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# MUTUALLY ORTHOGONAL LATIN SQUARES AND SELF-COMPLEMENTARY DESIGNS

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Abstract. Suppose that $n$ is even and a set of $\frac{n}{2}-1$ mutually orthogonal Latin squares of order $n$ exists. Then we can construct a strongly regular graph with parameters $(n^2, \frac{n}{2}(n-1), \frac{n}{2}(\frac{n}{2}-1), \frac{n}{2}(\frac{n}{2}-1))$, which is called a Latin square graph. In this paper, we give a sufficient condition of the Latin square graph for the existence of a projective plane of order $n$. For the existence of a Latin square graph under the condition, we will introduce and consider a self-complementary 2-design (allowing repeated blocks) with parameters $(n, \frac{n}{2}, \frac{n}{2}(\frac{n}{2}-1))$. For $n \equiv 2 \pmod 4$, we give a proof of the non-existence of the design.

## 1. Introduction

A Latin square of order $n$ is an $n \times n$ array with entries $1, \dots, n$ having the property that each element of $\{1, \dots, n\}$ occurs exactly once in each row and column. Two Latin squares $A = (a_{ij}), B = (b_{ij})$ of order $n$ are said to be orthogonal if, for any $x, y \in \{1, \dots, n\}$, there exists a unique position $(i, j)$ such that $a_{ij} = x$ and $b_{ij} = y$. Latin squares are said to be mutually orthogonal if every two of them are orthogonal. Let $N(n)$ denote the maximum number of mutually orthogonal Latin squares of order $n$ ($n \geq 2$).

The value of $N(n)$ has been studied by many mathematicians, and the following three theorems are well-known.

**Theorem 1.1.** $N(6) = 1$. *If* $n \neq 2, 6$, *then* $N(n) \geq 2$.

**Theorem 1.2.** $N(n) \leq n - 1$, *with equality if and only if there exists a projective plane of order* $n$.

**Theorem 1.3.** $N(n) = n - 1$, *if* $n$ *is a prime power number.*

In 1900, Tarry showed $N(6) = 1$ by a systematic enumeration. Also in 1984, Stinson [9] gave a short proof of the fact. In 1960, Bose, Shikhande and Parker [3] proved $N(n) \geq 2$ for all $n > 6$, demolishing Euler's conjecture. Theorem 1.1 is obtained from their results.

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The Bruck-Ryser-Chowla theorem shows that if a projective plane of order $n \equiv 1$ or $2 \pmod 4$ exists, then $n$ is the sum of two squares. As noted above, this theorem dose not preclude the existence of a projective plane of order 10. In 1989, the non-existence of such a plane was shown by Lam, Swiercz and Thiel [8].

If $n$ is not a prime power number, then there is no known example of a projective plane of order $n$. We consider the existence of a projective plane of order non-prime power number. We use the following theorem, (see Bose and Shrikhande [2], Cameron and Lint [6, Chapter 7 and 8]).

**Theorem 1.4.** *The existence of $k-2$ mutually orthogonal Latin squares of order $n$ is equivalent to the existence of:*

(1) *a transversal 2-design of order $n$, block size $k$, namely a $TD(k,n)$,*
(2) *a Latin square graph, namely an $L_k(n)$-graph.*

In this paper, we give a sufficient condition of the Latin square graph for the existence of a projective plane of order $n$. If $n$ is an even integer, we show that a 2-$(n, \frac{n}{2}, \frac{n}{2}(\frac{n}{2}-1))$ design $D$ (allowing repeated blocks) such that $D \cong \bar{D}$ is obtained from the Latin square graph under the condition, where $\bar{D}$ denotes the complementary design of $D$ and $D \cong \bar{D}$ means that two designs $D, \bar{D}$ are isomorphic (Theorem 4.7).

As a special case, we consider the existence of a self-complementary 2-design ($D = \bar{D}$) with parameters $(n, \frac{n}{2}, \frac{n}{2}(\frac{n}{2}-1))$. In the case $n \equiv 0 \pmod 4$, an $\frac{n}{2}$-repeated design of a Hadamard 3-$(n, \frac{n}{2}, \frac{n}{4}-1)$ design is an example of a self-complementary design. If $n \equiv 2 \pmod 4$, there exists no self-complementary design.

