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# ON HIGHER SYZYGIES OF PROJECTIVE TORIC VARIETIES

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Abstract. Let $A$ be an ample line bundle on a projective toric variety $X$ of dimension $n$ ($\geq 2$). It is known that the $d$-th tensor power $A^{\otimes d}$ embedds $X$ as a projectively normal variety in $\mathbb{P}^r := \mathbb{P}(H^0(X, L^{\otimes d}))$ if $d \geq n-1$. In this paper first we show that when $\dim X = 2$ the line bundle $A^{\otimes d}$ satisfies the property $N_p$ for $p \leq 3d-3$. Second we show that when $\dim X = n \geq 3$ the bundle $A^{\otimes d}$ satisfies the property $N_p$ for $p \leq d-n+2$ and $d \geq n-1$.

## Introduction

The purpose of this article is to study the minimal free resolution of homogeneous coordinate rings of toric varieties.

Let $X$ be a projective toric variety of dimension $n$ and $L$ a very ample line bundle on $X$. Since projective toric variety of dimension one is isomorphic to the projective line, we may assume that $n \geq 2$.

Koelman showed that an ample line bundle on a projective toric surface $X$ is very ample and embedds $X$ as a projectively normal variety in $\mathbb{P}^r := \mathbb{P}(H^0(X, L))$ [10], and obtained a criterion when the surface is defined by only quadrics [11]. When $n \geq 3$ an ample line bundle is not very ample in general. Ewald and Wessels [3] showed that for an ample line bundle $A$ on $X$ the $d$-th tensor power $L = A^{\otimes d}$ is very ample for $d \geq \dim X - 1$. Ogata and Nakagawa [13] showed that $L = A^{\otimes d}$ embedds $X$ as a projectively normal variety if $d \geq \dim X - 1$ and that the homogeneous ideal $I$ of $X$ in $\mathbb{P}^r := \mathbb{P}(H^0(X, L))$ is generated by quadrics if $d \geq \dim X$. In this paper, we study higher syzygies of the homogeneous ideal of $X$ in $\mathbb{P}^r$, especially the property $N_p$ introduced by Green and Lazarsfeld [7].

**Definition 1.** Let $X$ be a projective variety and $L$ a very ample line bundle on $X$ defining an embedding $X \hookrightarrow \mathbb{P}^r := \mathbb{P}(H^0(X, L))$. Denote by $S = \operatorname{Sym} H^0(X, L)$ the homogeneous coordinate ring of the projective space $\mathbb{P}^r$. Consider the graded $S$-module $R = R(L) = \oplus_{i \geq 0} H^0(X, L^{\otimes i})$, the homogeneous coordinate ring of $X$. Let $E_\bullet$ be a minimal graded free resolution of $R$:

$$\cdots \to E_2 \to E_1 \to E_0 \to R \to 0,$$

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where $E_i = \bigoplus_j S(-a_{ij})$. Then the line bundle $L$ satisfies *Property* $(N_0)$ if $E_0 = S$. For an integer $p \geq 1$ the line bundle $L$ satisfies *Property* $(N_p)$ if $E_0 = S$ and if $a_{ij} = i+1$ for $1 \leq i \leq p$.

Schenck and Smith [17] proved that for an ample line bundle $A$ on a projective toric variety of dimension $n$, the bundle $A^{\otimes d}$ satisfies Property $N_{d-n+1}$ for $d \geq n-1$.

This paper improves their results by separately considering the case $n = 2$ and $n \geq 3$.

**Theorem 0.1.** *Let $X$ be a projective toric surface and $A$ an ample line bundle on $X$. Then $A^{\otimes d}$ satisfies Property $N_p$ for $p \leq 3d-3$.*

This is given by Proposition 2.3.

**Theorem 0.2.** *Let $X$ be a projective toric variety of dimension $n$ $(\geq 3)$ and $A$ an ample line bundle on $X$. Then $A^{\otimes d}$ satisfies Property $N_p$ for $p \leq d-n+2$ and $d \geq n-1$.*

This is given by Proposition 3.3.

