Let $S^2$ be the unit sphere of $\mathbb{R}^3 = \{(u_1, u_2, u_3)\}$. Let $x_1, \ldots, x_{2p-1}$ be the points on $S^1 = \{u_3 = 0\} \cap S^2$ which are equally spaced:

$$x_l = \left(\cos \frac{2(l-1)\pi}{2p-1}, \sin \frac{2(l-1)\pi}{2p-1}, 0\right), \quad 1 \leq l \leq 2p-1.$$ 

Define the function $f(x)$ on $S^2$ by

$$f(x) = \sum_{i=1}^{2p-1} \text{dist}(x_i, x), \quad x \in S^2.$$ 

The aim of this appendix is to give a proof to the following theorem, which is Theorem 2.1 in Sochi’s paper [So].

**Theorem 1.** The function $f(x)$ takes its maximum at $x \in S^2$ when and only when $x = \bar{x}_l$ ($1 \leq l \leq 2p-1$).

Here, for any point $x \in S^2$, we denote by $\bar{x}$ its antipodal point. For the proof, we use the following isometries on $S^2$:

- $\phi_1$: the rotation with angle $\frac{2\pi}{2p-1}$ around the $u_3$-axis.
- $\phi_2$: the rotation with angle $\pi$ around the $u_3$-axis.
- $\sigma$: the reflection with respect to the plane $u_2 = 0$.
- $\tau$: the reflection with respect to the plane $u_3 = 0$.

**Lemma 1.**

1. $f(\phi_1(x)) = f(x)$.
2. $f(\phi_2(x)) = (2p-1)\pi - f(x)$.
3. $f(\sigma(x)) = f(x), \quad f(\tau(x)) = f(x)$.

**Proof.** Since $\phi_1, \sigma,$ and $\tau$ preserves the set $\{x_l\}$, (1) and (3) follows. (2) follows from the facts that $\phi_2(x) = \tau(\bar{x})$ and $\text{dist}(y, x) + \text{dist}(y, \bar{x}) = \pi$ for any $y \in S^2$. \qed

Let $\Omega$ be the geodesic triangle whose vertices are $x_1, \bar{x}_{p+1}$, and $N = (0, 0, 1)$, which is a fundamental domain of the group generated by $\phi_1, \sigma,$ and $\tau$. Let $\psi$ be the rotation around the $u_3$-axis with angle $\pi/(2p - 1)$. Then clearly $\psi^2 = \phi_1$. Moreover, we have the following

**Lemma 2.** $\psi = \phi_2 \circ \phi_1^p$.

**Proof.** Since $\frac{2p\pi}{2p-1} + \pi \equiv \frac{\pi}{2p-1}$ (mod $2\pi$), the lemma follows. \qed
The following corollary is an immediate consequence of the lemmas above.

**Corollary 1.** $f(\psi \circ \sigma(x)) = (2p - 1)\pi - f(x)$ for any $x \in S^2$.

Now, suppose that the function $f(x)$ takes its maximum at $x = z \in \Omega$. We shall prove that $z = \bar{x}_{p+1}$. First, we have

$$f(z) = \sum_{i=1}^{2p-1} \text{dist}(x_i, z) \geq f(\bar{x}_{p+1}) = \frac{2p^2 - 2p + 1}{2p - 1} \pi$$

by the definition of $z$. Put $z' = \psi(\sigma(\bar{z}))$. Note that $\psi \circ \sigma$ is the reflection with respect to the great circle passing through $N$ and the midpoint $y_1$ of $x_1$ and $\bar{x}_{p+1}$. Since the distance between $z$ and this great circle is equal to or less than $\pi / (2p - 1)$, so is the distance between $\bar{z}$ and this great circle. Therefore we have

$$(0.2) \quad \text{dist}(z', \bar{z}) \leq \frac{\pi}{2p - 1}$$

and

$$(0.3) \quad \text{dist}(z', z) = \pi - \text{dist}(z', \bar{z}) \geq \frac{2p - 2}{2p - 1} \pi,$$

and equality holds if and only if $z = x_1$ or $\bar{x}_{p+1}$. By Corollary 1 we also have

$$(0.4) \quad f(z') = (2p - 1)\pi - f(\bar{z}) = f(z).$$

Now, let us consider the sum of distances of two points in the $2p+1$ points $x_1, \cdots, x_{2p-1}, z, z'$:

$$\sum_{i<j} \text{dist}(x_i, x_j) + \sum_i \text{dist}(x_i, z) + \sum_i \text{dist}(x_i, z') + \text{dist}(z, z')$$

$$= p(p - 1)\pi + f(z) + f(z) + \text{dist}(z, z')$$

$$\geq p(p - 1)\pi + \frac{4p^2 - 4p + 2}{2p - 1} \pi + \frac{2p - 2}{2p - 1} \pi \quad \text{(by (1) and (3))}$$

$$= p(p + 1)\pi.$$

Since the value in the right-hand side in the above inequality is equal to $\binom{2p+1}{2} \text{xt}_{2p+1}(S^2)$, where $\text{xt}_{2p+1}(S^2)$ is the invariant of Grove-Markvosen (see Theorem 1.3 of [So]). Therefore equality holds in the above inequality, and we have $z = x_1$ or $\bar{x}_{p+1}$. Since

$$f(x_1) = \frac{2p^2 - 2p}{2p - 1} \pi < f(\bar{x}_{p+1}),$$

We consequently obtain $z = \bar{x}_{p+1}$. 

APPENDIX

REFERENCES


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(Received September 29, 2004)