MUTUALLY ORTHOGONAL LATIN SQUARES
AND
SELF-COMPLEMENTARY DESIGNS

HIROYUKI NAKASORA

Abstract. Suppose that \( n \) is even and a set of \( \frac{n^2}{2} - 1 \) mutually orthogonal Latin squares of order \( n \) exists. Then we can construct a strongly regular graph with parameters \((n^2, \frac{n^2}{2}(n-1), \frac{n^2}{2}(\frac{n^2}{2} - 1))\), which is called a Latin square graph. In this paper, we give a sufficient condition of the Latin square graph for the existence of a projective plane of order \( n \). For the existence of a Latin square graph under the condition, we will introduce and consider a self-complementary 2-design (allowing repeated blocks) with parameters \((n, \frac{n}{2}, \frac{n}{2}(\frac{n}{2} - 1))\). For \( n \equiv 2 \pmod{4} \), we give a proof of the non-existence of the design.

1. Introduction

A Latin square of order \( n \) is an \( n \times n \) array with entries 1, \ldots, \( n \) having the property that each element of \( \{1, \ldots, n\} \) occurs exactly once in each row and column. Two Latin squares \( A = (a_{ij}) \), \( B = (b_{ij}) \) of order \( n \) are said to be orthogonal if, for any \( x, y \in \{1, \ldots, n\} \), there exists a unique position \((i, j)\) such that \( a_{ij} = x \) and \( b_{ij} = y \). Latin squares are said to be mutually orthogonal if every two of them are orthogonal. Let \( N(n) \) denote the maximum number of mutually orthogonal Latin squares of order \( n \) \( (n \geq 2) \).

The value of \( N(n) \) has been studied by many mathematicians, and the following three theorems are well-known.

Theorem 1.1. \( N(6) = 1 \). If \( n \neq 2, 6 \), then \( N(n) \geq 2 \).

Theorem 1.2. \( N(n) \leq n - 1 \), with equality if and only if there exists a projective plane of order \( n \).

Theorem 1.3. \( N(n) = n - 1 \), if \( n \) is a prime power number.

In 1900, Tarry showed \( N(6) = 1 \) by a systematic enumeration. Also in 1984, Stinson [9] gave a short proof of the fact. In 1960, Bose, Shikhande and Parker [3] proved \( N(n) \geq 2 \) for all \( n > 6 \), demolishing Euler’s conjecture. Theorem 1.1 is obtained from their results.
The Bruck-Ryser-Chowla theorem shows that if a projective plane of order \( n \equiv 1 \) or \( 2 \) \( \pmod{4} \) exists, then \( n \) is the sum of two squares. As noted above, this theorem does not preclude the existence of a projective plane of order 10. In 1989, the non-existence of such a plane was shown by Lam, Swiercz and Thiel [8].

If \( n \) is not a prime power number, then there is no known example of a projective plane of order \( n \). We consider the existence of a projective plane of order non-prime power number. We use the following theorem, (see Bose and Shrikhande [2], Cameron and Lint [6, Chapter 7 and 8]).

**Theorem 1.4.** The existence of \( k - 2 \) mutually orthogonal Latin squares of order \( n \) is equivalent to the existence of:

1. a transversal 2-design of order \( n \), block size \( k \), namely a \( TD(k, n) \),
2. a Latin square graph, namely an \( L_k(n) \)-graph.

In this paper, we give a sufficient condition of the Latin square graph for the existence of a projective plane of order \( n \). If \( n \) is an even integer, we show that a \( 2-(n, \frac{n}{2}, \frac{n}{2}(\frac{n}{2} - 1)) \) design \( D \) (allowing repeated blocks) such that \( D \cong \bar{D} \) is obtained from the Latin square graph under the condition, where \( \bar{D} \) denotes the complementary design of \( D \) and \( D \cong \bar{D} \) means that two designs \( D, \bar{D} \) are isomorphic (Theorem 4.7).