## 1. Polarized Toric Varieties

First we mention the facts about toric varieties needed in this paper following Oda's book [14], or Fulton's book [5].

Let $N$ be a free $\mathbb{Z}$-module of rank $n$, $M$ its dual and $\langle , \rangle : M \times N \to \mathbb{Z}$ the canonical pairing. By scalar extension to the field $\mathbb{R}$ of real numbers, we have real vector spaces $N_{\mathbb{R}} := N \otimes_{\mathbb{Z}} \mathbb{R}$ and $M_{\mathbb{R}} := M \otimes_{\mathbb{Z}} \mathbb{R}$. Let $T_N := N \otimes_{\mathbb{Z}} \mathbb{C}^* \cong (\mathbb{C}^*)^n$ be the algebraic $n$-torus over the field $\mathbb{C}$ of complex numbers, where $\mathbb{C}^*$ is the multiplicative group of $\mathbb{C}$. Then $M = \mathrm{Hom}_{\mathrm{gr}}(T_N, \mathbb{C}^*)$ is the character group of $T_N$. For $m \in M$ we denote $\mathbf{e}(m)$ the corresponding character of $T_N$. Let $\Delta$ be a complete finite fan of $N$ consisting of strongly convex rational polyhedral cones $\sigma$, that is, there exist a finite number of elements $v_1, v_2, \ldots, v_s$ in $N$ such that

$$\sigma = \mathbb{R}_{\geq 0} v_1 + \cdots + \mathbb{R}_{\geq 0} v_s,$$

and $\sigma \cap \{-\sigma\} = \{0\}$. Then we have a complete toric variety $X = T_N\mathrm{emb}(\Delta) : = \cup_{\sigma \in \Delta} U_\sigma$ of dimension $n$ (see Section 1.2 [14], or Section 1.4 [5]). Here $U_\sigma = \mathrm{Spec}\ \mathbb{C}[\sigma^\vee \cap M]$ and $\sigma^\vee$ is the dual cone of $\sigma$ with respect to the pairing $\langle , \rangle$. For the origin $\{0\}$, the affine open set $U_{\{0\}} = \mathrm{Spec}\ \mathbb{C}[M]$ is the unique dense $T_N$-orbit. We note that a toric variety is always normal.

Let $L$ be an ample $T_N$-equivariant invertible sheaf on $X$. Then the polarized variety $(X, L)$ corresponds to an integral convex polytope $P$ in $M_{\mathbb{R}}$ of dimension $n$. We call the convex hull $\mathrm{Conv}\{u_0, u_1, \ldots, u_r\}$ in $M_{\mathbb{R}}$ of a finite subset $\{u_0, u_1, \ldots, u_r\} \subset M$ an *integral convex polytope* in $M_{\mathbb{R}}$. The correspondence is given by the isomorphism

$$H^0(X, L) \cong \bigoplus_{m \in P \cap M} \mathbb{C}\mathbf{e}(m), \tag{1.1}$$

where $\mathbf{e}(m)$ are considered as rational functions on $X$ because they are functions on the open dense subset $T_N$ of $X$ (see Section 2.2 [14], or Section 3.5 [5]).

Let $P_1$ and $P_2$ be integral convex polytopes in $M_{\mathbb{R}}$. Then we can consider the Minkowski sum $P_1 + P_2 := \{x_1 + x_2 \in M_{\mathbb{R}}; x_i \in P_i \quad (i = 1, 2)\}$ and the multiplication by scalars $rP_1 := \{rx \in M_{\mathbb{R}}; x \in P_1\}$ for a positive real number $r$. If $l$ is a natural number, then $lP_1$ coincides with the $l$ times sum of $P_1$, i.e., $lP_1 = \{x_1 + \cdots + x_l \in M_{\mathbb{R}}; x_1, \ldots, x_l \in P_1\}$. The $l$-th tensor power $L^{\otimes l}$ corresponds to the convex polytope $lP := \{lx \in M_{\mathbb{R}}; x \in P\}$. Moreover the multiplication map

