Math. J. Okayama Univ. **48** (2006), 33–42

ON REGULAR RINGS SATISFYING WEAK CHAIN CONDITION

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Abstract. In this paper, we shall study regular rings satisfying weak chain condition. As main results, we show that regular rings satisfying weak chain condition are unit-regular, and show that these rings have the unperforation and power cancellation properties for the family of finitely generated projective modules.

1. Introduction

There is an important problem for studying regular rings: When are directly finite regular rings unit-regular? For some comments on the history of the above problem, we can refer [7]. We notice that not all directly finite regular rings are unit-regular, as Goodearl's book [3, Example 5.10] and Ara et al. [2, Example 3.2] showed. On the other hand, Open Problem 3 in Goodearl's book [3] asks if a directly finite simple regular ring must be unit-regular. For the problem, O'Meara [7] gave the notion of weak comparability and proved that directly finite simple regular rings satisfying weak comparability are unit-regular.

In this paper, we treat a problem concerning when a directly finite regular rings is unit-regular. As a notion related with the idea of weak comparability, we newly define the notion of weak chain condition for regular rings, that is, a regular ring R satisfies weak chain condition if R cannot contain a chain $J_1 \geq J_2 \geq \cdots$ of nonzero principal right ideals such that $nJ_n \lesssim R_R$ for all positive integers n. And, we shall investigate properties for regular rings satisfying weak chain condition and give some interesting results for these rings, as follows.

First, we show that regular rings satisfying weak chain condition are unit-regular (Theorem 6). Next, we give the result that these rings have the unperforation (resp. the power cancellation) property for the family of finitely generated projective modules, i.e., if $nA \lesssim nB$ (resp. $nA \cong nB$) for some positive integer n and some finitely generated projective R-modules A and B, then $A \lesssim B$ (resp. $A \cong B$) (Theorem 10). We also prove that the property of weak chain condition for regular rings is Morita invariant (Theorem 13). Finally, we show that regular rings satisfying weak chain condition

Mathematics Subject Classification. Primary 16E50; Secondary 16D70.

Key words and phrases. Von Neumann regular rings, Weak chain condition, Unit-regularity, Unperforation property.

have the property that nP is directly finite for any positive integer n and any directly finite projective R-module P (Theorem 14).

Throughout this paper, R is a ring with identity and R-modules are unitary right R-modules. We begin with some notations and definitions.

Notation. For two R-modules M and N, we use $M \lesssim N$ (resp. $M \lesssim_\oplus N$) to mean that there exists an isomorphism from M to a submodule of N (resp. a direct summand of N). For a submodule M of an R-module N, $M \leq_\oplus N$ means that M is a direct summand of N. For a cardinal number k and an R-module M, kM denotes the direct sum of k-copies of M.

Definition. An R-module M is *directly finite* provided that M is not isomorphic to a proper direct summand of itself. If M is not directly finite, then M is said to be *directly infinite*. Note that every direct summand of a directly finite module is directly finite, and that every directly infinite module contains an infinite direct sum of nonzero pairwise isomorphic submodules ([3, Corollary 5.6]). A ring R is *directly finite* (resp. *directly infinite*) if the R-module R_R is directly finite (resp. directly infinite). It is well-known from [3, Lemma 5.1] that an R-module M is directly finite if and only if so is $End_R(M)$. A ring R is said to be (von Neumann) *regular* if for each $x \in R$, there exists an element y of R such that $xyx = x$, and R is said to be *unit-regular* if for each $x \in R$, there exists a unit element (i.e. an invertible element) u of R such that $xux = x$. It is well-known that a regular ring R is unit-regular if and only if $R_R = A \oplus B = A' \oplus C$ with $A \cong A'$ implies $B \cong C$ ([3, Theorem 4.1]).

We shall recall well-known elementary properties for regular rings and unit-regular rings:
(1) $End_R(P)$ is a regular ring for each finitely generated projective module P over a regular ring ([3, Theorem 1.7]).
(2) Let R be a regular ring, and let P be a projective R-module. Then

(a) Every finitely generated submodules of P is a direct summand of P ([3, Theorem 1.11]).

(b) P is a direct sum of cyclic submodules, each of which is isomorphic to a principal right ideal of R.

