

## SOME METRIC INVARIANTS OF SPHERES AND ALEXANDROV SPACES II

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ABSTRACT. A metric invariant  $a_k$  is defined, and we have that  $a_k(X) \leq a_k(S^n)$  holds in an Alexandrov space  $X$  with curvature  $\geq 1$  ([So]). And the borderline case when  $a_{2p-1}(X) = a_{2p-1}(S^n)$  and  $a_k(S^n)$  are studied.

### 1. INTRODUCTION

In the previous paper ([So]) we introduced a metric invariant  $a_k(X)$  for a compact metric space  $X$ , and studied the explicit value of  $a_k(S^n)$  for an  $n$ -dimensional round sphere  $S^n$  with radius 1. Furthermore, we studied the behavior of  $a_k(X)$  for an Alexandrov space  $X$  with curvature  $\geq 1$ . In the present paper we continue to study the above invariant  $a_k$  and give answers to some problems that are conjectured in [So]. We begin with recalling the definition of  $a_k$  and results obtained in [So]. The distance between  $x, y \in X$  will be denoted by  $dist(x, y)$ .

**Definition 1.1.** *For a positive integer  $k$ , we define the metric invariant  $a_k(X)$  of  $X$  as follows:*

$$(1.1) \quad a_k(X) = \min_{x_1, \dots, x_k \in X} \max_{x \in X} \frac{1}{k} \sum_{i=1}^k dist(x, x_i).$$

In the previous paper we were concerned with  $a_k(S^n)$  and got the explicit value of  $a_k(S^1)$ . A  $k$ -tuple  $(x_1, \dots, x_k)$  of points  $x_i (i = 1, \dots, k)$  of  $S^1$  located in counterclockwise order is called a configuration, where each  $x_i$  is called a vertex of the configuration.

**Theorem 1.1.** (1) *For  $k = 2p - 1$ , we have*

$$(1.2) \quad a_k(S^1) = \frac{2p^2 - 2p + 1}{(2p - 1)^2} \pi.$$

*$a_k(S^1)$  is realized if and only if a configuration  $(x_1, \dots, x_k)$  of  $k$  points is equally spaced in  $S^1$ , and  $\max_{x \in S^1} (1/k) \sum_{i=1}^k dist(x, x_i)$  is attained exactly at the antipodal points of  $x_i (1 \leq i \leq k)$ .*

(2) *For  $k = 2p$ , we have*

$$(1.3) \quad a_k(S^1) = \frac{1}{2} \pi.$$

$a_k(S^1)$  is realized if and only if  $(x_1, \dots, x_{2p})$  consists of pairs of antipodal points, and in the case we have  $(1/k) \sum_{i=1}^k \text{dist}(x, x_i) \equiv \pi/2$ .

Now in this paper we complete the following theorem for general dimension  $n$ .

**Theorem 1.2.** (1) For  $k = 2p - 1$ , we have

$$(1.4) \quad a_k(S^n) = a_k(S^1) = \frac{2p^2 - 2p + 1}{(2p - 1)^2} \pi.$$

$a_{2p-1}(S^n)$  is realized if and only if  $\{x_1, \dots, x_{2p-1}\}$  is located on a great circle  $S^1$  and gives an equally spaced configuration after rearranging the order of  $\{x_1, \dots, x_{2p-1}\}$ , and  $\max_{x \in S^n} (1/k) \sum_{i=1}^k \text{dist}(x, x_i)$  is attained exactly at the antipodal points of  $x_i (1 \leq i \leq k)$ .

(2) For  $k = 2p$ , we have

$$(1.5) \quad a_k(S^n) = \frac{1}{2} \pi.$$

Moreover,  $a_k(S^n)$  is realized if and only if the set  $\{x_1, \dots, x_{2p}\}$  consists of pairs of antipodal points, and in the case we have  $(1/k) \sum_{i=1}^k \text{dist}(x, x_i) \equiv \pi/2$ .