$$H^0(X, L^{\otimes l}) \otimes H^0(X, L) \to H^0(X, L^{\otimes (l+1)}) \tag{1.2}$$

transforms $\mathbf{e}(u_1) \otimes \mathbf{e}(u_2)$ for $u_1 \in lP \cap M$ and $u_2 \in P \cap M$ to $\mathbf{e}(u_1 + u_2)$ through the isomorphism (1.1). Therefore the equality $(lP \cap M) + (P \cap M) = (l+1)P \cap M$ is equivalent to the surjectivity of the multiplication map (1.2).

**Proposition 1.1** (Nakagawa-Ogata [13])**.** *Let $X$ be a projective toric variety of dimension $n$ and $L$ an ample line bundle on $X$. Then the multiplication map*

$$H^0(X, L^{\otimes i}) \otimes H^0(X, L) \to H^0(X, L^{\otimes (i+1)})$$

*is surjective for all $i \geq n - 1$.*

This implies that $L^{\otimes d}$ satisfies Property $N_0$ for $d \geq n-1$. By employing an analogous method of Mumford [12] we obtained that $L^{\otimes d}$ satisfies Property $N_1$ for $d \geq n$ (see Corollary 2.2 in [13]). Schenck and Smith [17] generalizes for $L^{\otimes d}$ to satisfy Property $N_p$ for $d \geq n - 1 + p$.

## 2. Toric Surfaces

Ogata[16] generalize the result of [11] to higher dimension by using the method of Fujita's regular ladder [4]. In this section we use the same method in the case of dimension two.

Let $X$ be a projective toric surface and $L$ an ample line bundle on $X$. We consider a general hyperplane section $C$ of the linear system $|L|$. Since $X$ is normal, we may assume that $C$ is nonsingular. Set $L_C = L|C$, the restriction of $L$ to the curve $C$. From easy calculation, we see that

$$h^0(C, L_C) = h^0(X, L) - 1 = {}^\sharp P \cap M - 1, \tag{2.1}$$

$$
\begin{aligned}
(2.2) \quad h^1(C, \mathcal{O}_C) &= h^2(X, L^{-1}) = h^0(X, \omega_X \otimes L) = {}^\sharp\mathrm{Int}\ P \cap M, \\
(2.3) \quad h^1(C, L_C) &= 0.
\end{aligned}
$$

Denote by $g = h^1(C, \mathcal{O}_C)$ the genus of $C$. Then from Riemann-Roch Theorem we have $\deg L_C = 2g - 2 + {}^\sharp\partial P \cap M$.

**Lemma 2.1** (Green [6], (4.a.1))**.** *Let $L$ be a line bundle of degree $2g+1+p$ $(p \geq 0)$ on a smooth irreducible projective curve of genus $g$. Then $L$ satisfies Property $N_p$.*

This is a generalization of Mumford [12] (the case $p = 0$) and of Fujita [4] (the case $p = 1$).

**Lemma 2.2** (Green [6], (3.b.7))**.** *Let $X$ be a compact complex manifold, $L$ a line bundle on $X$, $Y \subset X$ a connected hypersurface in the linear system $|L|$ and $L_Y$ denote the restriction of $L$ to $Y$. Assume that*

$$H^1(X, L^{\otimes q}) = 0 \quad \textit{for all } q \geq 0.$$

*Then $\mathcal{K}_{p,q}(X, L) \cong \mathcal{K}_{p,q}(Y, L_Y)$ for all $p, q$.*

Since $H^1(X, L^{\otimes q}) = 0$ for $q \geq 0$ and for any ample ample line bundle $L$ on a toric variety $X$, this lemma implies that if $L_Y$ satisfies Property $N_p$ for a general smooth hypersurface $Y$ in $|L|$, then $L$ also satisfies Property $N_p$.