(c) P satisfies the exchange property, where an R-module M *satisfies the exchange property* if for every R-module A and any decompositions $A = M' \oplus N = \oplus_{i \in I} A_i$ with $M' \cong M$, there exist submodules $A_i' \leq A_i$ such that $A = M' \oplus (\oplus_{i \in I} A_i')$.

(3) Let R be a unit-regular ring, and let A be a finitely generated projective R-module. If B and C are any right R-modules such that $A \oplus B \cong A \oplus C$, then $B \cong C$ ([3, Theorem 4.14]). Therefore any finitely generated projective R-module is directly finite.

All basic results concerning regular rings can be found in Goodearl's book [3].

2. Regular rings satisfying weak chain condition

We give a new definition as follows.

Definition. A regular ring R satisfies *weak chain condition* if R cannot contain a chain $J_1 \geq J_2 \geq \cdots$ of nonzero principal right ideals such that $nJ_n \lesssim R_R$ for all positive integers n.

We recall the following well-known result.

Lemma 1. *Let R be a ring, and let e, f be idempotents in R. Then the following conditions are equivalent*:

(a) $eR_R \cong fR_R$.

(b) *There exist $x \in eRf$ and $y \in fRe$ such that $xy = e$ and $yx = f$.*

(c) ${}_RRe \cong {}_RRf$.

Moreover, assume that R is a regular ring. Then the following conditions are equivalent:

(a') $eR_R \lesssim fR_R$.

(b') *There exist $x \in eRf$ and $y \in fRe$ such that $xy = e$.*

(c') ${}_RRe \lesssim {}_RRf$.

By Lemma 1, we see that the notion of weak comparability for regular rings is right-left symmetric. We also notice that every regular ring R satisfying weak chain condition is directly finite. Because, if R is directly infinite, then there exists a nonzero principal right ideal X of R such that $\aleph_0 X \lesssim R_R$. Put $J_n = X$ for each positive integer n, and hence $nJ_n \lesssim R_R$, which contradicts the assumption of weak chain condition for R. In particular, we see that every simple regular ring satisfying weak chain condition is artinian, as Lemma 2 below shows.

Lemma 2 ([1, Lemma 1.1])**.** *Let R be a non-artinian simple regular ring. Then, for each nonzero finitely generated projective R-module P and for all positive integers k, there exists a nonzero finitely generated projective R-module Q such that $kQ \lesssim P$.*

We shall investigate properties for regular rings satisfying weak chain condition. By Lemma 1, we obtain the following lemma.

Lemma 3. *Let $\{R_i\}_{i \in I}$ be a family of regular rings, and set $R = \prod_{i \in I} R_i$. Let $e = (e_i), f = (f_i)$ be idempotents of R. Then*

(1) *e and f are orthogonal if and only if so are e_i and f_i for all $i \in I$.*

(2) *$eR \cong fR$ as an R-module if and only if $e_iR_i \cong f_iR_i$ as an R_i-module for all $i \in I$.*

(3) *$eR \lesssim fR$ as an R-module if and only if $e_iR_i \lesssim f_iR_i$ as an R_i-module for all $i \in I$.*

From Lemma 3 and the definition of weak chain condition, we can easily prove the following proposition.

Proposition 4. *The class of regular rings satisfying weak chain condition is closed under finite direct products.*

Remark 1. We note that infinite direct products of regular rings satisfying weak chain condition do not satisfy weak chain condition in general. For example, take a field F, and for each positive integer n, let $R_n = M_{n!}(F)$ with the standard $n! \times n!$ matrix units e_{ij} (for $i, j = 1, \cdots n!$). Then R_n is simple artinian, whence R_n satisfies weak chain condition, and $X_n(s) := e_{1,1}R_n + \cdots + e_{n!/s!,1}R_n$ $(1 \leq s \leq n)$ is a principal right ideal of R_n. Put $R = \prod_{n=1}^{\infty} R_n$, and set $J_i = (\prod_{k=1}^{i-1} 0) \times (\prod_{k=i}^{\infty} X_k(i))$ for each positive integer i. Then R contains a chain $J_1 \geq J_2 \geq \cdots$ of nonzero principal right ideals such that $nJ_n \lesssim R$ for all positive integers n. Therefore R does not satisfy weak chain condition, as desired.

