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APPENDIX TO "SOME METRIC INVARIANTS OF SPHERES AND ALEXANDROV SPACES II"

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Let S^2 be the unit sphere of $\mathbb{R}^3 = \{(u_1, u_2, u_3)\}$. Let $x_1, \cdots, x_{2p-1}$ be the points on $S^1 = \{u_3 = 0\} \cap S^2$ which are equally spaced:

$$x_l = \left(\cos \frac{2(l-1)\pi}{2p-1}, \sin \frac{2(l-1)\pi}{2p-1}, 0\right), \quad 1 \le l \le 2p-1.$$

Define the function $f(x)$ on S^2 by

$$f(x) = \sum_{i=1}^{2p-1} dist(x_i, x), \quad x \in S^2$$

The aim of this appendix is to give a proof to the following theorem, which is Theorem 2.1 in Sochi's paper [So].

Theorem 1. *The function $f(x)$ takes its maximum at $x \in S^2$ when and only when $x = \bar{x}_l \quad (1 \le l \le 2p-1)$.*

Here, for any point $x \in S^2$, we denote by $\bar{x}$ its antipodal point. For the proof, we use the following isometries on S^2:

ϕ_1: the rotation with angle $\frac{2\pi}{2p-1}$ around the u_3-axis.

ϕ_2: the rotation with angle π around the u_3-axis.

σ: the reflection with respect to the plane $u_2 = 0$.

τ: the reflection with respect to the plane $u_3 = 0$.

Lemma 1.

(1) $f(\phi_1(x)) = f(x)$.
(2) $f(\phi_2(x)) = (2p-1)\pi - f(x)$.
(3) $f(\sigma(x)) = f(x), \quad f(\tau(x)) = f(x)$.

Proof. Since ϕ_1, σ, and τ preserves the set $\{x_l\}$, (1) and (3) follows. (2) follows from the facts that $\phi_2(x) = \tau(\bar{x})$ and $dist(y, x) + dist(y, \bar{x}) = \pi$ for any $y \in S^2$. $\square$

Let Ω be the geodesic triangle whose vertices are $x_1, \bar{x}_{p+1}$, and $N = (0, 0, 1)$, which is a fundamental domain of the group generated by ϕ_1, σ, and τ. Let ψ be the rotation around the u_3-axis with angle $\pi/(2p-1)$. Then clearly $\psi^2 = \phi_1$. Moreover, we have the following

Lemma 2. $\psi = \phi_2 \circ \phi_1^p$.

Proof. Since $\frac{2p\pi}{2p-1} + \pi \equiv \frac{\pi}{2p-1} \pmod{2\pi}$, the lemma follows. $\square$

The following corollary is an immediate consequence of the lemmas above.

Corollary 1. *$f(\psi \circ \sigma(x)) = (2p-1)\pi - f(x)$ for any $x \in S^2$.*

Now, suppose that the function $f(x)$ takes its maximum at $x = z \in \Omega$. We shall prove that $z = \bar{x}_{p+1}$. First, we have

$$f(z) = \sum_{i=1}^{2p-1} dist(x_i, z) \geq f(\bar{x}_{p+1}) = \frac{2p^2 - 2p + 1}{2p-1}\pi \tag{0.1}$$

by the definition of z. Put $z' = \psi(\sigma(\bar{z}))$. Note that $\psi \circ \sigma$ is the reflection with respect to the great circle passing through N and the midpoint y_1 of x_1 and $\bar{x}_{p+1}$. Since the distance between z and this great circle is equal to or less than $\pi/2(2p-1)$, so is the distance between $\bar{z}$ and this great circle. Therefore we have

$$dist(z', \bar{z}) \leq \frac{\pi}{2p-1} \tag{0.2}$$

and

$$dist(z', z) = \pi - dist(z', \bar{z}) \geq \frac{2p-2}{2p-1}\pi, \tag{0.3}$$

and equality holds if and only if $z = x_1$ or $\bar{x}_{p+1}$. By Corollary 1 we also have

$$f(z') = (2p-1)\pi - f(\bar{z}) = f(z). \tag{0.4}$$

Now, let us consider the sum of distances of two points in the $2p+1$ points $x_1, \cdots, x_{2p-1}, z, z'$:

$$\begin{aligned}
&\sum_{i<j} dist(x_i, x_j) + \sum_i dist(x_i, z) + \sum_i dist(x_i, z') + dist(z, z') \\
&= p(p-1)\pi + f(z) + f(z) + dist(z, z') \\
&\geq p(p-1)\pi + \frac{4p^2 - 4p + 2}{2p-1}\pi + \frac{2p-2}{2p-1}\pi \quad \text{(by (1) and (3))} \\
&= p(p+1)\pi.
\end{aligned}$$

Since the value in the right-hand side in the above inequality is equal to $\binom{2p+1}{2} xt_{2p+1}(S^2)$, where $xt_{2p+1}(S^2)$ is the invariant of Grove-Markvosen (see Theorem 1.3 of [So]). Therefore equality holds in the above inequality, and we have $z = x_1$ or $\bar{x}_{p+1}$. Since

$$f(x_1) = \frac{2p^2 - 2p}{2p-1}\pi < f(\bar{x}_{p+1}),$$

We consequently obtain $z = \bar{x}_{p+1}$.

References

[So] N. Sochi. *Some metric invariants of spheres and Alexandrov spaces II*, to appear in Math. J. Okayama Univ. **47**.

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