**Proposition 2.3.** *Let $A$ be an ample line bundle on a projective toric surface $X$ corresponding to an integral convex polygon $Q$ in $M_{\mathbb{R}}$ given by the isomorphism*

$$H^0(X, A) \cong \bigoplus_{m \in Q \cap M} \mathbb{C}\mathbf{e}(m).$$

*Then the $d$-th tensor power $A^{\otimes d}$ satisfies Property $N_p$ for $p \leq d\ {}^\sharp\mathrm{Int}\ Q \cap M - 3$. In particular, $A^{\otimes d}$ satisfies Property $N_p$ for $p \leq 3d - 3$.*

*Proof.* Set $L = A^{\otimes d}$. Let $C$ be a general hyperplane section of $|L|$. Set $L_C = L|C$. Denote by $g$ the genus of $C$. Then we have

$$
\begin{aligned}
\deg L_C &= 2g - 2 + d\ {}^\sharp\mathrm{Int}\ Q \cap M \\
&= 2g + 1 + (d\ {}^\sharp\mathrm{Int}\ Q \cap M - 3).
\end{aligned}
$$

From Lemma 2.1, the bundle $L_C$ satisfies Property $N_p$ for $p \leq d\ {}^\sharp\mathrm{Int}\ Q \cap M - 3$. From Lemma 2.2 we obtain a proof of Proposition. □

This is a generalization of the case $(X, A) = (\mathbb{P}^2, \mathcal{O}(1))$ treated by Birkenhake in [1].

## 3. Higher Dimension

**Lemma 3.1** (Ogata [15]). *Let $A$ be an ample line bundle on a projective toric variety of dimension $n$ ($n \geq 3$). Then $A^{\otimes d}$ satisfies Property $N_1$ for $d \geq n-1$.*

A very ample invertible sheaf $L$ on a projective variety $X$ defines an embedding $\Phi_L : X \to \mathbb{P}(H^0(X,L)) = \mathbb{P}^r$. Set $M_L := \Phi_L^* \Omega^1_{\mathbb{P}^r}(1)$ so that there exists the following exact sequence of vector bundles

$$0 \to M_L \to H^0(X,L) \otimes_{\mathbb{C}} \mathcal{O}_X \to L \to 0. \tag{3.1}$$

**Lemma 3.2** (Ein-Lazarsfeld [2]). *Assume that $L$ is very ample and that $H^1(X, L^{\otimes k}) = 0$ for all $k \geq 1$. Then $L$ satisfies Property $N_p$ if and only if*

$$H^1(X, \wedge^a M_L \otimes L^{\otimes b}) = 0 \quad \text{for } 1 \leq a \leq p+1 \text{ and } b \geq 1.$$

Since in characteristic zero $\wedge^a M_L$ is a direct summand of $M_L^{\otimes a}$, we have only to show the vanishing of $H^1(X, M_L^{\otimes a} \otimes L^{\otimes b})$ as in [2] and [8].

**Proposition 3.3.** *Let $p \geq 2$ be an integer. Let $A$ be an ample line bundle on a projective toric variety $X$ of dimension $n$ ($n \geq 3$). Then $A^{\otimes d}$ satisfies Property $N_p$ for $p \leq n-2-d$.*

For a proof we have to show the vanishing of $H^1(X, M_L^{\otimes a} \otimes L^{\otimes b})$ for $1 \leq a \leq p+1$ and $b \geq 1$ with $L = A^{\otimes d}$. We need a lemma.

**Lemma 3.4** (Mumford [12]). *Let $F$ be a coherent sheaf on a projective algebraic variety $X$. Let $A$ be a line bundle on $X$ generated by global sections. If $H^i(X, F \otimes A^{\otimes(-i)}) = 0$ for all $i \geq 1$, then the multiplication map*

$$H^0(X, F \otimes A^{\otimes j}) \otimes H^0(X, A) \to H^0(X, F \otimes A^{\otimes(j+1)})$$

*is surjective for all $j \geq 0$.*

For a proof see Theorem 2 in [12].