We show that every regular ring satisfying weak chain condition is unit-regular. To see this, we need the following lemma.

Lemma 5. *Let R be a regular ring, and let $A \oplus C' \leq_{\oplus} B \oplus C$ with an isomorphism f from C to C' for some finitely generated projective R-modules B and C. Then there exist decompositions $B = B_1 \oplus B_1^*$ and $B_n^* = B_{n+1} \oplus B_{n+1}^* (n \geq 1); C = C_1 \oplus C_1^*$ and $C_n^* = C_{n+1} \oplus C_{n+1}^* (n \geq 1)$ such that $A \cong B_1 \oplus C_1$, $C_n \cong B_{n+1} \oplus C_{n+1} (n \geq 1)$ and $B \oplus C = A \oplus fC_1 \oplus \cdots \oplus fC_n \oplus B_{n+1}^* \oplus C_{n+1}^*$ for each positive integer n.*

Proof. Using the exchange property for A, there exist decompositions $B = B_1 \oplus B_1^*$ and $C = C_1 \oplus C_1^*$ such that $B \oplus C = A \oplus B_1^* \oplus C_1^*$, and hence $A \cong B_1 \oplus C_1$. Since $A \oplus fC_1 \leq_{\oplus} B \oplus C = A \oplus B_1^* \oplus C_1^*$, there exist decompositions $B_1^* = B_2 \oplus B_2^*$ and $C_1^* = C_2 \oplus C_2^*$ such that $B \oplus C = A \oplus fC_1 \oplus B_2^* \oplus C_2^*$, and hence $C_1 \cong fC_1 \cong B_2 \oplus C_2$. Note that $C_1 \cap C_2 = 0$, and so we have that $A \oplus fC_1 \oplus fC_2 \leq_{\oplus} B \oplus C = A \oplus fC_1 \oplus B_2^* \oplus C_2^*$. Then there exist decompositions $B_2^* = B_3 \oplus B_3^*$ and $C_2^* = C_3 \oplus C_3^*$ such that $B \oplus C = A \oplus fC_1 \oplus fC_2 \oplus B_3^* \oplus C_3^*$, and hence $C_2 \cong fC_2 \cong B_3 \oplus C_3$. Continuing the above procedure, we have decompositions $B_n^* = B_{n+1} \oplus B_{n+1}^*$ and $C_n^* = C_{n+1} \oplus C_{n+1}^*$ such that $B \oplus C = A \oplus fC_1 \oplus \cdots \oplus fC_n \oplus B_{n+1}^* \oplus C_{n+1}^*$ and $C_n \cong B_{n+1} \oplus C_{n+1}$. The proof is complete. □

Theorem 6. *Every regular ring satisfying weak chain condition is unit-regular.*

Proof. Let $R_R = A \oplus C' = B \oplus C$ with $C' \cong C$, and let f be an isomorphism from C to C'. We claim that $A \cong B$. We may assume with no loss of generality that $A \neq 0$, by the direct finiteness of R. Since $A \oplus C' \leq_{\oplus} B \oplus C$, using Lemma 5, there exist decompositions $B = B_1 \oplus B_1^*$ and $B_n^* = B_{n+1} \oplus B_{n+1}^* (n \geq 1)$; $C = C_1 \oplus C_1^*$ and $C_n^* = C_{n+1} \oplus C_{n+1}^* (n \geq 1)$ such that $A \cong B_1 \oplus C_1$ and $C_n \cong B_{n+1} \oplus C_{n+1}$ for each positive integer n. Hence R has a sequence $C_1, C_2, \cdots$ of principal right ideals such that $C_{n+1} \lesssim C_n$ and $nC_n \lesssim C \leq R_R$ for all positive integers n. Since R satisfies weak chain condition, there exists a positive integer m such that $C_m = 0$. Then we have that $A \cong B_1 \oplus C_1 \cong B_1 \oplus B_2 \oplus C_2 \cong \cdots \cong B_1 \oplus \cdots \oplus B_{m-1} \oplus C_{m-1} \cong B_1 \oplus \cdots \oplus B_m \leq B$, whence $A \lesssim B$. Similarly, we have $B \lesssim A$. Hence $A \cong B$ by the direct finiteness of A and B. Therefore R is unit-regular as desired. □

Definition. A regular ring R is said to satisfy that *its primitive factor rings are artinian* if R/P is artinian for all right (or left) primitive ideals P of R, or equivalently, R/P is artinian for all prime ideals P of R ([3, Theorem 6.2]).