Theorem 1.2(2) is already proved in section 4 of [So]. Theorem 1.2(1) for  $k = 3$  is also proved in section 3 of [So]. We give a proof of Theorem 1.2(1) for all  $k = 2p - 1$  in section 2 of this paper. In our proof another metric invariant  $k$ -extent  $xt_k(X)$  introduced by Grove and Markvorsen plays an important role. Recall that for an integer  $k \geq 2$ ,  $xt_k(X)$  is defined as follows:

$$(1.6) \quad xt_k(X) = \max_{x_1, \dots, x_k \in X} \binom{k}{2}^{-1} \sum_{i < j} \text{dist}(x_i, x_j).$$

Now we give some results of [G-M] on  $xt_k(S^n)$  which are necessary for our proof.

**Theorem 1.3** (Grove-Markvorsen). For all  $n \geq 1$  and  $k \geq 2$  we have

$$(1.7) \quad xt_k(S^n) = xt_k(S^1) = \pi / \left( 2 - \left[ \frac{k+1}{2} \right]^{-1} \right).$$

Those points that realize  $xt_k(S^n)$  all lie on a great circle except for antipodal pairs.

More specifically,  $xt_k(S^1)$  is given as follows:

$$(1.8) \quad xt_{2p}(S^1) = \frac{p}{2p-1} \pi,$$

$$(1.9) \quad xt_{2p-1}(S^1) = \frac{p}{2p-1}\pi.$$

$xt_{2p}(S^1)$  is realized if and only if  $\{x_1, \dots, x_{2p}\}$  consists of pairs of antipodal points.

Now we explain the separation condition as follows. A configuration  $(x_1, \dots, x_k)$  in  $S^1$  which does not contain any pair of antipodal points is said to satisfy the separation condition if the following is satisfied: For any  $x_i$  the line through  $x_i$  and the origin separates  $(x_1, \dots, x_k) \setminus \{x_i\}$  into sets of equal cardinality. Note that  $k$  is odd in this case. Then  $xt_{2p-1}(S^1)$  is realized if and only if  $\{x_1, \dots, x_{2p-1}\}$  consists of pairs of antipodal points together with a configuration of points satisfying the separation condition.

For  $k = 2p - 1$ ,  $a_k(S^n)$  is related to  $xt_k(S^n)$  by an inequality

$$(1.10) \quad a_k(S^n) \geq \pi - 2k^{-2} \binom{k}{2} xt_k(S^n) = \frac{2p^2 - 2p + 1}{(2p - 1)^2} \pi,$$

as is shown in section 2.

Our proof of Theorem 1.2 depends also on a theorem of K. Kiyohara in [K] : For an equally spaced configuration  $(x_1, \dots, x_n)$  of a great circle  $S^1$  of  $S^2$ , a function  $f_{x_1, \dots, x_k}(x) = \sum_{i=1}^k dist(x, x_i)$  on  $S^2$  attains its maximum when and only when  $x = \bar{x}_l (l = 1, \dots, k)$ .

Let  $X$  be an Alexandrov space with curvature  $\geq 1$  and we want to compare  $a_k(X)$  with  $a_k(S^n)$ . In the previous paper we got the following theorem by using the generalized Toponogov comparison theorem([Pe]).

**Theorem 1.4.** *Let  $X$  be an  $n$ -dimensional Alexandrov space with curvature  $\geq 1$ , then we have*

$$(1.11) \quad a_k(X) \leq a_k(S^n).$$

*Especially, we have*

$$(1.12) \quad a_{2p}(X) \leq a_{2p}(S^n) = \frac{\pi}{2}.$$

In the case where equality holds in (1.11) for  $k = 3$  we showed that  $X$  is isometric to a double spherical suspension, where the generalized Toponogov comparison theorem played an important role(see section 5 of [So]). In the present paper we improve Theorem 1.4 and show the following theorem.

**Theorem 1.5.** *Let  $X$  be an  $n$ -dimensional Alexandrov space with curvature  $\geq 1$ . Suppose  $a_{2p-1}(X) = a_{2p-1}(S^n)$ . Then  $X$  is isometric to the unit sphere  $S^n$ .*