*Proof of Proposition 3.3.* Let $q \geq 2$ be an integer and $L = A^{\otimes d}$ with $d \geq n+q-2$. We want to show that

$$H^1(X, M_L^{\otimes q} \otimes A^{\otimes j}) = 0 \quad \text{for } j \geq n+q-3, \tag{3.2}$$

$$H^i(X, M_L^{\otimes q} \otimes A^{\otimes j}) = 0 \quad \text{for } i \geq 2 \text{ and } j \geq 0. \tag{3.3}$$

From Proposition 1.1 and the exact sequence (3.1) we see that

$$H^i(X, M_L \otimes A^{\otimes j}) = 0 \quad \text{for } i \geq 1,\ j \geq 0 \text{ and } d \geq n-1. \tag{3.4}$$

First we shall show the vanishing of (3.2) for $q = 2$. Taking tensor product of (3.1) with $M_L \otimes A^{\otimes j}$ we obtain an exact sequence

$$0 \to M_L^{\otimes 2} \otimes A^{\otimes j} \to H^0(X,L) \otimes_{\mathbb{C}} M_L \otimes A^{\otimes j} \to M_L \otimes L \otimes A^{\otimes j} \to 0. \tag{3.5}$$

From (3.4) and (3.5) we have

$$H^i(X, M_L^{\otimes 2} \otimes A^{\otimes j}) = 0 \quad \text{for } i \geq 2,\ j \geq 0 \text{ and } d \geq n-1. \tag{3.6}$$

Second we shall show the vanishing of (3.2) for $q = 2$. From Lemmas 3.1 and 3.2, we see that $H^1(X, \wedge^2 M_L \otimes L^{\otimes b}) = 0$ for $b \geq 1$ and $d \geq n-1$. Taking wedge product in (3.1) and twisting by $L^{\otimes b}$, we obtain an exact sequence

$$0 \to \wedge^2 M_L \otimes L^{\otimes b} \to \wedge^2 H^0(X, L) \otimes_{\mathbb{C}} L^{\otimes b} \to M_L \otimes L^{\otimes (b+1)} \to 0. \tag{3.7}$$

The vanishing of the first cohomology group implies the surjectivity of the map

$$\wedge^2 H^0(X, L) \otimes H^0(X, L^{\otimes b}) \to H^0(X, M_L \otimes L^{\otimes (b+1)}). \tag{3.8}$$

Taking global sections of (3.7) with $b = 0$ we see that the surjective map (3.8) factors through $\wedge^2 H^0(X, L) \to H^0(X, M_L \otimes L)$. Thus we have that

$$H^1(X, M_L^{\otimes 2} \otimes L^{\otimes b}) = 0 \quad \text{for } b \geq 1 \text{ and } d \geq n-1 \tag{3.9}$$

from the exact sequence (3.5) replacing $A$ by $L$. For line bundles $L_1$ and $L_2$, denote by $R(L_1, L_2)$ the kernel of the multiplication map $H^0(X, L_1) \otimes H^0(X, L_2) \to H^0(X, L_1 \otimes L_2)$. By taking a global section of the exact sequence

$$0 \to M_L \otimes A \to \Gamma(X, L) \otimes_{\mathbb{C}} A \to L \otimes A \to 0,$$

we have $H^0(X, M_L \otimes A) = R(L, A)$. Then we can rewrite the sequence (3.5) after taking its global sections as

$$\begin{array}{ccc} H^0(X, L) \otimes H^0(X, M_L \otimes A^{\otimes j}) & \longrightarrow & H^0(X, M_L \otimes L \otimes A^{\otimes j}) \\ \| & & \| \\ H^0(X, L) \otimes R(A^{\otimes j}, L) & \longrightarrow & R(L \otimes A^{\otimes j}, L). \end{array}$$