As a special case, we consider the existence of a self-complementary 2-design \( (D = \bar{D}) \) with parameters \( (n, \frac{n}{2}, \frac{n}{2}(\frac{n}{2} - 1)) \). In the case \( n \equiv 0 \) \( \pmod{4} \), an \( \frac{n}{2} \)-repeated design of a Hadamard \( 3-(n, \frac{n}{2}, \frac{n}{4} - 1) \) design is an example of a self-complementary design. If \( n \equiv 2 \) \( \pmod{4} \), there exists no self-complementary design.

### 2. Transversal 2-designs

**Definition 2.1.** Let \( k \geq 2, n \geq 1 \). A transversal 2-design of order \( n \), block size \( k \), is a triple \( (X, \mathcal{G}, \mathcal{B}) \) satisfying the following three conditions, and is denoted by \( TD(k, n) \).

1. \( X \) is a set of \( kn \) points.
2. \( \mathcal{G} = \{G_1, G_2, \ldots, G_k\} \) is a partition of \( X \) into \( k \) subsets \( G_i \) (called groups), each containing \( n \) points.
3. \( \mathcal{B} \) is a class of subsets of \( X \) (called blocks) such that each block \( B \in \mathcal{B} \) contains precisely one point from each group and each pair \( x, y \) of points not contained in the same group occur together in precisely one block \( B \).

**Proposition 2.2.** Let \( (X, \mathcal{G}, \mathcal{B}) \) be a \( TD(k, n) \). Then the followings hold.

1. Each block contains \( k \) points.
2. Each point occurs in \( n \) blocks.
(3) For any \( B, B' \in \mathcal{B} \ (B \neq B') \), \(|B \cap B'| = 0 \) or 1.

(4) \(|\mathcal{B}| = n^2\).

The following theorem is due to Bose and Shirikhande [2] (also see R. M. Wilson [12]). By this theorem, we have \( 2 \leq k \leq n + 1 \).

**Theorem 2.3.** (Bose-Shrikhande) The existence of a set of \( k - 2 \) mutually orthogonal Latin squares of order \( n \) is equivalent to the existence of a \( TD(k,n) \).

Now, we will make preparations for the normalized incidence matrix of a \( TD(k,n) \). At first we give a normalized Latin square.

Let \( A = (a_{ij}) \) be a Latin square of order \( n \), and set \( \Omega = \{1,2,\ldots,n\} \).

Take a bijection \( \sigma : \Omega \to \Omega \), and define \( \sigma(a_{ii}) = i \), for \( i = 1,2,\ldots,n \). Then,

\[
\sigma(A) = \begin{pmatrix}
1 & 2 & \cdots & n \\
\vdots & \ddots & \vdots & \vdots \\
\vdots & \cdots & \ddots & \vdots \\
\end{pmatrix}.
\]

**Lemma 2.4.** Let \( A \) and \( B \) be mutually orthogonal Latin squares of order \( n \). For any permutations \( \sigma, \tau \) on \( \Omega \), \( \sigma(A) \) and \( \tau(B) \) also are orthogonal.

By (2.1) and Lemma 2.4, we can put the first rows of mutually orthogonal Latin squares the integers 1,2,\ldots,n.

**Definition 2.5.** Let \((X,G,\mathcal{B})\) be a \( TD(k,n) \) with

\( X = \{x_1, x_2, \ldots, x_{kn}\} \), \( \mathcal{B} = \{B_1, B_2, \ldots, B_{n^2}\} \).

The incidence matrix of a \( TD(k,n) \) is the \( n^2 \times kn \) matrix \( A = (a_{ij}) \) defined by

\[
a_{ij} = \begin{cases}
1 & \text{if } x_i \in B_j \\
0 & \text{if } x_i \notin B_j.
\end{cases}
\]

Then we have the following proposition.