## 2. PROOF OF THEOREM 1.2(1)

In this section we give a proof of Theorem 1.2(1). First we will show for  $k = 2p - 1$

$$(2.1) \quad a_k(S^n) \geq \frac{2p^2 - 2p + 1}{(2p - 1)^2} \pi.$$

In the proof we apply a result of Grove-Markvorsen on  $k$ -extent  $xt_k(S^n)$  of the sphere (see Theorem 1.3). Recall that for  $k = 2p - 1$  the  $k$ -extent  $xt_k(S^n) = 1/\binom{k}{2} \max_{y_1, \dots, y_k \in S^n} \sum_{i < j} \text{dist}(y_i, y_j)$  of the sphere is equal to  $p(p-1)\pi/\binom{2p-1}{2} = p\pi/(2p-1)$ . Let  $x_1, \dots, x_k$  be points on  $S^n$  that realize  $a_k(S^n)$ , i.e.,  $a_k(S^n) = \max_{x \in S^n} 1/k \sum_{i=1}^k \text{dist}(x, x_i)$ . We denote by  $\bar{x}$  the antipodal point of  $x \in S^n$ . Then for any  $\bar{x}_j (1 \leq j \leq k)$  we have

$$(2.2) \quad ka_k(S^n) \geq \sum_{i=1}^k \text{dist}(x_i, \bar{x}_j).$$

Adding (2.2) with respect to  $j$  we obtain

$$(2.3) \quad \begin{aligned} k^2 a_k(S^n) &\geq \sum_{j=1}^k \sum_{i=1}^k \text{dist}(x_i, \bar{x}_j) \\ &= k^2 \pi - 2 \sum_{i < j} \text{dist}(x_i, x_j) \\ &\geq k^2 \pi - 2 \max_{y_1, \dots, y_k \in S^n} \sum_{i < j} \text{dist}(y_i, y_j) \\ &= k^2 \pi - 2 \binom{k}{2} xt_k(S^n) = (2p^2 - 2p + 1)\pi. \end{aligned}$$

Hence we have (2.1). To complete the proof of Theorem 1.2(1) we have to show the following assertions:

- (1)  $a_k(S^n) = (2p^2 - 2p + 1)\pi/(2p - 1)^2$ .
- (2) If  $\{x_1, \dots, x_k\}$  realizes  $a_k(S^n)$ , then  $\{x_1, \dots, x_k\}$  is located on a great circle  $S^1$  and a configuration obtained by rearranging  $\{x_1, \dots, x_k\}$  is equally spaced on  $S^1$ .
- (3) For an equally spaced configuration  $(x_1, \dots, x_k)$ ,  $f_{x_1, \dots, x_k}(x) = \sum_{i=1}^k \text{dist}(x, x_i)$ ,  $x \in S^n$  takes the maximum exactly at  $x = \bar{x}_j$  for all  $j$ .

To show these assertions we need the following theorem.

**Theorem 2.1.** *Let  $(x_1, \dots, x_k)$  be an equally spaced configuration on a great circle  $S^1$  in  $S^n$ . The function  $f(x) = \sum_{i=1}^k \text{dist}(x, x_i)$  takes its maximum at  $x \in S^n$  when and only when  $x = \bar{x}_l (1 \leq l \leq k = 2p - 1)$ .*

Indeed, making use of Theorem 2.1 we may show (1), (2), and (3) as follows. First consider a configuration  $(x_1, \dots, x_k)$  as in Theorem 2.1, then we have  $\max_{x \in S^n} f_{x_1, \dots, x_k} = \sum_{i=1}^k \text{dist}(x_i, \bar{x}_i) = (2p^2 - 2p + 1)\pi / (2p - 1)$ . It follows that  $a_k(S^n) \leq (2p^2 - 2p + 1)\pi / (2p - 1)^2$  and the assertion (1) holds. Next we show the assertion (2). Suppose  $\{x_1, \dots, x_k\}$  realizes  $a_k(S^n)$ , i.e.,  $\max_{x \in S^n} f_{x_1, \dots, x_k} = (2p^2 - 2p + 1)\pi / (2p - 1)$ . Then equality holds in the above inequalities (2.3),  $\{x_1, \dots, x_k\}$  also realizes  $xt_k(S^n)$ , i.e., they are lying on a great circle  $S^1$  except for some antipodal pairs by Theorem 1.3. Note, however, that  $a_{2p-1}(S^n)$  cannot be realized if  $\{x_1, \dots, x_k\}$  contains an antipodal pair. Indeed, suppose that  $\{x_1, \dots, x_{2m-1}\} (1 \leq m < p)$  is on  $S^1$  and  $\{x_{2m}, \dots, x_{2p-1}\}$  consists of  $(p - m)$  antipodal pairs. Then we have