From the proof of [3, Theorem 6.6], we see that every regular ring whose primitive factor rings are artinian satisfies weak chain condition. Hence we have the following corollary.

Corollary 7 ([3, Theorem 6.10])**.** *Every regular ring whose primitive factor rings are artinian is unit-regular.*

We also recall the definition of weak comparability for regular rings.

Definition ([7])**.** A regular ring R *satisfies weak comparability* if for each nonzero $x \in R$, there exists a positive integer $n = n(x)$ such that $n(yR) \lesssim R$ implies $yR \lesssim xR$ for all $y \in R$.

Remark 2. Every regular ring whose primitive factor rings are artinian satisfies weak chain condition as above, but it does not satisfy weak comparability in general, using [7, Proposition 2] and [3, Example 6.5]. Also, there exists a directly finite simple regular ring with weak comparability which does not satisfy weak chain condition, by [7, Corollary 2], [3, Example 8.1] and the statement before Lemma 2. Therefore we see that, for a regular ring, weak chain condition does not imply weak comparability, and vice versa.

Next, we shall show that regular rings satisfying weak chain condition have the unperforation property for the family of finitely generated projective modules. To see this, we need the following lemmas.

Lemma 8 ([1, Lemma 3.3])**.** *Let R be a regular ring, and let P, Q be finitely generated projective R-modules with $P \lesssim nQ$ for some positive integer n.*

Then there exists a decomposition $P = P_1 \oplus P_2 \oplus \cdots \oplus P_n$ *such that* $P_n \lesssim \cdots \lesssim P_1 \lesssim Q$.

Lemma 9. *Let* R *be a unit-regular ring, and let* A, B *be finitely generated projective* R*-modules such that* $nA \lesssim nB$ *for some positive integer* n (≥ 2). *Assume that* $kC \lesssim kD$ *implies* $C \lesssim D$ *for any positive integer* k $(< n)$ *and any finitely generated projective* R*-modules* C *and* D. *Then we have decompositions* $\bar{\bar{A}}_i = \bar{A}_{i+1} \oplus \bar{\bar{A}}_{i+1}$ *and* $\bar{\bar{B}}_i = \bar{B}_{i+1} \oplus \bar{\bar{B}}_{i+1}$ *such that* $\bar{A}_{i+1} \cong \bar{B}_{i+1}, 2\bar{\bar{A}}_{i+1} \lesssim \bar{\bar{A}}_i, \bar{\bar{A}}_{i+1} \lesssim (n-1)\bar{A}_{i+1}$ *and* $n\bar{\bar{A}}_{i+1} \lesssim n\bar{\bar{B}}_{i+1}$ *for each* $i = 0, 1, 2, \cdots$, *where* $\bar{\bar{A}}_0 = A$ *and* $\bar{\bar{B}}_0 = B$.

