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A NOTE ON QUOTIENTS OF ORTHOGONAL GROUPS

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Abstract. We discuss the mod 2 cohomology of the quotient of a compact classical Lie group by its maximal 2-torus. In particular, the case of the orthogonal group is treated. The case of the spinor group is not included.

1. Introduction.

Let G be a compact simple Lie group. It is well known that for classical G, the cohomology modulo 2 of BG does not have higher 2-torsion ([7]).

According to Adams [1], a subgroup of G is called a 2-torus when it is isomorphic to an elementary abelian 2-group. Let V be a 2-torus of the maximal rank. The rank of V is called 2-rank of G. These two notions are used to study the cohomology ring of a compact Lie group (for instance, [2], [3], [4], [6], [11]). G/V is, for example, connected with calculation of 2-roots, *i.e.* the eigenvalues as functions associated with the restriction of the adjoint representation to V. When $G = SU(n), U(n)$ or $Sp(n)$, it is known by some topologists that G/V does not have higher 2-torsion. But the case of $O(n)$ does not seem so obvious. The purpose of this paper is to show the following theorem.

Theorem 1.1. *$H^*(O(n)/V;\boldsymbol{Z})$ has no higher 2-torsion.*

The corresponding result also holds when one replace $O(n)$ with $SO(n)$. The other classical cases above are also verified similarly to our proof for $O(n)$. The case of $Spin(n)$ seems much complicated. In this paper we will make use of the method of [3], [8] and [5]. We denote the mod 2 cohomology of a space X simply by H^*X.

2. Sq^1-cohomology and the proof.

As is well known, the Serre spectral sequence for the fibration

$$O(n)/V \to BV \to BO(n)$$

collapses with respect to the mod 2 cohomology, and the image of $H^*(BO(n))$ is generated by the elementary symmetric polynomials, i.e. the Stiefel-Whitney classes. Thus let $H^*(BV) = \boldsymbol{F}_2[\,t_1, \ldots, t_n\,]$, and then $H^*(O(n)/V) = \boldsymbol{F}_2[\,t_1, \ldots, t_n\,]\,/\,(w_1, \ldots, w_n)$, where we abuse the same

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symbol t_i for its image. The method of [5] is applicable for computing the Sq^1-cohomology.

Let A_0 be $\boldsymbol{F}_2[t_1, \ldots, t_n]/(w_1)$, which also admits the Sq^1-action as a differential. We sketch the program here. We consider the Sq^1-cohomology of successive quotients of A_0 in a slightly different order so as to regard the multiplication by w_i as a monomorphic Sq^1-cochain map. The multiplication by w_3 on A_0 commutes with the Sq^1-action, since $Sq^1(w_{2i-1}x) = w_{2i-1}Sq^1x$. And since $Sq^1(w_{2i}x) = w_{2i+1}x + w_{2i}Sq^1x$ in A_0, the multiplication by w_2 is a cochain map on $A_1 = A_0/(w_3)$ with respect to Sq^1. Note that if one consider $A_0/(w_2)$ instead of A_1 above, Sq^1 cannot act on it since $Sq^1w_2 = w_3$ in A_0, that is, the ideal in A_0 is not closed under the Sq^1-action. Thus we define elements of A_0 as follows: $g_1 = w_3,\ g_2 = w_2,\ g_3 = w_5,\ g_4 = w_4$ and so on. If n is odd, this definition goes well for all g_k. If n is even, let $g_{n-1} = w_n$. Let A_k be $A_0/(g_1, \ldots, g_k)$, on which the multiplication by g_{k+1} acts as a cochain map. A_{n-1} is isomorphic to $H^*(O(n)/V)$.

Now we begin to calculate $H^*(A_k)$. First, it is immediate to see that $H^*(A_0) = \boldsymbol{F}_2$ and $H^*(A_1) = \bigwedge(w_2)$. Define α_{4i-1} by $\displaystyle\sum_{j_1<j_2<\cdots<j_{2i}} t_{j_1}{t_{j_2}}^2 \cdots {t_{j_{2i}}}^2$ in A_0. This element satisfies $Sq^1\alpha_{4i-1} = {w_{2i}}^2$. We assert

Lemma 2.1.

$$H^*(A_k) = \begin{cases} \bigwedge(\alpha_3, \alpha_7, \ldots, \alpha_{4m-1}) & (k = 2m) \\ \bigwedge(\alpha_3, \alpha_7, \ldots, \alpha_{4m-1}, g_{k+1}) & (k = 2m+1) \end{cases}$$

except for the case n is even and $k = n-1$.

Proof. Note that $g_{k+1} = w_{k+1}$ in the above. We proceed by induction. We have an exact sequence

$$0 \longrightarrow A_{k-1} \xrightarrow{\cdot g_k} A_{k-1} \longrightarrow A_k \longrightarrow 0$$

and hence the resulting long exact sequence

$$\cdots \longrightarrow H^*(A_{k-1}) \xrightarrow{\cdot [g_k]} H^*(A_{k-1}) \longrightarrow H^*(A_k) \longrightarrow H^*(A_{k-1}) \longrightarrow \cdots.$$

If k is odd, $g_k = w_{k+2}$ is 0 in the cohomology because $w_{k+2} = Sq^1w_{k+1}$ in A_{k-1}. Thus the long exact sequence splits into short ones

$$0 \longrightarrow H^*(A_{k-1}) \longrightarrow H^*(A_k) \longrightarrow H^*(A_{k-1}) \longrightarrow 0.$$

It is easy to check that $H^*(A_k) = H^*(A_{k-1}) \otimes \bigwedge(w_{k+1})$, and the inductive step is proved in this case.