## 2. Transversal 2-designs

**Definition 2.1.** Let $k \geq 2, n \geq 1$. A transversal 2-design of order $n$, block size $k$, is a triple $(X, \mathcal{G}, \mathcal{B})$ satisfying the following three conditions, and is denoted by $TD(k,n)$.

(1) $X$ is a set of $kn$ points.
(2) $\mathcal{G} = \{G_1, G_2, \ldots, G_k\}$ is a partition of $X$ into $k$ subsets $G_i$ (called groups), each containing $n$ points.
(3) $\mathcal{B}$ is a class of subsets of $X$ (called blocks) such that each block $B \in \mathcal{B}$ contains precisely one point from each group and each pair $x, y$ of points not contained in the same group occur together in precisely one block $B$.

**Proposition 2.2.** *Let $(X, \mathcal{G}, \mathcal{B})$ be a $TD(k,n)$. Then the followings hold.*

(1) *Each block contains $k$ points.*
(2) *Each point occurs in $n$ blocks.*

(3) *For any* $B, B' \in \mathcal{B}$ $(B \neq B')$, $|B \cap B'| = 0$ *or* 1.
(4) $|\mathcal{B}| = n^2$.

The following theorem is due to Bose and Shirikhande [2] (also see R. M. Wilson [12]). By this theorem, we have $2 \leq k \leq n+1$.

**Theorem 2.3.** *(Bose-Shrikhande) The existence of a set of* $k-2$ *mutually orthogonal Latin squares of order* $n$ *is equivalent to the existence of a TD(k,n).*

Now, we will make preparations for the normalized incidence matrix of a $TD(k, n)$. At first we give a normalized Latin square.

Let $A = (a_{ij})$ be a Latin square of order $n$, and set $\Omega = \{1, 2, \dots, n\}$. Take a bijection $\sigma : \Omega \to \Omega$, and define $\sigma(a_{1i}) = i$, for $i = 1, 2, \dots, n$. Then,

$$\sigma(A) = \begin{pmatrix} 1 & 2 & \cdots & n \\ & \cdots & \cdots & \\ & \cdots & \cdots & \end{pmatrix}. \tag{2.1}$$

**Lemma 2.4.** *Let* $A$ *and* $B$ *be mutually orthogonal Latin squares of order* $n$. *For any permutations* $\sigma, \tau$ *on* $\Omega$, $\sigma(A)$ *and* $\tau(B)$ *also are orthogonal.*

By (2.1) and Lemma 2.4, we can put the first rows of mutually orthogonal Latin squares the integers $1, 2, \dots, n$.

**Definition 2.5.** Let $(X, \mathcal{G}, \mathcal{B})$ be a $TD(k, n)$ with

$$X = \{x_1, x_2, \dots, x_{kn}\}, \mathcal{B} = \{B_1, B_2, \dots, B_{n^2}\}.$$

The incidence matrix of a $TD(k, n)$ is the $n^2 \times kn$ matrix $A = (a_{ij})$ defined by

$$a_{ij} = \begin{cases} 1 & \text{if } x_i \in B_j \\ 0 & \text{if } x_i \notin B_j. \end{cases}$$

Then we have the following proposition.

**Proposition 2.6.** *The incidence matrix of a* $TD(k, n)$ *can be normalized as*

$$\left(\begin{array}{c|c|ccc} H_1 & I & I & \dots & I \\ \hline H_2 & I & & & \\ \vdots & \vdots & & \dots & \\ H_n & I & & & \end{array}\right),$$

*where* $I$ *is the identity matrix of size* $n$, *and* $H_i$ $(1 \leq i \leq n)$ *is an* $n \times n$ *matrix with every entry* 1 *of* $i$ *th column, otherwise* 0.