$$\begin{aligned} \max_{x \in S^n} f_{x_1, \dots, x_{2p-1}}(x) &\geq \max_{x \in S^1} f_{x_1, \dots, x_{2p-1}}(x) \\ &\geq (p - m)\pi + (2m - 1)a_{2m-1}(S^1) \\ &= (p - m)\pi + \frac{2m^2 - 2m + 1}{2m - 1}\pi \\ &= \frac{(2p - 1)m + 1 - p}{2m - 1}\pi \\ &> \frac{2p^2 - 2p + 1}{2p - 1}\pi. \end{aligned}$$

It follows that  $\{x_1, \dots, x_k\}$  is located on a great circle  $S^1$ . We may also assume that  $(x_1, \dots, x_k)$  is a configuration in  $S^1$ . Then  $f_{x_1, \dots, x_k}(x)$  attains the maximum  $ka_k(S^1) = \frac{2p^2 - 2p + 1}{2p - 1}\pi$  at  $x = \bar{x}_j \in S^1$  by (2.3). By restricting  $f_{x_1, \dots, x_k}$  to  $S^1$  we see that  $(x_1, \dots, x_k)$  is equally spaced in  $S^1$  by Theorem 1.1. Then assertion (3) also follows from Theorem 2.1.

Now we give some remarks about a proof of Theorem 2.1. It suffices to consider the case  $n = 2$  to prove Theorem 2.1. By virtue of Theorem 1.1 we need only to show that  $f_{x_1, \dots, x_k}(x)$  cannot take a maximum at a point  $x \in S^2 \setminus S^1$ . For  $k = 3$  we gave a proof of the theorem by showing that  $f_{x_1, \dots, x_k}(x)$  admits no critical points in  $S^2 \setminus S^1$  (see section 3 of [So]). But for  $k = 2p - 1 > 3$  the behavior of critical points of  $f_{x_1, \dots, x_k}(x)$  is rather complicated and it is not so clear whether the above approach works for general  $k = 2p - 1$ . Then K. Kiyohara gave a simple and ingenious proof of theorem 2.1 which will be presented in the appendix.

### 3. PROOF OF THEOREM 1.5

Let  $X$  be an  $n$ -dimensional Alexandrov space with curvature  $\geq 1$ . Recall that we have an inequality

$$(3.1) \quad a_{2p-1}(X) \leq a_{2p-1}(S^n) = \frac{2p^2 - 2p + 1}{(2p-1)^2} \pi$$

by Theorem 1.2 and Theorem 1.4. In this section we show that  $X$  is isometric to the round sphere  $S^n$  of radius 1 when equality holds in (3.1). First we recall the notion of the spherical suspension and the spherical join ([B-G-P]).

**Definition 3.1.** *The spherical suspension of a metric space  $Y$  is the quotient space*

$$(3.2) \quad \Sigma_1 Y = Y \times [0, \pi] / \sim,$$

where the equivalence relation  $\sim$  is given by  $(x_1, a_1) \sim (x_2, a_2) \Leftrightarrow x_1 = x_2, 0 < a_1 = a_2 < \pi$  or  $a_1 = a_2 = 0$  or  $a_1 = a_2 = \pi$ , and is equipped with the canonical metric

$$(3.3) \quad \cos \operatorname{dist}(\hat{x}_1, \hat{x}_2) = \cos a_1 \cos a_2 + \sin a_1 \sin a_2 \cos \operatorname{dist}(x_1, x_2),$$

where we set  $\hat{x}_1 = (x_1, a_1)$ ,  $\hat{x}_2 = (x_2, a_2)$ .