Proof. Let A, B be finitely generated projective R-modules such that $nA \lesssim nB$ for some positive integer n (≥ 2). First, we claim that there exist decompositions $A = \bar{\bar{A}}_0 = \bar{A}_1 \oplus \bar{\bar{A}}_1$ and $B = \bar{\bar{B}}_0 = \bar{B}_1 \oplus \bar{\bar{B}}_1$ such that $\bar{A}_1 \cong \bar{B}_1, 2\bar{\bar{A}}_1 \lesssim \bar{\bar{A}}_0, \bar{\bar{A}}_1 \lesssim (n-1)\bar{A}_1$ and $n\bar{\bar{A}}_1 \lesssim n\bar{\bar{B}}_1$. Since $A \lesssim nB$, we have a decomposition $A = A_{11} \oplus \cdots \oplus A_{1n}$ such that $A_{1n} \lesssim \cdots \lesssim A_{11} \lesssim B$ by Lemma 8. Setting that $A_1^* = A_{11}$ and $A_1^{**} = A_{12} \oplus \cdots \oplus A_{1n}$, we have $A = A_1^* \oplus A_1^{**}$. Noting that $A_1^* \lesssim_{\oplus} B$, we have a decomposition $B = B_1^* \oplus B_1^{**}$ such that $A_1^* \cong B_1^*, A_1^{**} \lesssim (n-1)A_1^*$ and $nA_1^{**} \lesssim nB_1^{**}$, because note that $nA \lesssim nB$ and nA_1^* is finitely generated projective, and hence $nA_1^{**} \lesssim nB_1^{**}$ using Theorem 6. Next, since $nA_1^{**} \lesssim nB_1^{**}$, we have that $A_1^{**} \lesssim nB_1^{**}$, and hence there exists a decomposition $A_1^{**} = A_{21} \oplus \cdots \oplus A_{2n}$ such that $A_{2n} \lesssim \cdots \lesssim A_{21} \lesssim B_1^{**}$. Setting that $A_2^* = A_{21}$ and $A_2^{**} = A_{22} \oplus \cdots \oplus A_{2n}$, we have that $A_1^{**} = A_2^* \oplus A_2^{**}$ and $A_2^* \lesssim_{\oplus} B_1^{**}$. Then we have a decomposition $B_1^{**} = B_2^* \oplus B_2^{**}$ such that $A_2^* \cong B_2^*, A_2^{**} \lesssim (n-1)A_2^*$ and $nA_2^{**} \lesssim nB_2^{**}$. Continuing the above procedure $(n-2)$ times, we have decompositions $A_{n-1}^{**} = A_n^* \oplus A_n^{**}$ and $B_{n-1}^{**} = B_n^* \oplus B_n^{**}$ such that $A_n^* \cong B_n^*, A_n^{**} \lesssim (n-1)A_n^*$ and $nA_n^{**} \lesssim nB_n^{**}$, where $A_n^* = A_{n1}$ and $A_n^{**} = A_{n2} \oplus \cdots \oplus A_{nn}$. Now we put $\bar{A}_1 = A_1^* \oplus \cdots \oplus A_n^*, \bar{\bar{A}}_1 = A_n^{**}, \bar{B}_1 = B_1^* \oplus \cdots \oplus B_n^*$ and $\bar{\bar{B}}_1 = B_n^{**}$. Then we see that $A = \bar{A}_1 \oplus \bar{\bar{A}}_1$ and $B = \bar{B}_1 \oplus \bar{\bar{B}}_1$ such that $\bar{A}_1 \cong \bar{B}_1$, $\bar{\bar{A}}_1 = A_n^{**} \lesssim (n-1)A_n^* \lesssim (n-1)\bar{A}_1$ and $n\bar{\bar{A}}_1 \lesssim n\bar{\bar{B}}_1$. Also, we have that $2\bar{\bar{A}}_1 \lesssim A$. To see this, we notice that $(n-1)\bar{\bar{A}}_1 = (n-1)A_n^{**} \lesssim A_1^{**} \oplus \cdots \oplus A_{n-1}^{**} \lesssim (n-1)(A_1^* \oplus \cdots \oplus A_{n-1}^*)$. Using the assumption for $k = n-1$, we have that $\bar{\bar{A}}_1 = A_n^{**} \lesssim A_1^* \oplus \cdots \oplus A_{n-1}^*$, and so $2\bar{\bar{A}}_1 \cong \bar{\bar{A}}_1 \oplus \bar{\bar{A}}_1 \lesssim (A_1^* \oplus \cdots \oplus A_{n-1}^*) \oplus \bar{\bar{A}}_1 \lesssim \bar{A}_1 \oplus \bar{\bar{A}}_1 = \bar{\bar{A}}_0 = A$. Therefore the first claim is proved.

Secondly, noting that $n\bar{\bar{A}}_1 \lesssim n\bar{\bar{B}}_1$, from the above claim, we have decompositions $\bar{\bar{A}}_1 = \bar{A}_2 \oplus \bar{\bar{A}}_2$ and $\bar{\bar{B}}_1 = \bar{B}_2 \oplus \bar{\bar{B}}_2$ such that $\bar{A}_2 \cong \bar{B}_2, 2\bar{\bar{A}}_2 \lesssim \bar{\bar{A}}_1$, $\bar{\bar{A}}_2 \lesssim (n-1)\bar{A}_2$ and $n\bar{\bar{A}}_2 \lesssim n\bar{\bar{B}}_2$. Continuing the above procedure, we have desired decompositions. The proof is complete. $\square$

Using Theorem 6 and Lemma 9, we can prove that regular rings satisfying weak chain condition have the unperforation property and the power cancellation property for the family of finitely generated projective modules, as follows.