**Example 2.7.** The following pair $(A, B)$ is an example of the pair of mutually orthogonal Latin squares of order 3:

$$A = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \\ 3 & 1 & 2 \end{pmatrix}, B = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \\ 2 & 3 & 1 \end{pmatrix}.$$

The incidence matrix of the corresponding $TD(4, 3)$ is given by

$$\left(\begin{array}{ccc|ccc|ccc|ccc} 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 \\ \hline 0 & 1 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 1 & 0 \\ \hline 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 0 \end{array}\right).$$

## 3. Latin square graphs

**Definition 3.1.** Let $(X, \mathcal{G}, \mathcal{B})$ be a $TD(k, n)$. The Latin square graph $\Gamma = (V, E)$ is defined as follows and is denoted by an $L_k(n)$-graph.

(1) $V = \mathcal{B}$.
(2) Two vertices $B, B' \in \mathcal{B}$ are adjacent if and only if $|B \cap B'| = 1$.

The following proposition is well-known, (see Cameron and Lint [6, Chapter 8]).

**Proposition 3.2.** *Let $\Gamma$ be an $L_k(n)$-graph. Then*

($a$) *If $n + 1 > k \geq 2$, then $\Gamma$ is a strongly regular graph with parameters $\big(n^2, (n-1)k, n + k(k-3), k(k-1)\big)$;*
($b$) *If $k = n + 1$, then $\Gamma$ is isomorphic to $K_{n^2}$, where $K_{n^2}$ is a complete graph with $n^2$ vertices.*

**Definition 3.3.** For $n+1 > k \geq 1$, a pseudo Latin square graph is a strongly regular graph with parameters $\big(n^2, (n-1)k, n + k(k-3), k(k-1)\big)$. Such a graph is denoted by a $PL_k(n)$-graph.

It is well-known that the complement of a strongly regular graph is strongly regular (see Cameron and Lint [6, Chapter 2]). Therefore, we have the following propositon.

**Proposition 3.4.** *The complement of a $PL_k(n)$-graph is a $PL_{n+1-k}(n)$-graph.*

Clearly, an $L_k(n)$-graph is a $PL_k(n)$-graph. However the converse does not hold. We will give a criterion whether a $PL_k(n)$-graph is an $L_k(n)$-graph or not. Let $\Gamma$ be a $PL_k(n)$-graph. By the definition of the Latin square graph, we can easily see that if $\Gamma$ is an $L_k(n)$-graph, then every edge is contained in a clique of size $n$, where a clique is an induced complete subgraph with $n$ vertices and is denoted by $\mathcal{C}_n$. The following lemma is due to Bruck [4].

**Lemma 3.5.** *(Bruck) Let $\Gamma$ be a $PL_k(n)$-graph, and $(n-1)k \le \frac{n^2}{2}$. Then $\Gamma$ is an $L_k(n)$-graph if and only if every edge is contained in a unique clique of size $n$.*

**Example 3.6.** Let $\Gamma$ be the *Hall-Janko graph* such that $\mathrm{Aut}\ \Gamma = \mathrm{Aut}\ J_2$. Then $\Gamma$ and the complementary graph $\bar{\Gamma}$ are pseudo Latin square graphs (a $PL_4(10)$-graph and a $PL_7(10)$-graph) with parameters $(100, 36, 14, 12)$ and $(100, 63, 38, 42)$, respectively. In 1968, M. Suzuki [10] stated that $\Gamma$ and $\bar{\Gamma}$ are not Latin square graphs. Here, we will give a simple proof.

*Claim 1.* $\Gamma$ is not an $L_4(10)$-graph.

*Proof.* Let $\infty$ be a vertex of $\Gamma$. Set $V(\Gamma) = \{\infty\} \cup X \cup Y$,

$$X = \{x \in V(\Gamma) : (\infty, x) \in E(\Gamma)\},$$
$$Y = \{y \in V(\Gamma) : (\infty, y) \notin E(\Gamma)\},$$

where $V(\Gamma)$ is the vertex set of $\Gamma$ and $E(\Gamma)$ is the edge set of $\Gamma$.