**Proposition 2.6.** The incidence matrix of a \( TD(k,n) \) can be normalized as

\[
\begin{pmatrix}
H_1 & I & I & \cdots & I \\
H_2 & I & & & \cdots \\
\vdots & & \ddots & & \vdots \\
H_n & I & & & \ddots
\end{pmatrix},
\]

where \( I \) is the identity matrix of size \( n \), and \( H_i \ (1 \leq i \leq n) \) is an \( n \times n \) matrix with every entry 1 of \( i \) th column, otherwise 0.
Example 2.7. The following pair \((A, B)\) is an example of the pair of mutually orthogonal Latin squares of order 3:

\[
A = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \\ 3 & 1 & 2 \end{pmatrix}, \quad B = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \\ 2 & 3 & 1 \end{pmatrix}.
\]

The incidence matrix of the corresponding \(TD(4, 3)\) is given by

\[
\begin{pmatrix}
1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 \\
1 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 \\
1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 \\
0 & 1 & 0 & 1 & 0 & 0 & 0 & 1 & 0 \\
0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 \\
0 & 1 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \\
0 & 0 & 1 & 1 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 1 \\
0 & 0 & 1 & 0 & 0 & 1 & 0 & 1 & 0 \\
\end{pmatrix}.
\]

3. Latin Square Graphs

Definition 3.1. Let \((X, \mathcal{G}, \mathcal{B})\) be a \(TD(k, n)\). The Latin square graph \(\Gamma = (V, E)\) is defined as follows and is denoted by an \(L_k(n)\)-graph.

1. \(V = \mathcal{B}\).
2. Two vertices \(B, B' \in \mathcal{B}\) are adjacent if and only if \(|B \cap B'| = 1\).

The following proposition is well-known, (see Cameron and Lint [6, Chapter 8]).

Proposition 3.2. Let \(\Gamma\) be an \(L_k(n)\)-graph. Then

(a) If \(n + 1 > k \geq 2\), then \(\Gamma\) is a strongly regular graph with parameters \((n^2, (n-1)k, n + k(k-3), k(k-1))\);

(b) If \(k = n + 1\), then \(\Gamma\) is isomorphic to \(K_{n^2}\), where \(K_{n^2}\) is a complete graph with \(n^2\) vertices.

Definition 3.3. For \(n+1 > k \geq 1\), a pseudo Latin square graph is a strongly regular graph with parameters \((n^2, (n-1)k, n + k(k-3), k(k-1))\). Such a graph is denoted by a \(PL_k(n)\)-graph.

It is well-known that the complement of a strongly regular graph is strongly regular (see Cameron and Lint [6, Chapter 2]). Therefore, we have the following proposition.

Proposition 3.4. The complement of a \(PL_k(n)\)-graph is a \(PL_{n+1-k}(n)\)-graph.
Clearly, an $L_k(n)$-graph is a $PL_k(n)$-graph. However the converse does not hold. We will give a criterion whether a $PL_k(n)$-graph is an $L_k(n)$-graph or not. Let $\Gamma$ be a $PL_k(n)$-graph. By the definition of the Latin square graph, we can easily see that if $\Gamma$ is an $L_k(n)$-graph, then every edge is contained in a clique of size $n$, where a clique is an induced complete subgraph with $n$ vertices and is denoted by $C_n$. The following lemma is due to Bruck [4].

Lemma 3.5. (Bruck) Let $\Gamma$ be a $PL_k(n)$-graph, and $(n-1)k \leq \frac{n^2}{2}$. Then $\Gamma$ is an $L_k(n)$-graph if and only if every edge is contained in a unique clique of size $n$.

Example 3.6. Let $\Gamma$ be the Hall-Janko graph such that $\text{Aut } \Gamma = \text{Aut } J_2$. Then $\Gamma$ and the complementary graph $\bar{\Gamma}$ are pseudo Latin square graphs (a $PL_4(10)$-graph and a $PL_7(10)$-graph) with parameters $(100, 36, 14, 12)$ and $(100, 63, 38, 42)$, respectively. In 1968, M. Suzuki [10] stated that $\Gamma$ and $\bar{\Gamma}$ are not Latin square graphs. Here, we will give a simple proof.

Claim 1. $\Gamma$ is not an $L_4(10)$-graph.

Proof. Let $\infty$ be a vertex of $\Gamma$. Set $V(\Gamma) = \{\infty\} \cup X \cup Y$,

$$X = \{x \in V(\Gamma) : (\infty, x) \in E(\Gamma)\},$$

$$Y = \{y \in V(\Gamma) : (\infty, y) \notin E(\Gamma)\},$$

where $V(\Gamma)$ is the vertex set of $\Gamma$ and $E(\Gamma)$ is the edge set of $\Gamma$.