Theorem 10. *Let R be a regular ring satisfying weak chain condition, and let A, B be finitely generated projective R-modules.*

(1) *If $nA \lesssim nB$ for some positive integer n, then $A \lesssim B$.*

(2) *If $nA \cong nB$ for some positive integer n, then $A \cong B$.*

Proof. (1) We shall prove the result using induction on n. Let n (≥ 2) be a positive integer, and let A, B be finitely generated projective R-modules such that $nA \lesssim nB$. Then we may assume with no loss of generality that A is nonzero cyclic. Because, let $A = A_1 \oplus \cdots \oplus A_m$ be a cyclic decomposition of A. Since $nA \lesssim nB$, we have that $nA_1 \lesssim nB$. Noting that A_1 is cyclic projective, we see that $A_1 \lesssim B$ by the assumption, and so there exists a decomposition $B = B_1 \oplus B_1^*$ such that $A_1 \cong B_1$. Since $n(A_1 \oplus \cdots \oplus A_m) \lesssim n(B_1 \oplus B_1^*)$ and $nA_1 \cong nB_1$ are finitely generated projective R-modules, we have that $n(A_2 \oplus \cdots \oplus A_m) \lesssim nB_1^*$ by Theorem 6. Continuing the above procedure, there exists a decomposition $B_1^* = B_2 \oplus B_2^*$ such that $A_2 \cong B_2$ and $n(A_3 \oplus \cdots \oplus A_m) \lesssim nB_2^*$. Therefore we have that $nA_m \lesssim nB_{m-1}^*$, and so $A_m \lesssim B_{m-1}^*$ by the assumption. Then we have a decomposition $B_{m-1}^* = B_m \oplus B_m^*$ such that $A_m \cong B_m$. Thus $A = A_1 \oplus \cdots \oplus A_m \cong B_1 \oplus \cdots \oplus B_m \leq B$, as desired.

Since $nA \lesssim nB$, by the induction hypothesis and Lemma 9, there exist decompositions $\bar{\bar{A}}_i = \bar{A}_{i+1} \oplus \bar{\bar{A}}_{i+1}$ and $\bar{\bar{B}}_i = \bar{B}_{i+1} \oplus \bar{\bar{B}}_{i+1}$ such that $\bar{A}_{i+1} \cong \bar{B}_{i+1}$ and $2\bar{\bar{A}}_{i+1} \lesssim \bar{\bar{A}}_i$ for each $i = 0, 1, 2, \cdots$, where $\bar{\bar{A}}_0 = A$ and $\bar{\bar{B}}_0 = B$. Note that $(i+1)\bar{\bar{A}}_{i+1} \lesssim 2^{i+1}\bar{\bar{A}}_{i+1} \lesssim 2^i\bar{\bar{A}}_i \lesssim \cdots \lesssim \bar{\bar{A}}_0 = A$. Hence we have a chain $\bar{\bar{A}}_1 \geq \bar{\bar{A}}_2 \geq \cdots$ of cyclic submodules of A such that $A \lesssim R_R$ and $n\bar{\bar{A}}_n \lesssim A$ for all positive integers n. Since R satisfies weak chain condition, there exists a positive integer k such that $\bar{\bar{A}}_k = 0$. Then $A = \bar{\bar{A}}_0 = \bar{A}_1 \oplus \bar{\bar{A}}_1 = \cdots = \bar{A}_1 \oplus \cdots \oplus \bar{A}_k \cong \bar{B}_1 \oplus \cdots \oplus \bar{B}_k \leq B$. Therefore $A \lesssim B$.

(2) follows from (1) and Theorem 6. $\square$

Remark 3. Goodearl [4] constructed simple unit-regular rings R which do not have the power cancellation property for the family of finitely generated projective R-modules, i.e., there exist finitely generated projective R-modules A, B and a positive integer n such that $nA \cong nB$ and $A \not\cong B$. Hence Theorem 10(2) does not hold for simple unit-regular rings in general, from which Theorem 10(1) also does not hold for these rings in general.