**Definition 3.2.** *The spherical join of  $X$  and  $Y$  is defined as*

$$(3.4) \quad X * Y = X \times Y \times [0, \pi/2] / \sim,$$

where  $(x_1, y_1, a_1) \sim (x_2, y_2, a_2) \Leftrightarrow x_1 = x_2, y_1 = y_2, 0 < a_1 = a_2 < \pi/2$  or  $a_1 = a_2 = 0$ ,  $x_1 = x_2$  or  $a_1 = a_2 = \pi/2$ ,  $y_1 = y_2$ , and is equipped with the canonical metric

$$(3.5) \quad \begin{aligned} & \cos \operatorname{dist}((x_1, y_1, a_1), (x_2, y_2, a_2)) \\ &= \cos a_1 \cos a_2 \cos \operatorname{dist}(x_1, x_2) + \sin a_1 \sin a_2 \cos \operatorname{dist}(y_1, y_2). \end{aligned}$$

Further, we define  $\Sigma_k Y = \Sigma_{k-1}(\Sigma_1 Y)$  to be the  $k$ -times repeated spherical suspension. Then for an Alexandrov space  $X$  with curvature  $\geq 1$  we have  $X = \Sigma_k Y$  for some Alexandrov space  $Y$  with curvature  $\geq 1$  if and only if  $S^{k-1}$  is isometrically embedded in  $X$  ([G-W]). Hence the  $k$ -times repeated spherical suspension  $\Sigma_k Y$  is isometric to the spherical join  $S^{k-1} * Y$ .

In the previous paper, in the case of  $k = 3$  we showed that  $X$  is isometric to  $\Sigma_2 Z$  for some Alexandrov space  $Z$  with curvature  $\geq 1$ . First we show that  $X$  is isometric to the spherical suspension  $\Sigma_1 Y$  in the same manner as in the case of  $k = 3$  for completeness.

**Lemma 3.1.** *Let  $X$  be an  $n$ -dimensional Alexandrov space with curvature  $\geq 1$ . Suppose  $a_{2p-1}(X) = a_{2p-1}(S^n) = \frac{2p^2 - 2p + 1}{(2p-1)^2} \pi$ . Then  $X$  is isometric to the spherical suspension  $\Sigma_1 Y$ , where  $Y$  is an  $(n-1)$ -dimensional Alexandrov space with curvature  $\geq 1$ .*

*Proof.* By the maximal diameter theorem([G-P2]), it suffices to show that  $diam X$  is equal to  $\pi$ . Let  $\tilde{x}_1, \tilde{x}_2, \dots, \tilde{x}_{2p-1}$  be points on  $S^n$  that realize  $a_{2p-1}(S^n)$ . We may assume that  $(\tilde{x}_1, \tilde{x}_2, \dots, \tilde{x}_{2p-1})$  is an equally spaced configuration in a great circle  $S^1$ . Take a point  $\tilde{p} \in S^n$  different from the antipodals of  $\tilde{x}_i (i = 1, 2, \dots, 2p-1)$ . Take a regular point  $p \in X$ . We denote by  $S_p$  the space of directions of  $X$  at  $p$ . Then  $\Sigma_1 S_p$  is isometric to  $S^n$ , and we identify  $\Sigma_1 S_p$  (resp.  $S_p$ ) with  $S^n = \Sigma_1 S_{\tilde{p}}$  (resp.  $S_{\tilde{p}} = S^{n-1}$ ). Let  $x_i \in X$  be a point  $\exp_p \bar{c}_{v_i}(\text{dist}(\tilde{p}, \tilde{x}_i)) = c_{v_i}(\text{dist}(\tilde{p}, \tilde{x}_i))$ , where  $\bar{c}_{v_i}$  is a minimal geodesic in  $S^n$  emanating from  $\tilde{p}$  to  $\tilde{x}_i$  with initial direction  $v_i \in S_{\tilde{p}} = S_p = S^{n-1} (i = 1, 2, \dots, 2p-1)$  and  $c_{v_i}$  is a quasigeodesic in  $X$  emanating from  $p$  with initial direction  $v_i$ (see section 5 of [So]). Take a point  $x_0 \in X$  such that

$$\begin{aligned} a_{2p-1}(x_1, x_2, \dots, x_{2p-1}) &:= \max_{x \in X} \frac{1}{2p-1} \sum_{i=1}^{2p-1} \text{dist}(x, x_i) \\ &= \frac{1}{2p-1} \sum_{i=1}^{2p-1} \text{dist}(x_0, x_i). \end{aligned}$$