Suppose that $\Gamma$ is an $L_4(10)$-graph. Then, for any $(a, b) \in E(\Gamma)$, $(a, b) \in C_{10}$. Therefore

\begin{equation}
X \supset C_9.
\end{equation}

(3.1)

Here, we use a construction of the Hall-Janko graph. The following chain of groups is called the Suzuki chain. These groups are the full automorphism groups of strongly regular graphs.

$$S_4 \subset PGL(2, 7) \subset G_2(2) \subset \text{Aut } J_2 \subset \text{Aut } G_2(4) \subset \text{Aut } Sz$$

It is known that $\text{Aut } X = G_2(2)$ and $X$ is a strongly regular graph with parameters $(n, k, \lambda, \mu) = (36, 14, 4, 6)$. By (3.1), we have $7 \leq \lambda = 4$, a contradiction. \qed

Claim 2. $\bar{\Gamma}$ is not an $L_7(10)$-graph.

Proof. Suppose that $\bar{\Gamma}$ is an $L_7(10)$-graph. Then $\bar{\Gamma}$ must have a pair of cliques $C_{10}, C'_{10}$ such that $|C_{10} \cap C'_{10}| = 1$ (see Bruck [4]). It is known that $|C_{10} \cap C'_{10}| = 0$ or 2, for any distinct cliques $C_{10}, C'_{10}$ (see Chigira-Harada-Kitazume [7]), a contradiction. \qed
Thus, the above claims complete a proof of the fact that $\Gamma$ and $\bar{\Gamma}$ are not Latin square graphs.

**Proposition 3.7.** Suppose that $3 \leq k \leq n+1$. Let $(X, G, B)$ be a $TD(k, n)$. For $1 \leq i \leq k$, define a triple $(X', G', B')$ by

\[
X' = X \setminus G_i \\
G' = \{G_1, G_2, \ldots, G_k\} \setminus G_i \\
B' = \{B \setminus (B \cap G_i) : B \in B\}.
\]

Then $(X', G', B')$ is a $TD(k-1, n)$.

**Proof.** Suppose that $3 \leq k \leq n+1$. The following facts are easily verified.

1. $X'$ is a set of $(k-1)n$ points.
2. $G' = \{G_1, G_2, \ldots, G_k\} \setminus G_i$ is a partition of $X'$ into $k-1$ groups, each containing $n$ points.
3. $B' = \{B \setminus (B \cap G_i) : B \in B\}$ is a class of subsets of $X'$ such that each block $B' \in B'$ contains precisely one point from each group and each pair $x, y$ of points not contained in the same group occur together in precisely one block $B'$.

So, the triple $(X', G', B')$ is a $TD(k-1, n)$. \qed

**Proposition 3.8.**

1. Let $n$ be an odd integer. Suppose that an $L_{n+1/2}(n)$-graph $\Gamma$ exists. If $\bar{\Gamma} \cong \Gamma$, then $N(n) = n - 1$.

2. Let $n$ be an even integer. Suppose that an $L_{n+2/2}(n)$-graph $\Gamma$ exists. Then $\Gamma$ has a subgraph $C$ which is a disjoint union of $n$ cliques of size $n$. (We denote such a subgraph by $n \cdot C_n$.) Moreover, if $\bar{\Gamma} \cong \Gamma \setminus E(C)$, then $N(n) = n - 1$.

**Proof.** (1) Let $\Gamma = (V, E)$ be an $L_{n+1/2}(n)$-graph. We have $(n - 1)\frac{n+1}{2} < \frac{n^2}{2}$ and by Lemma 3.5, for any edge $(x, y) \in E$, there exists a unique clique $C_n$ such that $(x, y) \in C_n$. Suppose that $\bar{\Gamma} = (V, \bar{E})$ and $\Gamma \cong \bar{\Gamma}$. Then there exists a bijection $\sigma : V \to V$ such that any edge $(x, y) \in E$ implies $(\sigma(x), \sigma(y)) \in \bar{E}$. Thus, for any edge of $\bar{\Gamma}$, there exists a unique clique. By Proposition 3.4, $\bar{\Gamma}$ is a $PL_{n+1/2}(n)$-graph. Also by Lemma 3.5, $\Gamma$ is an $L_{n+1/2}(n)$-graph.