Now we study the endomorphism rings of finitely generated projective modules over regular rings satisfying weak chain condition. For the purpose of this, we need a more general definition of weak chain condition for finitely generated projective modules over regular rings, as follows.

Definition. A finitely generated projective module P over a regular ring satisfies *weak chain condition* if P cannot contain a chain $P_1 \geq P_2 \geq \cdots$ of nonzero finitely generated submodules such that $nP_n \lesssim P$ for all positive integers n. Clearly, weak chain condition for finitely generated projective modules over regular rings is inherited by direct summands. Also, we note that a regular ring R satisfies weak chain condition if and only if so does the R-module R_R.

Proposition 11. *Let R be a regular ring. Then the following conditions* (a) *through to* (d) *are equivalent*:

(a) *R satisfies weak chain condition.*

(b) *Every finitely generated projective R-module satisfies weak chain condition.*

(c) *nR satisfies weak chain condition for all positive integers n.*

(d) *nR satisfies weak chain condition for some positive integer n.*

Proof. (b) $\Rightarrow$ (d) is obvious.

(c) $\Rightarrow$ (b) and (d) $\Rightarrow$ (a) follow from the definition of weak chain condition.

(a) $\Rightarrow$ (c). Let $P_1 \geq P_2 \geq \cdots$ be a chain of finitely generated submodules of nR such that $kP_k \lesssim nR_R$ for all positive integers k. Then, we see from Theorem 10 that $mnP_{mn} \lesssim nR_R$ implies $mP_{mn} \lesssim R_R$ for each positive integer m. Also, we notice that $P_{mn} \lesssim R_R$. Since R satisfies weak chain condition, we have that $P_{mn} = 0$ for some positive integer m. Therefore nR satisfies weak chain condition. The proof is complete. $\square$

For an R-module M_R, we put $add(M_R) = \{$an R-module $N \mid N \lesssim_{\oplus} nM$ for some positive integer $n\}$. Then, the following lemma follows from equivalences of the Hom and Tensor functors by $Hom_R({}_SM_R, -)$ and $- \otimes_S {}_SM_R$ between the categories $add(M_R)$ and $add(S_S)$, where $S = End_R(M)$ (see [8, 46.7]).

Lemma 12. *Let M be a finitely generated projective R-module over a regular ring R, and set $S = End_R(M)$. Hence M is an (S,R)-bimodule, and M is flat as a left S-module. Then M satisfies weak chain condition if and only if so does S as an S-module.*

Using Proposition 11 and Lemma 12, we see that the property of weak chain condition for regular rings is inherited by matrix rings, as follows.

Theorem 13. *Let R be a regular ring. Then the following conditions* (a) *through to* (e) *are equivalent*:

(a) *R satisfies weak chain condition.*

(b) *For each finitely generated projective R-module P, $End_R(P)$ satisfies weak chain condition.*

(c) *Every ring S which is Morita equivalent to R satisfies weak chain condition.*

(d) *For all positive integers n, $M_n(R)$ satisfies weak chain condition.*

(e) *There exists a positive integer n such that $M_n(R)$ satisfies weak chain condition.*

Finally, we give a theorem about the direct finiteness of projective modules over regular rings satisfying weak chain condition, as follows.

Theorem 14. *Let R be a regular ring satisfying weak chain condition. Then*

(1) *For a projective R-module P, the following conditions* (a) *through to* (c) *are equivalent*:

(a) *P is directly infinite.*

(b) *There exists a nonzero R-module X such that $\aleph_0 X \lesssim P$.*

(c) *There exists a nonzero R-module X such that $\aleph_0 X \lesssim_{\oplus} P$.*

(2) *If P is a directly finite projective R-module, then so is nP for each positive integer n.*

Proof. (1) follows from a similar proof of one of [6, Theorem 1.3], using Theorem 6 and Proposition 11.

(2) follows from [5, Theorem 1.5], using Theorems 6 and 10. □

From Theorem 14(1), we have the following corollary.

Corollary 15. *Let R be a regular ring satisfying weak chain condition, and let P be a projective R-module. Then P is directly finite if and only if all submodules of P are directly finite.*

Acknowledgement

The author exresses his gratitude to the referee for useful comments on the first version of this paper.

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(*Received May 6, 2005*)