Let  $\gamma_0 : [0, \text{dist}(p, x_0)] \rightarrow X$  be a minimal geodesic from  $p$  to  $x_0$ , and set  $\tilde{x}_0 = \exp_{\tilde{p}}^{S^n}(\text{dist}(p, x_0)\dot{\gamma}_0(0))$ . Then by the generalized Toponogov comparison theorem for  $\Delta p x_i x_0$  and  $\Delta \tilde{p} \tilde{x}_i \tilde{x}_0$  ([Pe], see also [So]), we have

$$\text{dist}(x_0, x_i) \leq \text{dist}(\tilde{x}_0, \tilde{x}_i) \quad (i = 1, 2, \dots, 2p-1).$$

Then we have

$$\begin{aligned} (3.6) \quad a_{2p-1}(X) &\leq a_{2p-1}(x_1, x_2, \dots, x_{2p-1}) = \frac{1}{2p-1} \sum_{i=1}^{2p-1} \text{dist}(x_0, x_i) \\ &\leq \frac{1}{2p-1} \sum_{i=1}^{2p-1} \text{dist}(\tilde{x}_0, \tilde{x}_i) \leq a_{2p-1}(\tilde{x}_1, \tilde{x}_2, \dots, \tilde{x}_{2p-1}) \\ &= a_{2p-1}(S^n) = a_{2p-1}(X). \end{aligned}$$

It follows that

$$(3.7) \quad a_{2p-1}(X) = a_{2p-1}(S^n) = \frac{1}{2p-1} \sum_{i=1}^{2p-1} \text{dist}(\tilde{x}_0, \tilde{x}_i),$$

and we obtain for any  $i$

$$(3.8) \quad \text{dist}(x_0, x_i) = \text{dist}(\tilde{x}_0, \tilde{x}_i) \quad (i = 1, 2, \dots, 2p-1).$$

Then from Theorem 1.1  $\tilde{x}_0$  is the antipodal point of some  $\tilde{x}_i$ , namely, we have  $\text{dist}(\tilde{x}_0, \tilde{x}_i) = \pi$ , and hence  $\text{dist}(x_0, x_i) = \pi$  for some  $x_i (1 \leq i \leq 2p-1)$ .  $\square$

The following lemma is given in the previous paper ([So]).

**Lemma 3.2.** *Suppose  $X = \Sigma_1 Y$ , where  $Y$  is an  $(n-1)$ -dimensional Alexandrov space with curvature  $\geq 1$  and  $\text{diam} Y < \pi$  and  $n \geq 2$ . Let  $x_1, x_2 \in X$  be the pole points of the spherical suspension  $X = \Sigma_1 Y$ . Then there is no pair of points whose distance is  $\pi$  except for  $x_1, x_2$ .*

Next we show that  $X$  is isometric to  $\Sigma_2 Z$  if  $\text{dim} X \geq 2$  as in the case of  $k = 3$ . By Lemma 3.2 we have the following lemma.

**Lemma 3.3.** *Let  $X$  be an  $n$ -dimensional Alexandrov space with curvature  $\geq 1$  and  $n \geq 2$ . Suppose  $a_{2p-1}(X) = a_{2p-1}(S^n) = \frac{2p^2-2p+1}{(2p-1)^2} \pi$ . Then  $X$  is isometric to  $\Sigma_2 Z$ , where  $Z$  is an  $(n-2)$ -dimensional Alexandrov space with curvature  $\geq 1$ .*

*Proof.* By Lemma 3.1 we may write  $X = \Sigma_1 Y$ . Suppose  $\text{diam} Y < \pi$ . In the proof of Lemma 3.1 we may take a point  $p$  as an arbitrary regular point of  $X$ , and the set of regular points is dense in  $X$ . If the base point  $p \in X$  is shifted, the points  $x_1, x_2, \dots, x_{2p-1}$  that realize  $a_{2p-1}(X)$  can be moved. Then there exists another pair of points  $x_0, x_i (i = 1, 2, \dots, 2p-1)$  whose distance is equal to  $\pi$ . This contradicts Lemma 3.2. Therefore, we have  $\text{diam} Y = \pi$  and  $X = \Sigma_2 Z$ .  $\square$

We prepare one more lemma.