Suppose that $\Gamma$ is an $L_4(10)$-graph. Then, for any $(a, b) \in E(\Gamma)$, $(a, b) \in \mathcal{C}_{10}$. Therefore

$$X \supset \mathcal{C}_9. \tag{3.1}$$

Here, we use a construction of the *Hall-Janko graph*. The following chain of groups is called the *Suzuki chain*. These groups are the full automorphism groups of strongly regular graphs.

$$S_4 \subset PGL(2,7) \subset G_2(2) \subset \mathrm{Aut}\ J_2 \subset \mathrm{Aut}\ G_2(4) \subset \mathrm{Aut}\ Sz$$

It is known that $\mathrm{Aut}\ X = G_2(2)$ and $X$ is a strongly regular graph with parameters $(n, k, \lambda, \mu) = (36, 14, 4, 6)$. By (3.1), we have $7 \le \lambda = 4$, a contradiction. □

*Claim 2.* $\bar{\Gamma}$ is not an $L_7(10)$-graph.

*Proof.* Suppose that $\bar{\Gamma}$ is an $L_7(10)$-graph. Then $\bar{\Gamma}$ must have a pair of cliques $\mathcal{C}_{10}, \mathcal{C}'_{10}$ such that $|\mathcal{C}_{10} \cap \mathcal{C}'_{10}| = 1$ (see Bruck [4]). It is known that $|\mathcal{C}_{10} \cap \mathcal{C}'_{10}| = 0$ or $2$, for any distinct cliques $\mathcal{C}_{10}, \mathcal{C}'_{10}$ (see Chigira-Harada-Kitazume [7]), a contradiction. □

Thus, the above claims complete a proof of the fact that $\Gamma$ and $\bar{\Gamma}$ are not Latin square graphs.

**Proposition 3.7.** *Suppose that* $3 \leq k \leq n+1$. *Let* $(X, \mathcal{G}, \mathcal{B})$ *be a* $TD(k, n)$. *For* $1 \leq i \leq k$, *define a triple* $(X', \mathcal{G}', \mathcal{B}')$ *by*

$$\begin{aligned} X' &= X \setminus G_i \\ \mathcal{G}' &= \{G_1, G_2, \ldots, G_k\} \setminus G_i \\ \mathcal{B}' &= \{B \setminus (B \cap G_i) : B \in \mathcal{B}\}. \end{aligned}$$

*Then* $(X', \mathcal{G}', \mathcal{B}')$ *is a* $TD(k-1, n)$.

*Proof.* Suppose that $3 \leq k \leq n+1$. The following facts are easily verified.

(1) $X'$ is a set of $(k-1)n$ points.
(2) $\mathcal{G}' = \{G_1, G_2, \ldots, G_k\} \setminus G_i$ is a partition of $X'$ into $k-1$ groups, each containing $n$ points.
(3) $\mathcal{B}' = \{B \setminus (B \cap G_i) : B \in \mathcal{B}\}$ is a class of subsets of $X'$ such that each block $B' \in \mathcal{B}'$ contains precisely one point from each group and each pair $x, y$ of points not contained in the same group occur together in precisely one block $B'$.

So, the triple $(X', \mathcal{G}', \mathcal{B}')$ is a $TD(k-1, n)$. □

**Proposition 3.8.** (1) *Let* $n$ *be an odd integer.*
*Suppose that an* $L_{\frac{n+1}{2}}(n)$*-graph* $\Gamma$ *exists. If* $\bar{\Gamma} \cong \Gamma$, *then* $N(n) = n-1$.
(2) *Let* $n$ *be an even integer.*
*Suppose that an* $L_{\frac{n+2}{2}}(n)$*-graph* $\Gamma$ *exists. Then* $\Gamma$ *has a subgraph* $C$ *which is a disjoint union of* $n$ *cliques of size* $n$. *(We denote such a subgraph by* $n \cdot \mathcal{C}_n$.*)*
*Moreover, if* $\bar{\Gamma} \cong \Gamma \setminus E(C)$, *then* $N(n) = n-1$.

