\Sigma^2 i_{3,5}, \Sigma^2 i_{1,5}] \Sigma^2 s = (\Sigma^2 i_{3,5}) [\iota \Sigma^2 \pi^3, \Sigma^2 i_{1,3}] \Sigma^2 s$$

$$= (\Sigma^2 i_{3,5}) [s, \Sigma^2 i_{1,3}] + (\Sigma^2 i_{2,5}) [p_M, i_4] \Sigma^2 s.$$}

By the fact that $[\pi_M, i_4] \in [\Sigma^4 \pi^3, M^4] = (\Sigma^4 \pi^3, [\Sigma^4 \pi^3, M^4]) \oplus (\Sigma^2 \pi^3, [\Sigma^2 \pi^3, M^4])$, we obtain

$$(\Sigma^2 i_{2,5}) [p_M, i_4] \Sigma^2 s \in \Sigma^2 i_{2,5} \circ (\pi_7(M^4) \circ (\Sigma^4 i_{3,4}) \circ \Sigma^4 i_{3,4}) \circ \Sigma^2 \pi^3 + [M^6, M^4] \circ \Sigma^2 \pi^3 \circ \Sigma^2 s$$

$$= \Sigma^2 i_{2,5} \circ \pi_7(M^4).$$

We recall from [7, Lemma 2.2] that $\pi_7(M^4) = \mathbf{Z}_2 \{\lambda_2 \eta_6\} \oplus \mathbf{Z}_2 \{\eta_3 \eta_6^2\}$. Since $(\Sigma^2 i_{2,4}) \lambda_2 \eta_6 = 0$ [8, the proof of Lemma 2.2] and by (2.1), the group $\Sigma^2 i_{2,5} \circ \pi_7(M^4)$ is 0. Then,

$$[\Sigma^2 \gamma_5, \Sigma^2 i_{1,5}]p_9 = [\Sigma^2 i_{3,5}, \Sigma^2 i_{1,5}] (\Sigma^2 s) \eta_7 = (\Sigma^2 i_{3,5}) [s, \Sigma^2 i_{1,3}] \eta_7.$$

Here we consider an element $[\Sigma^2 \gamma_4, \Sigma^2 i_{2,4}] \in [M^9, \Sigma^2 \pi^4]$. Since $2 \iota_M^4 = i_4 \eta_3 p_4$ [11], $(\Sigma^2 i_{2,4}) \lambda_2 \eta_6 = 0$, $(\Sigma^2 i_{3,4}) [s, \Sigma^2 i_{2,3}] \eta_7 = 0$ and $\eta_2 \land \iota_M^2 = i_4 \eta_3 + \eta_3 p_5$, we obtain

$$[\Sigma^2 \gamma_4, \Sigma^2 i_{2,4}] = (\Sigma^2 i_{3,4}) [s \eta_5, \Sigma^2 i_{2,3}] + (\Sigma^2 i_{2,4}) [2 \lambda_2, \iota_M^4]$$

$$= (\Sigma^2 i_{3,4}) [s, \Sigma^2 i_{2,3} \circ \Sigma(\eta_4 \land \iota_M^3) + (\Sigma^2 i_{2,4}) [\lambda_2, 2 \iota_M^4]$$

where we use $\Sigma^2 \gamma_5 = \Sigma^2 i_{3,5} \circ \pi_9 (\Sigma^4 \pi^4) \circ p_9$.
Lemma 3.1.

\[ (\Sigma^2 i_{3,4})[s, \Sigma^2 i_{2,3}] \circ \Sigma (i_7 \eta_6 + \widetilde{\eta}_6 p_8) + (\Sigma^2 i_{2,4})[\lambda_2, i_4 \eta_3 p_4] \]
\[ = (\Sigma^2 i_{3,4})[s, \Sigma^2 i_{1,3}] \eta_7 + (\Sigma^2 i_{2,4})[\lambda_2 \eta_6, i_4] p_9 \]
\[ = (\Sigma^2 i_{3,4})[s, \Sigma^2 i_{1,3}] \eta_7. \]

Thus, we get that
\[ [\Sigma^2 \gamma_5, \Sigma^2 i_{1,5}] p_9 = (\Sigma^2 i_{3,5})[s, \Sigma^2 i_{1,3}] \eta_7 = (\Sigma^2 i_{4,5})[\Sigma^2 \gamma_4, \Sigma^2 i_{2,4}] = 0. \]

By use of the cofibre sequence \( S^8 \xrightarrow{i_9} M^9 \xrightarrow{p_9} S^9 \xrightarrow{2\mu_9} S^9 \), we have
\[ [\Sigma^2 \gamma_5, \Sigma^2 i_{1,5}] \in 2\pi_9(\Sigma^2 P^5) = \mathbb{Z}_2 \{2\eta_6^2\}. \]

Let \( l_1 : P^4/P^3 = S^4 \rightarrow P^5/P^3 = S^4 \vee S^5 \) be the canonical inclusion map. By Lemma 2.7 and by the relations \( p_{5,3} \circ i_{4,5} = l_1 \circ p_{4,3} \) and \( p_{5,3} \circ i_{1,5} = 0 \), we obtain
\[ \Sigma^2 p_{5,3} \circ 2\eta_6^2 = \Sigma^2 p_{5,3} \circ \Sigma^2 i_{4,5} \circ \eta_6^3 \]
\[ = \Sigma^2 l_1 \circ \Sigma^2 p_{4,3} \circ \eta_6^3 \]
\[ = (\Sigma^2 l_1) \eta_6^3 \neq 0 \in \pi_9(S^6 \vee S^7) \cong \mathbb{Z}_{24} \oplus \mathbb{Z}_2 \]
and
\[ \Sigma^2 p_{5,3} \circ [\Sigma^2 \gamma_5, \Sigma^2 i_{1,5}] = 0. \]

Therefore we have \([\Sigma^2 \gamma_5, \Sigma^2 i_{1,5}] = 0\) and the proof of Theorem 1.2 is complete.