Thus the union of $\Gamma$ and $\bar{\Gamma}$ gives a set of complete mutually orthogonal Latin squares of order $n$. So, $N(n) = n - 1$.

(2) Let $\Gamma$ be an $L_{n+2/2}(n)$-graph. Then there exists a $TD(\frac{n+2}{2}, n)$. Let $(X, G, B)$ be a $TD(\frac{n+2}{2}, n)$. By Proposition 3.7, $(X', G', B')$ is a $TD(\frac{n}{2}, n)$.
So, there exists an $L_2(n)$-graph $\Gamma'$. For $(B, B') \in E(\Gamma)$, if $B \cap B' = x \in G_i$, then $(B, B') \notin E(\Gamma')$. By Proposition 2.2 (2) and $|G_i| = n$, we have $E(\Gamma) \setminus E(\Gamma') = E(C)$, where $C = n \cdot C_n$. Also, we have $V(\Gamma) \setminus V(C) = B' = V(\Gamma')$. It follows that $\Gamma' = \Gamma \setminus E(C)$. By Proposition 3.4, $\Gamma$ is a $PL_2(n)$-graph.

Suppose that $\bar{\Gamma} \cong \Gamma \setminus E(C)$. Lemma 3.5 and the fact $(n - 1)\frac{n}{2} < \frac{n^2}{2}$ show that $\bar{\Gamma}$ is an $L_2(n)$-graph by using the similar argument of the proof in (1).

Hence, $N(n) = n - 1$. 

\section{Latin Square Graphs and Self-Complementary 2-Designs}

In this section, we consider the normalized incidence matrix of a $TD(k, n)$. Let $(X, G, \mathcal{B})$ be a $TD(k, n)$ and $G = \{G_1, G_2, \ldots, G_k\}$. For $(i, j) \in G_1 \times G_2 = \{1, \ldots, n\} \times \{1, \ldots, n\}$, we put $\mathcal{B} = \{B_{i,j} : 1 \leq i \leq n, 1 \leq j \leq n\}$.

The following two propositions are easily seen by the definition of transversal 2-designs and Latin square graphs.

\begin{proposition}
(1) $|B_{i,j} \cap B_{i,j'}| = 1$, $(j \neq j')$.

(2) $|B_{i,j} \cap B_{i',j}| = 1$, $(i \neq i')$.

(3) For $B_{i,j} \in \mathcal{B}$, there are $k - 2$ blocks $B_{i',j'}$ such that $|B_{i,j} \cap B_{i',j'}| = 1$ $(i \neq i', j \neq j')$.
\end{proposition}

\begin{proposition}
Let $\Gamma$ be an $L_k(n)$-graph and let $A(\Gamma)$ be the adjacency matrix of $\Gamma$. Then

$$A(\Gamma) = \begin{pmatrix}
J - I & A_{1,2} & A_{1,3} & \cdots & \cdots & A_{1,n} \\
A_{2,1} & J - I & A_{2,3} & \cdots & \cdots & A_{2,n} \\
A_{3,1} & A_{3,2} & J - I & \vdots & & \vdots \\
& \vdots & & \ddots & \ddots & \vdots \\
& & & & \ddots & A_{n-1,n} \\
A_{n,1} & A_{n,2} & \cdots & \cdots & A_{n,n-1} & J - I
\end{pmatrix},
$$

where $I$ is the identity matrix of size $n$, $J$ is the $n \times n$ all-1 matrix, $A_{i,j}$ is an $n \times n$ matrix whose $k - 1$ entries are equal to 1 in each row or column and satisfies $A_{i,j} = A_{j,i}^\top$ where $A_{j,i}^\top$ denotes the transposed matrix of $A_{j,i}$.