*Proof.* (1) Let $\Gamma = (V, E)$ be an $L_{\frac{n+1}{2}}(n)$-graph. We have $(n-1)\frac{n+1}{2} < \frac{n^2}{2}$ and by Lemma 3.5, for any edge $(x, y) \in E$, there exists a unique clique $\mathcal{C}_n$ such that $(x, y) \in \mathcal{C}_n$. Suppose that $\bar{\Gamma} = (V, \bar{E})$ and $\Gamma \cong \bar{\Gamma}$. Then there exists a bijection $\sigma : V \to V$ such that any edge $(x, y) \in E$ implies $(\sigma(x), \sigma(y)) \in \bar{E}$. Thus, for any edge of $\bar{\Gamma}$, there exists a unique clique. By Propositon 3.4, $\bar{\Gamma}$ is a $PL_{\frac{n+1}{2}}(n)$-graph. Also by Lemma 3.5, $\bar{\Gamma}$ is an $L_{\frac{n+1}{2}}(n)$-graph.

Thus the union of $\Gamma$ and $\bar{\Gamma}$ gives a set of complete mutually orthogonal Latin squares of order $n$. So, $N(n) = n-1$.

(2) Let $\Gamma$ be an $L_{\frac{n+2}{2}}(n)$-graph. Then there exists a $TD(\frac{n+2}{2}, n)$. Let $(X, \mathcal{G}, \mathcal{B})$ be a $TD(\frac{n+2}{2}, n)$. By Proposition 3.7, $(X', \mathcal{G}', \mathcal{B}')$ is a $TD(\frac{n}{2}, n)$.

So, there exists an $L_{\frac{n}{2}}(n)$-graph $\Gamma'$. For $(B, B') \in E(\Gamma)$, if $B \cap B' = x \in G_i$, then $(B, B') \notin E(\Gamma')$. By Proposition 2.2 (2) and $|G_i| = n$, we have $E(\Gamma) \setminus E(\Gamma') = E(C)$, where $C = n \cdot \mathcal{C}_n$. Also, we have $V(\Gamma) \setminus V(C) = \mathcal{B}' = V(\Gamma')$. It follows that $\Gamma' = \Gamma \setminus E(C)$. By Propositon 3.4, $\bar{\Gamma}$ is a $PL_{\frac{n}{2}}(n)$-graph. Suppose that $\bar{\Gamma} \cong \Gamma \setminus E(C)$. Lemma 3.5 and the fact $(n-1)\frac{n}{2} < \frac{n^2}{2}$ show that $\bar{\Gamma}$ is an $L_{\frac{n}{2}}(n)$-graph by using the similar argument of the proof in (1). Hence, $N(n) = n - 1$. $\square$

## 4. Latin square graphs and self-complementary 2-designs

In this section, we consider the normalized incidence matrix of a $TD(k, n)$. Let $(X, \mathcal{G}, \mathcal{B})$ be a $TD(k, n)$ and $\mathcal{G} = \{G_1, G_2, \ldots, G_k\}$. For $(i, j) \in G_1 \times G_2 = \{1, \ldots, n\} \times \{1, \ldots, n\}$, we put $\mathcal{B} = \{B_{i,j} : 1 \le i \le n, 1 \le j \le n\}$.

The following two propositions are easily seen by the definition of transversal 2-designs and Latin square graphs.

**Proposition 4.1.** (1) $|B_{i,j} \cap B_{i,j'}| = 1,\ (j \neq j')$.

(2) $|B_{i,j} \cap B_{i',j}| = 1,\ (i \neq i')$.