By [8, the proof of Lemma 2.5], we have a relation \((\Sigma^2 \gamma_5) \eta_7 = 0\) and we can define a coextension \( \widetilde{\eta}_7 \in \pi_9(\Sigma^2 P^6) \) of \( \eta_7 \) as follows:
\[ \widetilde{\eta}_7 \in \{\Sigma^2 i_{5,6}, \Sigma^2 \gamma_5, \eta_7\}. \]

Since \( 2\widetilde{\eta}_7 \in \{\Sigma^2 i_{5,6}, \Sigma^2 \gamma_5, \eta_7\} \circ 2\mu_9 = (\Sigma^2 i_{5,6} \circ \{\Sigma^2 \gamma_5, \eta_7, 2\mu_8\}) \) and
\[ \Sigma^2 p_{5,4} \circ \{\Sigma^2 \gamma_5, \eta_7, 2\mu_8\} \subset \{2\mu_7, \eta_7, 2\mu_8\} = \eta_7^2, \]
we obtain \( 2\widetilde{\eta}_7 \equiv (\Sigma^2 i_{5,6}) \eta_6^2 \mod \Sigma^2 i_{5,6} \circ 2\pi_9(\Sigma^2 P^5) + \Sigma^2 i_{4,6} \circ \pi_9(\Sigma^2 P^4) = \Sigma^2 i_{4,6} \circ \pi_9(\Sigma^2 P^4) \). From the exact sequence of a pair \((\Sigma^2 P^6, \Sigma^2 P^5)\) and by Theorem 1.2, we see that \((\Sigma^2 i_{5,6})^* : \pi_9(\Sigma^2 P^5) \rightarrow \pi_9(\Sigma^2 P^6)\) is a monomorphism. Thus, \( \widetilde{\eta}_7 \) is of order 8 and the group structure of \( \pi_9(\Sigma^2 P^6) \) is given as follows.

Lemma 3.1.

\[ \pi_9(\Sigma^2 P^6) = \mathbb{Z}_8 \{\widetilde{\eta}_7\} \oplus \mathbb{Z}_2 \{(\Sigma^2 i_{2,6}) \lambda_2 \nu_6\} \oplus \mathbb{Z}_2 \{(\Sigma^2 i_{3,6}) s \nu_5 \eta_8\} \]
\[ \oplus \mathbb{Z}_2 \{(\Sigma^2 i_{3,6}) \{[\Sigma^2 i_{1,3}, s], \Sigma^2 i_{1,3}\}\},\]

where \( 2\widetilde{\eta}_7 = (\Sigma^2 i_{5,6}) \eta_6^2 \) for a suitable choice of \( \eta_6^2 \).
We denote by $s_1 : S^9 \to \Sigma^2 P^7 = \Sigma^2 P^6 \lor S^9$ the inclusion map to the second factor and by $q_1 : \Sigma^2 P^7 = \Sigma^2 P^6 \lor S^9 \to \Sigma^2 P^6$ the map collapsing $S^9$ to one point. Finally we obtain the following.

**Theorem 3.2.**

$[\Sigma^2 P^7, \Sigma^2 P^7] = \mathbb{Z}\{s_1\Sigma^2 p_{7,6}\} \oplus \mathbb{Z}_8\{(\Sigma^2 i_{6,7})q_1\} \oplus \mathbb{Z}_8\{(\Sigma^2 i_{6,7})\eta_7\Sigma^2 p_{7,6}\} \oplus \mathbb{Z}_2\{(\Sigma^2 i_{3,7})[s, \Sigma^2 p_{6,4}]q_1\} \oplus \mathbb{Z}_2\{(\Sigma^2 i_{2,7})\lambda_2(\Sigma^2 p_{4,3})q_1\} \oplus \mathbb{Z}_2\{(\Sigma^2 i_{2,7})[\lambda_2, i_4](\Sigma^2 p_{6,5})q_1\} \oplus \mathbb{Z}_2\{(\Sigma^2 i_{2,7})\lambda_2\nu_0\Sigma^2 p_{7,6}\} \oplus \mathbb{Z}_2\{(\Sigma^2 i_{3,7})\nu_5(\Sigma^2 p_{6,5})q_1\} \oplus \mathbb{Z}_2\{(\Sigma^2 i_{3,7})\lambda_2\nu_0\Sigma^2 p_{7,6}\} \oplus \mathbb{Z}_2\{(\Sigma^2 i_{3,7})\nu_5\eta_8\Sigma^2 p_{7,6}\} \oplus \mathbb{Z}_2\{(\Sigma^2 i_{3,7})[\Sigma^2 i_{1,3}, s, \Sigma^2 i_{1,3}]\Sigma^2 p_{7,6}\}$.

**References**


TOSHIYUKI MIYAUCHI
GRADUATE SCHOOL OF MATHEMATICS
KYUSHU UNIVERSITY
FUKUOKA, 812-8581 JAPAN

*e-mail address*: miyauchi@math.kyushu-u.ac.jp

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