**Lemma 3.4.** *Let  $X$  be an  $n$ -dimensional Alexandrov space with curvature  $\geq 1$ . Suppose  $a_{2p-1}(X) = a_{2p-1}(S^n) = \frac{2p^2-2p+1}{(2p-1)^2} \pi$ . If  $X$  is isometric to  $S^{k-1} * Y^{n-k} (1 \leq k \leq n-1)$ , where  $Y^{n-k}$  is an  $(n-k)$ -dimensional Alexandrov space with curvature  $\geq 1$ , then  $\text{diam} Y^{n-k} = \pi$ .*

*Proof.* Suppose  $\text{diam} Y^{n-k} < \pi$ . Take any points  $(x, y, t_1), (x', y', t_2) \in X (x, x' \in S^{k-1}, y, y' \in Y^{n-k}, 0 \leq t_1, t_2 \leq \pi/2)$ , and set

$$l = \text{dist}((x, y, t_1), (x', y', t_2)).$$

Then we will show that  $l = \pi$  holds exactly when  $t_1 = t_2 = 0$  and  $\text{dist}(x, x') = \pi$ , namely,  $(x, y, t_1), (x', y', t_2)$  are antipodal pair of  $S^{k-1}$ . Indeed, by the distance formula (3.5) we have

$$\begin{aligned} -1 &= \cos l \\ (3.9) \quad &= \cos t_1 \cos t_2 \cos \text{dist}(x, x') + \sin t_1 \sin t_2 \cos \text{dist}(y, y') \\ &\geq \cos(\pi + t_1 - t_2) \geq -1. \end{aligned}$$

Then we have  $t_1 = t_2 = 0$  because of  $\cos(\text{dist}(y, y')) > -1$ , and also  $\text{dist}(x, x') = \pi$  holds.

Now in the proof of Lemma 3.1 we may choose a point  $p$  arbitrarily as long as  $p$  is regular. Since  $\text{rad} X = a_1(X) \geq a_{2p-1}(X) = (2p^2 - 2p + 1)\pi / (2p - 1)^2 > \pi/2$ ,  $X$  is homeomorphic to  $S^n$  by the radius sphere theorem, and



the set  $S$  of non regular points is a closed set of dimension  $\leq n - 2$  ([G-P1]). Also we may choose identification between  $S_p$  and  $S_{\bar{p}} = S^{n-1}$  up to isometries of  $S^{n-1}$ . If the point  $p \in X$  is shifted or  $v_1, v_2, \dots, v_{2p-1}$  are rotated around  $p$  in  $S_p = S^{n-1}$ , the points  $x_1, x_2, \dots, x_{2p-1}$  that realize  $a_{2p-1}(X)$  can be moved outside of  $S^{k-1}$ . Then there exists another pair of points  $x_0, x_i (i = 1, 2, \dots, 2p - 1)$  with  $dist(x_0, x_i) = \pi$  such that they are not antipodal pair in  $S^{k-1}$ . This contradicts  $diam Y^{n-k} < \pi$ . It follows that  $diam Y^{n-k} = \pi$  holds.  $\square$

Now we show that  $X$  is isometric to  $S^n$ .

*Proof of Theorem 1.5.* By Lemma 3.1 and Lemma 3.3 we have  $X = S^0 * Y^{n-1} = S^1 * Y^{n-2}$ . Next we assume that  $X = S^{k-1} * Y^{n-k}$  holds for  $k (1 \leq k \leq n - 1)$ . By Lemma 3.4 we have  $diam Y^{n-k} = \pi$ . It follows that  $X = S^k * Y^{n-k-1}$ . By induction on  $k$  we see that  $X$  is isometric to  $S^{n-2} * Y^1$ . Since  $rad X = rad(\Sigma_1 Y^{n-1}) = rad Y^{n-1}$ , we have  $rad Y^{n-1} > \pi/2$  (see [G-P1]). It follows that  $rad Y^1 > \pi/2$  and  $Y^1$  is homeomorphic to the circle  $S^1$ . By Lemma 3.4  $diam Y^1 = \pi$  and therefore  $Y^1$  is isometric to  $S^1$ . It follows that  $X$  is isometric to  $S^n$ . This completes the proof of Theorem 1.5.  $\square$

**Remark 3.1.** *If  $a_{2p-1}(X)$  is close to  $a_{2p-1}(S^n) = (2p^2 - 2p + 1)\pi / (2p - 1)^2 > \pi/2$ , then  $X$  is homeomorphic to  $S^n$  since  $rad X > \pi/2$ .*

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