(3) *For $B_{i,j} \in \mathcal{B}$, there are $k-2$ blocks $B_{i',j'}$ such that $|B_{i,j} \cap B_{i',j'}| = 1\ (i \neq i', j \neq j')$.*

**Proposition 4.2.** *Let $\Gamma$ be an $L_k(n)$-graph and let $A(\Gamma)$ be the adjacency matrix of $\Gamma$. Then*

$$A(\Gamma) = \begin{pmatrix} J-I & A_{1,2} & A_{1,3} & \cdots & \cdots & A_{1,n} \\ A_{2,1} & J-I & A_{2,3} & \cdots & \cdots & A_{2,n} \\ A_{3,1} & A_{3,2} & J-I & & & \vdots \\ \vdots & \vdots & & \ddots & & \vdots \\ \vdots & \vdots & & & \ddots & A_{n-1,n} \\ A_{n,1} & A_{n,2} & \cdots & \cdots & A_{n,n-1} & J-I \end{pmatrix},$$

*where $I$ is the identity matrix of size $n$, $J$ is the $n \times n$ all-1 matrix, $A_{i,j}$ is an $n \times n$ matrix whose $k-1$ entries are equal to 1 in each row or column and satisfies $A_{i,j} = A_{j,i}^\top$ where $A_{j,i}^\top$ denotes the transposed matrix of $A_{j,i}$.*

**Definition 4.3.** Let $\Gamma = (\mathcal{B}, E)$ be an $L_k(n)$-graph. We define the incidence structure $D = (P, Q)$ as follows.

(1) $P = \{B_{1,h} \in \mathcal{B} : 1 \le h \le n\}$ is a set of points,
(2) $Q = \{B_{i,j} \in \mathcal{B} : 2 \le i \le n, 1 \le j \le n\}$ is a set of blocks,
(3) $B_{1,h} \in P$ and $B_{i,j} \in Q$ are incident if and only if $(B_{1,h}, B_{i,j}) \in E$.

By this definition, the incidence matrix of $D$ is

$$\begin{pmatrix} A_{2,1} \\ A_{3,1} \\ \vdots \\ A_{n,1} \end{pmatrix}.$$

**Example 4.4.** The following matrix is an example of the adjacency matrix of $L_3(4)$-graphs.

$$\left(\begin{array}{cccc|cccc|cccc|cccc}
0&1&1&1&1&1&0&0&1&0&1&0&1&0&0&1\\
1&0&1&1&1&1&0&0&0&1&0&1&0&1&1&0\\
1&1&0&1&0&0&1&1&1&0&1&0&0&1&1&0\\
1&1&1&0&0&0&1&1&0&1&0&1&1&0&0&1\\
\hline
1&1&0&0&0&1&1&1&1&0&0&1&1&0&1&0\\
1&1&0&0&1&0&1&1&0&1&1&0&0&1&0&1\\
0&0&1&1&1&1&0&1&0&1&1&0&1&0&1&0\\
0&0&1&1&1&1&1&0&1&0&0&1&0&1&0&1\\
\hline
1&0&1&0&1&0&0&1&0&1&1&1&1&1&0&0\\
0&1&0&1&0&1&1&0&1&0&1&1&1&1&0&0\\
1&0&1&0&0&1&1&0&1&1&0&1&0&0&1&1\\
0&1&0&1&1&0&0&1&1&1&1&0&0&0&1&1\\
\hline
1&0&0&1&1&0&1&0&1&1&0&0&0&1&1&1\\
0&1&1&0&0&1&0&1&1&1&0&0&1&0&1&1\\
0&1&1&0&1&0&1&0&0&0&1&1&1&1&0&1\\
1&0&0&1&0&1&0&1&0&0&1&1&1&1&1&0
\end{array}\right).$$

The incidence matrix of $D$ obtained from the example of $L_3(4)$-graphs is given by

$$\begin{pmatrix}
1&1&0&0\\
1&1&0&0\\
0&0&1&1\\
0&0&1&1\\
1&0&1&0\\
0&1&0&1\\
1&0&1&0\\
0&1&0&1\\
1&0&0&1\\
0&1&1&0\\
0&1&1&0\\
1&0&0&1
\end{pmatrix}.$$