If k is even, $H^*(A_{k-1}) = H^*(A_{k-2}) \otimes \bigwedge(g_k)$ and whence the following sequence is exact.

$$0 \longrightarrow H^*(A_{k-2}) \longrightarrow H^*(A_k) \longrightarrow g_k \cdot H^*(A_{k-2}) \longrightarrow 0$$

Then diagram chasing shows $H^*(A_k) = H^*(A_{k-2}) \otimes \bigwedge(\alpha_{4m-1})$. Therefore the lemma is proved. $\square$

Finally we deal with the case n is even and $k = n-1$. In this case $g_k\,(= w_n)$ is a trivial cocycle. To see this, we note $w_i = w'_i + t_n w'_{i-1}$, where w'_i is the i-th elementary symmetric polynomial in $t_1, \ldots, t_{n-1}$. Thus in A_0, $w_n = w'_{n-1}w'_1 = Sq^1 w'_{n-1}$. Moreover, it is easy to see $w'_i = (w'_1)^i = {t_n}^i$ and hence $w'_{n-1} = {t_n}^{n-1}$ in A_{n-2}. We can reason similarly if we take t_i instead of t_n for any i.

Since the multiplication by w_n induces a null homomorphism on cohomology, we have a short exact sequence

$$0 \longrightarrow H^*(A_{n-2}) \longrightarrow H^*(A_{n-1}) \longrightarrow H^*(A_{n-2}) \longrightarrow 0,$$

where $H^*(A_{n-2}) = \bigwedge(\alpha_3, \alpha_7, \ldots, \alpha_{4m-5})$. Again by diagram chasing, we obtain $H^*(A_{n-1}) = H^*(A_{n-2}) \otimes \bigwedge({t_n}^{n-1})$. Summing up all, we have obtained

Proposition 2.2.

$$H^*(H^*(O(n)/V);\, Sq^1) = \begin{cases} \bigwedge(\alpha_3, \alpha_7, \ldots, \alpha_{4m-1}) & (n = 2m+1) \\ \bigwedge(\alpha_3, \alpha_7, \ldots, \alpha_{4m-5}, \beta) & (n = 2m), \end{cases}$$

where β is represented by t_i^{n-1} for arbitrary i.

Here in $H^*(BV)$, $Sq^1(\alpha_{4i-1}) = {w_{2i}}^2$ and $Sq^1(\beta) = t_i^n = w_n$ when n is even, both of which has the image null in $H^*(O(n)/V)$.

In $H^*(H^*(O(n)/V);\, Sq^1)$ the degree of the generators are as follows: $\deg \alpha_{4i-1} = 4i-1$ and $\deg \beta = 2k-1\ (= n-1)$. On the other hand, the rational cohomology of $O(n)/V$ is of the same form. Therefore the Bockstein spectral sequence collapses and $O(n)/V$ does not have higher torsion. It is immediate to see the similar result holds for $SO(n)$.

As in [7], we describe $H^*(O(n)/V; \boldsymbol{Z})$ as a graded module as follows. Put

$$f(t) = \prod_{i=1}^{n} \frac{1-t^i}{1-t},$$

$$g^+(t) = (1 + t^{n-1}) \prod_{i=1}^{k} (1 + t^{4i-1}) = \sum_i g_i^+ t^i \ (g_i^+ \in \boldsymbol{Z}),$$

$$g^-(t) = \prod_{i=1}^{k} (1 + t^{4i-1}) = \sum_i g_i^- t^i \ (g_i^- \in \boldsymbol{Z}),$$

where $k = \max\left\{ i \in \boldsymbol{Z} \mid i \leq \dfrac{n-1}{2} \right\}$. Proposition 3 deduces that these three are the Poincaré polynomials of $H^*(O(n)/V; \boldsymbol{F}_2)$, $H^*(O(n)/V; \boldsymbol{Q})$ for even n, and $H^*(O(n)/V;\boldsymbol{Q})$ for odd n, respectively. There then exist polynomials $r^+(t)=\sum_i r_i^+t^i$ ($r_i^+\in\boldsymbol{Z}$) and $r^-(t)=\sum_i r_i^-t^i$ ($r_i^-\in\boldsymbol{Z}$) such that $f(t)-g^+(t)=\left(1+\frac{1}{t}\right)r^+(t)$ for even n and $f(t)-g^-(t)=\left(1+\frac{1}{t}\right)r^-(t)$ for odd n. (Note that a factorization into monic polynomials in rational coefficients can be realized already in integral coefficients since $\boldsymbol{Z}$ is integrally closed.) Thus we obtain the next corollary.

Corollary 2.3.

$$H^i(O(n)/V;\boldsymbol{Z})=\begin{cases}\boldsymbol{Z}^{g_i^+}\oplus(\boldsymbol{Z}/2\boldsymbol{Z})^{r_i^+} & (n:\mathit{even}),\\ \boldsymbol{Z}^{g_i^-}\oplus(\boldsymbol{Z}/2\boldsymbol{Z})^{r_i^-} & (n:\mathit{odd}).\end{cases}$$

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