For simplicity we denote $A^{\otimes i}$ as $A^i$ and $H^0(X, L)$ as $\Gamma(L)$. From Corollary 2.2 in [13], we have that

$$\Gamma(A^i) \otimes R(A^d, A^j) \to R(A^{d+i}, A^j)$$

is surjective for $d \geq n$, $i \geq 1$ and $j \geq 1$. Hence we showed the vanishing of (3.2) for $j \geq n$. We remain to show the vanishing of $H^1(X, M_L^{\otimes 2} \otimes A^{\otimes (n-1)})$

for $d \geq n$. Consider the diagram

$$
\begin{tikzcd}
 & & 0 \arrow[d] & 0 \arrow[d] & \\
 & & \begin{matrix} R(A^{n-1}, A^{n-1}) \\ \otimes \\ \Gamma(A^d) \end{matrix} \arrow[r, "\alpha"] \arrow[d] & R(A^{n-1}, A^{d+n-1}) \arrow[d] & \\
0 \arrow[r] & \begin{matrix} \Gamma(A^{n-1}) \\ \otimes \\ R(A^{n-1}, A^d) \end{matrix} \arrow[r] \arrow[d, "\beta"] & \begin{matrix} \Gamma(A^{n-1})^{\otimes 2} \\ \otimes \\ \Gamma(A^d) \end{matrix} \arrow[r] \arrow[d] & \begin{matrix} \Gamma(A^{n-1}) \\ \otimes \\ \Gamma(A^{d+n-1}) \end{matrix} \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & R(A^{2n-2}, A^d) \arrow[r] & \begin{matrix} \Gamma(A^{2n-2}) \\ \otimes \\ \Gamma(A^d) \end{matrix} \arrow[r] \arrow[d] & \Gamma(A^{d+2n-2}) \arrow[r] \arrow[d] & 0 \\
 & & 0 & 0 &
\end{tikzcd}
$$

If $\alpha$ is surjective, then $\beta$ is also surjective. The vanishing (3.9) implies the surjectivity of

$$R(A^{n-1}, A^{n-1}) \otimes \Gamma(A^{n-1}) \to R(A^{n-1}, A^{2n-2}).$$

Since $\Gamma(A^i) \otimes \Gamma(A^{n-1}) \to \Gamma(A^{n+i-1})$ is surjective for $i \geq 1$, the map $\alpha$ is surjective, hence,

$$\Gamma(A^d) \otimes R(A^{n-1}, A^d) \to R(A^{d+n-1}, A^d)$$

is surjective. This map is the same as

$$\Gamma(L) \otimes \Gamma(M_L \otimes A^{n-1}) \to \Gamma(M_L \otimes L \otimes A^{n-1})$$

with $L = A^d$. Thus we obtain that $H^1(X, M_L^{\otimes 2} \otimes A^{n-1}) = 0$.

For $q \geq 2$ and $L = A^d$ with $d \geq n + q - 2$, if we have the equalities (3.2) and (3.3), then from the exact sequence

$$0 \to M_L^{\otimes(q+1)} \otimes A^j \to \Gamma(L) \otimes M_L^{\otimes q} \otimes A^j \to M_L^{\otimes q} \otimes L \otimes A^j \to 0 \tag{3.10}$$

we have

$$H^i(X, M_L^{\otimes(q+1)} \otimes A^j) = 0 \quad \text{for } i \geq 2 \text{ and } j \geq 0,$$

and from Lemma 3.4 we see the surjectivity of

$$\Gamma(M_L^{\otimes q} \otimes A^j) \otimes \Gamma(A) \to \Gamma(M_L^{\otimes q} \otimes A^{j+1})$$

for $j \geq n+q-2$. From this and the exact sequence (3.10) we have

$$H^1(X, M_L^{\otimes(q+1)} \otimes A^j) = 0 \quad \text{for } j \geq n+q-2.$$

In particular, we have

$$H^1(X, M_L^{\otimes(q+1)} \otimes L^{\otimes b}) = 0$$

for $b \geq 1$ and $L = A^{\otimes d}$ with $d \geq n+q-2$. □

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