**Proposition 4.5.** *The pair $D = (P, Q)$ is a 2-$(n, k-1, (k-1)(k-2))$ design (allowing repeated blocks).*

*Proof.* By Definiton 4.3, we have $|P| = n$. By Proposition 4.1 (2) and (3), $Q$ is a collection of $(k-1)$-element subsets of $P$. Here, $\Gamma$ is a strongly regular graph with parameters $\big(n^2, (n-1)k, n+k(k-3), k(k-1)\big)$. Since any two vertices $B_{1,h}, B_{1,h'} \in P$ $(h \neq h')$ are adjacent, the number of common neighbours of $B_{1,h}$ and $B_{1,h'}$ in the sets of $Q$ is $n+k(k-3)-(n-2) = (k-1)(k-2)$. It follows that the pair $(P, Q)$ is a 2-$(n, k-1, (k-1)(k-2))$ design. □

*Remark.* In this paper, we normally allow repeated blocks. An isomorphism from $(P, Q)$ to $(P', Q')$ is a pair of bijections from $P$ to $P'$ and from $Q$ to $Q'$, preserving incidence and non-incidence.

Here, we introduce a self-complementary 2-design.

**Definition 4.6.** A 2-design $D = (X, \mathcal{B})$ is called self-complementary, and denoted by $D = \bar{D}$ if, for any $B \in \mathcal{B}$,

$$\big|\{B' \in \mathcal{B} : B = B' \text{ as a set}\}\big| = \big|\{B'' \in \mathcal{B} : B'' = X \setminus B \text{ as a set}\}\big|.$$

In particular, $B \in \mathcal{B}$ if and only if $X \setminus B \in \mathcal{B}$.

Let $D = (X, \mathcal{B})$ be a self-complementary 2-design. It is clear that $|X|$ is even and the block size is $\frac{|X|}{2}$. In Example 4.4, we give a self-complementary 2-$(4, 2, 2)$ design obtained from an $L_3(4)$-graph.

**Theorem 4.7.** *Let $\Gamma$ be an $L_{\frac{n+2}{2}}(n)$-graph and $C$ be a disjoint union of $n$ cliques of size $n$. If $\bar{\Gamma} \cong \Gamma \setminus E(C)$, then there exists a 2-$(n, \frac{n}{2}, \frac{n}{2}(\frac{n}{2}-1))$ design $D$ such that $D \cong \bar{D}$.*

*Proof.* By Definition 4.3 and Proposition 4.5, $D = (P, Q)$ is a 2-$(n, \frac{n}{2}, \frac{n}{2}(\frac{n}{2} - 1))$ design. Suppose that $\bar{\Gamma} \cong \Gamma \setminus E(C)$ and we put $\Gamma' = \Gamma \setminus E(C)$. Then there exists a bijection $\sigma : V(\bar{\Gamma}) \to V(\Gamma')$ such that $(x, y) \in E(\bar{\Gamma})$ implies $(\sigma(x), \sigma(y)) \in E(\Gamma')$.

Set

$$P' = \{\sigma(B_{1,h}) : 1 \leq h \leq n\} \subset \mathcal{B},$$
$$Q' = \{\sigma(B_{i,j}) : 2 \leq i \leq n, 1 \leq j \leq n\} \subset \mathcal{B}.$$

Define the incidence structure $D' = (P', Q')$ by $\sigma(B_{1,h}) \in P'$ and $\sigma(B_{i,j}) \in Q'$ are incident if and only if $(\sigma(B_{1,h}), \sigma(B_{i,j})) \in E(\Gamma')$.

For $B_{i,j} \in Q$, there are $\frac{n}{2}$ vertices $B_{1,t} \in P$ such that $(B_{1,t}, B_{i,j}) \in E(\bar{\Gamma})$. If $(B_{1,t}, B_{i,j}) \in E(\bar{\Gamma})$, then $(\sigma(B_{1,t}), \sigma(B_{i,j})) \in E(\Gamma')$. Therefore, $\sigma$ is a pair of bijections from $P$ to $P'$ and from $\bar{Q}$ to $Q'$, preserving incidence and non-incidence. Hence, we have $D' \cong \bar{D}$.