\end{proposition}

\begin{definition}
Let $\Gamma = (\mathcal{B}, E)$ be an $L_k(n)$-graph. We define the incidence structure $D = (P, Q)$ as follows.

(1) $P = \{B_{1,h} \in \mathcal{B} : 1 \leq h \leq n\}$ is a set of points,

(2) $Q = \{B_{i,j} \in \mathcal{B} : 2 \leq i \leq n, 1 \leq j \leq n\}$ is a set of blocks,

(3) $B_{1,h} \in P$ and $B_{i,j} \in Q$ are incident if and only if $(B_{1,h}, B_{i,j}) \in E$.
\end{definition}
By this definition, the incidence matrix of $D$ is
\[
\begin{pmatrix}
A_{2,1} \\
A_{3,1} \\
\vdots \\
A_{n,1}
\end{pmatrix}.
\]

**Example 4.4.** The following matrix is an example of the adjacency matrix of $L_3(4)$-graphs.

\[
\begin{pmatrix}
0 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 0 & 1 \\
1 & 0 & 1 & 1 & 1 & 1 & 0 & 0 & 1 & 0 & 1 & 0 & 1 & 1 & 1 & 0 & 0 & 1 \\
1 & 1 & 0 & 1 & 0 & 0 & 1 & 1 & 1 & 1 & 0 & 1 & 0 & 0 & 1 & 1 & 0 & 1 \\
1 & 1 & 1 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 0 & 1 & 0 & 1 & 1 & 0 & 1 & 0 \\
1 & 1 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 0 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 \\
0 & 0 & 1 & 1 & 1 & 1 & 0 & 1 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 \\
0 & 0 & 1 & 1 & 1 & 1 & 1 & 0 & 1 & 0 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 \\
1 & 1 & 0 & 1 & 0 & 1 & 1 & 1 & 1 & 0 & 0 & 1 & 1 & 1 & 1 & 0 & 0 & 1 \\
0 & 1 & 0 & 1 & 0 & 1 & 1 & 1 & 1 & 1 & 0 & 1 & 1 & 0 & 1 & 0 & 1 & 1 \\
1 & 0 & 1 & 0 & 0 & 1 & 1 & 1 & 1 & 0 & 0 & 1 & 1 & 1 & 1 & 0 & 0 & 1 \\
0 & 1 & 1 & 0 & 0 & 1 & 0 & 1 & 1 & 0 & 0 & 1 & 1 & 0 & 1 & 1 & 0 & 1 \\
0 & 1 & 1 & 0 & 1 & 0 & 1 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 0 & 1 & 0 & 1 \\
1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 1 & 1
\end{pmatrix}.
\]

The incidence matrix of $D$ obtained from the example of $L_3(4)$-graphs is given by
\[
\begin{pmatrix}
1 & 1 & 0 & 0 \\
1 & 1 & 0 & 0 \\
0 & 0 & 1 & 1 \\
0 & 0 & 1 & 1 \\
1 & 0 & 1 & 0 \\
0 & 1 & 0 & 1 \\
1 & 0 & 1 & 0 \\
0 & 1 & 0 & 1 \\
1 & 0 & 0 & 1 \\
0 & 1 & 1 & 0 \\
0 & 1 & 1 & 0 \\
1 & 0 & 0 & 1
\end{pmatrix}.
\]

**Proposition 4.5.** The pair $D = (P, Q)$ is a $2-(n, k-1, (k-1)(k-2))$ design (allowing repeated blocks).
Proof. By Definition 4.3, we have $|P| = n$. By Proposition 4.1 (2) and (3), $Q$ is a collection of $(k - 1)$-element subsets of $P$. Here, $\Gamma$ is a strongly regular graph with parameters $(n^2, (n - 1)k, n + k(k - 3), k(k - 1))$. Since any two vertices $B_{1,h}, B_{1,h'} \in P \ (h \neq h')$ are adjacent, the number of common neighbours of $B_{1,h}$ and $B_{1,h'}$ in the sets of $Q$ is $n + k(k - 3) - (n - 2) = (k - 1)(k - 2)$.