For any $h$ and $h'$ $(1 \leq h, h' \leq n)$, since $(B_{1,h}, B_{1,h'}) \notin E(\bar{\Gamma})$, then we have $(\sigma(B_{1,h}), \sigma(B_{1,h'})) \notin E(\Gamma')$. Here, we have $E(\Gamma') \cup E(C) \cup E(\bar{\Gamma}) = E(K_{n^2})$. Thus, we have $(\sigma(B_{1,h}), \sigma(B_{1,h'})) \in E(C)$, hence $(\sigma(B_{1,h}), \sigma(B_{1,h'})) \in E(\Gamma)$, for any $h$ and $h'$ $(1 \leq h, h' \leq n)$. So, there exists a bijection $\tau : \Gamma \to \Gamma$ such that $\tau(P') = P$. Also, since $(\sigma(B_{1,h}), \sigma(B_{i,j})) \notin E(C)$, we have $(\sigma(B_{1,h}), \sigma(B_{i,j})) \in E(\Gamma')$ if and only if $(\sigma(B_{1,h}), \sigma(B_{i,j})) \in E(\Gamma)$. For $\sigma(B_{i,j}) \in Q'$, there are $\frac{n}{2}$ vertices $\sigma(B_{1,s}) \in P'$ such that $(\sigma(B_{1,s}), \sigma(B_{i,j})) \in E(\Gamma')$. If $(\sigma(B_{1,s}), \sigma(B_{i,j})) \in E(\Gamma')$, then $(\tau\sigma(B_{1,s}), \tau\sigma(B_{i,j})) \in E(\Gamma)$. Therefore, $\tau$ is a pair of bijections from $P'$ to $P$ and from $Q'$ to $Q$, preserving incidence and non-incidence.

So, we have $D' \cong D$. Hence, $D \cong \bar{D}$.

□

We consider the special case that $\sigma : P \to P'$ is given by $\sigma(B_{1,h}) = B_{1,h}$. Then we get a self-complementary design $D = \bar{D}$. If $n = 2^e$ $(e > 1)$, there exists an example of a self-complementary 2-$(n, \frac{n}{2}, \frac{n}{2}(\frac{n}{2} - 1))$ design obtained from an $L_{\frac{n+2}{2}}(n)$-graph $\Gamma$ such that $\bar{\Gamma} \cong \Gamma \setminus E(C)$. Therefore, we introduce a self-complementary design and consider the existence of the design.

The following theorem is known [11, Theorem 1.7.14. of Chapter 1]

**Theorem 4.8.** *If $D$ is a $t$-$(2k, k, \lambda)$ design with an even integer $t$ and self-complementary ($D = \bar{D}$), then $D$ is also a $(t+1)$-$(2k, k, \mu)$ design with $\mu = \lambda(k - t)/(2k - t)$.*

Let $D$ be a self-complementary 2-design with parameters $(n, \frac{n}{2}, \frac{n}{2}(\frac{n}{2} - 1))$. In the case $n \equiv 0 \pmod 4$, we give an example.

**Proposition 4.9.** *The $2m$-repeated design of a Hadamard 3-$(4m, 2m, m-1)$ design is a self-complementary 2-$(4m, 2m, 2m(2m-1))$ design.*

*Proof.* Since a Hadamard 3-$(4m, 2m, m-1)$ design is a self-complementary 2-design with parameters $(4m, 2m, 2m-1)$, the $2m$-repeated of the design is also a self-complementary design. □

*Remark.* It is known that there exists a Hadamard matrix of order $4m$ if and only if there exists a Hadamard 3-$(4m, 2m, m-1)$ design.

In the case $n \equiv 2 \pmod 4$, we give the following proposition.

**Proposition 4.10.** *There exists no self-complementary 2-$(4m+2, 2m+1, 2m(2m+1))$ design.*

*Proof.* By Theorem 4.8, if $D$ is a self-complementary 2-design with parameters $(4m+2, 2m+1, 2m(2m+1))$, then $D$ is also a 3-$(4m+2, 2m+1, \mu)$ design. Since $\mu = 2m(2m+1)(2m-1)/4m = (2m+1)(2m-1)/2$ is not an integer number, there is no 3-$(4m+2, 2m+1, \mu)$ design.

□

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