It follows that the pair $(P, Q)$ is a 2-$(n, k - 1, (k - 1)(k - 2))$ design. \hfill $\square$

Remark. In this paper, we normally allow repeated blocks. An isomorphism from $(P, Q)$ to $(P', Q')$ is a pair of bijections from $P$ to $P'$ and from $Q$ to $Q'$, preserving incidence and non-incidence.

Here, we introduce a self-complementary 2-design.

**Definition 4.6.** A 2-design $D = (X, \mathcal{B})$ is called self-complementary, and denoted by $D = \bar{D}$ if, for any $B \in \mathcal{B}$,

$$|\{B' \in \mathcal{B} : B = B' \text{ as a set}\}| = |\{B'' \in \mathcal{B} : B'' = X \setminus B \text{ as a set}\}|.$$  

In particular, $B \in \mathcal{B}$ if and only if $X \setminus B \in \mathcal{B}$.

Let $D = (X, \mathcal{B})$ be a self-complementary 2-design. It is clear that $|X|$ is even and the block size is $\frac{|X|}{2}$. In Example 4.4, we give a self-complementary 2-$(4, 2, 2)$ design obtained from an $L_3(4)$-graph.

**Theorem 4.7.** Let $\Gamma$ be an $L_{n+2}(n)$-graph and $C$ be a disjoint union of $n$ cliques of size $n$. If $\bar{\Gamma} \cong \Gamma \setminus E(C)$, then there exists a 2-$(n, \frac{n}{2}, \frac{n}{2}(\frac{n}{2} - 1))$ design $D$ such that $D \cong \bar{D}$.

Proof. By Definition 4.3 and Proposition 4.5, $D = (P, Q)$ is a 2-$(n, \frac{n}{2}, \frac{n}{2}(\frac{n}{2} - 1))$ design. Suppose that $\bar{\Gamma} \cong \Gamma \setminus E(C)$ and we put $\Gamma' = \Gamma \setminus E(C)$. Then there exists a bijection $\sigma : V(\bar{\Gamma}) \rightarrow V(\Gamma')$ such that $(x, y) \in E(\bar{\Gamma})$ implies $(\sigma(x), \sigma(y)) \in E(\Gamma')$.

Set

$$P' = \{\sigma(B_{1,h}) : 1 \leq h \leq n\} \subset \mathcal{B},$$

$$Q' = \{\sigma(B_{i,j}) : 2 \leq i \leq n, 1 \leq j \leq n\} \subset \mathcal{B}.$$  

Define the incidence structure $D' = (P', Q')$ by $\sigma(B_{1,h}) \in P'$ and $\sigma(B_{i,j}) \in Q'$ are incident if and only if $(\sigma(B_{1,h}), \sigma(B_{i,j})) \in E(\Gamma')$.

For $B_{i,j} \in Q$, there are $\frac{n}{2}$ vertices $B_{1,t} \in P$ such that $(B_{1,t}, B_{i,j}) \in E(\bar{\Gamma})$. If $(B_{1,t}, B_{i,j}) \in E(\bar{\Gamma})$, then $(\sigma(B_{1,t}), \sigma(B_{i,j})) \in E(\Gamma')$. Therefore, $\sigma$ is a pair of bijections from $P$ to $P'$ and from $Q$ to $Q'$, preserving incidence and non-incidence. Hence, we have $\Gamma' \cong \bar{D}$.

For any $h$ and $h' \ (1 \leq h, h' \leq n)$, since $(B_{1,h}, B_{1,h'}) \notin E(\bar{\Gamma})$, then we have $(\sigma(B_{1,h}), \sigma(B_{1,h'})) \notin E(\Gamma')$. Hence, we have $E(\Gamma') \cup E(C) \cup E(\bar{\Gamma}) = E(K_{n+2})$.

Thus, we have $(\sigma(B_{1,h}), \sigma(B_{1,h'})) \in E(C)$, hence $(\sigma(B_{1,h}), \sigma(B_{1,h'})) \in E(\Gamma)$,
for any $h$ and $h'$ ($1 \leq h, h' \leq n$). So, there exists a bijection $\tau : \Gamma \rightarrow \Gamma$ such that $\tau(P') = P$. Also, since $(\sigma(B_{1,h}), \sigma(B_{i,j})) \notin E(C)$, we have $(\sigma(B_{1,h}), \sigma(B_{i,j})) \in E(\Gamma')$ if and only if $(\sigma(B_{1,h}), \sigma(B_{i,j})) \in E(\Gamma)$. For $\sigma(B_{i,j}) \in Q'$, there are $\frac{n}{2}$ vertices $\sigma(B_{1,s}) \in P'$ such that $(\sigma(B_{1,s}), \sigma(B_{i,j})) \in E(\Gamma')$. Therefore, $\tau$ is a pair of bijections from $P'$ to $P$ and from $Q'$ to $Q$, preserving incidence and non-incidence.

So, we have $D' \cong D$. Hence, $D \cong D'$. $$\square$$

We consider the special case that $\sigma : P \rightarrow P'$ is given by $\sigma(B_{1,h}) = B_{1,h}$. Then we get a self-complementary design $D = D'$. If $n = 2^e$ ($e > 1$), there exists an example of a self-complementary $2$-$(2k, k, \lambda)$ design with parameters $(2m, \frac{n}{2}, \frac{n}{2}(\frac{n}{2} - 1))$ design obtained from an $L_{\frac{n}{2} + 2}(n)$-graph $\Gamma$ such that $\bar{\Gamma} \cong \Gamma \setminus E(C)$. Therefore, we introduce a self-complementary design and consider the existence of the design.

The following theorem is known [11, Theorem 1.7.14. of Chapter 1]

**Theorem 4.8.** If $D$ is a $t$-$(2k, k, \lambda)$ design with an even integer $t$ and self-complementary ($D = \bar{D}$), then $D$ is also a $(t + 1)$-$(2k, k, \mu)$ design with $\mu = \lambda(k - t)/(2k - t)$.

Let $D$ be a self-complementary $2$-design with parameters $(n, \frac{n}{2}, \frac{n}{2}(\frac{n}{2} - 1))$. In the case $n \equiv 0 \pmod{4}$, we give an example.

**Proposition 4.9.** The $2m$-repeated design of a Hadamard $3$-$(4m, 2m, m - 1)$ design is a self-complementary $2$-$(4m, 2m, 2m(2m - 1))$ design.

**Proof.** Since a Hadamard $3$-$(4m, 2m, m - 1)$ design is a self-complementary $2$-design with parameters $(4m, 2m, 2m - 1)$, the $2m$-repeated of the design is also a self-complementary design. $\square$

**Remark.** It is known that there exists a Hadamard matrix of order $4m$ if and only if there exists a Hadamard $3$-$(4m, 2m, m - 1)$ design.

In the case $n \equiv 2 \pmod{4}$, we give the following proposition.

**Proposition 4.10.** There exists no self-complementary $2$-$(4m + 2, 2m + 1, 2m(2m + 1))$ design.

**Proof.** By Theorem 4.8, if $D$ is a self-complementary $2$-design with parameters $(4m + 2, 2m + 1, 2m(2m + 1))$, then $D$ is also a $3$-$(4m + 2, 2m + 1, \mu)$ design. Since $\mu = 2m(2m + 1)(2m - 1)/4m = (2m + 1)(2m - 1)/2$ is not an integer number, there is no $3$-$(4m + 2, 2m + 1, \mu)$ design. $\square$
ACKNOWLEDGMENTS

The author is grateful to Takashi Tasaka for useful advice and comments. The author would like to thank Masaaki Kitazume for discussion and useful advice. The author would also like to thank Hiro-Fumi Yamada for helpful advice.

REFERENCES


HIROYUKI NAKASORA
GRADUATE SCHOOL OF NATURAL SCIENCE AND TECHNOLOGY
OKAYAMA UNIVERSITY
OKAYAMA, 700-8530 JAPAN

*e-mail address: nakasora@math.okayama-u.ac.jp*

*(Received March 14, 2005)*