A NOTE ON QUOTIENTS OF ORTHOGONAL GROUPS

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Abstract. We discuss the mod 2 cohomology of the quotient of a compact classical Lie group by its maximal 2-torus. In particular, the case of the orthogonal group is treated. The case of the spinor group is not included.

1. Introduction.

Let $G$ be a compact simple Lie group. It is well known that for classical $G$, the cohomology modulo 2 of $BG$ does not have higher 2-torsion ([7]).

According to Adams [1], a subgroup of $G$ is called a 2-torus when it is isomorphic to an elementary abelian 2-group. Let $V$ be a 2-torus of the maximal rank. The rank of $V$ is called 2-rank of $G$. These two notions are used to study the cohomology ring of a compact Lie group (for instance, [2], [3], [4], [6], [11]). $G/V$ is, for example, connected with calculation of 2-roots, i.e. the eigenvalues as functions associated with the restriction of the adjoint representation to $V$. When $G = SU(n), U(n)$ or $Sp(n)$, it is known by some topologists that $G/V$ does not have higher 2-torsion. But the case of $O(n)$ does not seem so obvious. The purpose of this paper is to show the following theorem.

Theorem 1.1. $H^*(O(n)/V; Z)$ has no higher 2-torsion.

The corresponding result also holds when one replace $O(n)$ with $SO(n)$. The other classical cases above are also verified similarly to our proof for $O(n)$. The case of $Spin(n)$ seems much complicated. In this paper we will make use of the method of [3], [8] and [5]. We denote the mod 2 cohomology of a space $X$ simply by $H^*X$.

2. $Sq^1$-cohomology and the proof.

As is well known, the Serre spectral sequence for the fibration

$$O(n)/V \rightarrow BV \rightarrow BO(n)$$

collapses with respect to the mod 2 cohomology, and the image of $H^*(BO(n))$ is generated by the elementary symmetric polynomials, i.e. the Stiefel-Whitney classes. Thus let $H^*(BV) = F_2[t_1, \ldots, t_n]$, and then $H^*(O(n)/V) = F_2[t_1, \ldots, t_n] / (w_1, \ldots, w_n)$, where we abuse the same

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symbol \( t_i \) for its image. The method of [5] is applicable for computing the \( Sq^1 \)-cohomology.

Let \( A_0 \) be \( F_2[t_1, \ldots, t_n]/(w_1) \), which also admits the \( Sq^1 \)-action as a differential. We sketch the program here. We consider the \( Sq^1 \)-cohomology of successive quotients of \( A_0 \) in a slightly different order so as to regard the multiplication by \( w_i \) as a monomorphic \( Sq^1 \)-cochain map. The multiplication by \( w_3 \) on \( A_0 \) commutes with the \( Sq^1 \)-action, since \( Sq^1(w_{2i-1}x) = w_{2i-1}Sq^1x \). And since \( Sq^1(w_{2i}x) = w_{2i+1}x + w_{2i}Sq^1x \) in \( A_0 \), the multiplication by \( w_2 \) is a cochain map on \( A_1 = A_0/(w_3) \) with respect to \( Sq^1 \). Note that if one consider \( A_0/(w_2) \) instead of \( A_1 \) above, \( Sq^1 \) cannot act on it since \( Sq^1w_2 = w_3 \) in \( A_0 \), that is, the ideal in \( A_0 \) is not closed under the \( Sq^1 \)-action. Thus we define elements of \( A_0 \) as follows: \( g_1 = w_3, g_2 = w_2, g_3 = w_5, g_4 = w_4 \) and so on. If \( n \) is odd, this definition goes well for all \( g_k \). If \( n \) is even, let \( g_{n-1} = w_n \). Let \( A_k \) be \( A_0/(g_1, \ldots, g_k) \), on which the multiplication by \( g_{k+1} \) acts as a cochain map. \( A_{n-1} \) is isomorphic to \( H^*(O(n)/V) \).

Now we begin to calculate \( H^*(A_k) \). First, it is immediate to see that \( H^*(A_0) = F_2 \) and \( H^*(A_1) = \text{Ann}(w_2) \). Define \( \alpha_{4i-1} \) by \( \sum_{j_1 < j_2 < \cdots < j_{2i}} t_{j_1}t_{j_2}^2 \cdots t_{j_{2i}}^2 \) in \( A_0 \). This element satisfies \( Sq^1\alpha_{4i-1} = w_{2i}^2 \). We assert

**Lemma 2.1.**

\[
H^*(A_k) = \begin{cases} \\
\wedge(\alpha_3, \alpha_7, \ldots, \alpha_{4m-1}) & (k = 2m) \\
\wedge(\alpha_3, \alpha_7, \ldots, \alpha_{4m-1}, g_{k+1}) & (k = 2m + 1)
\end{cases}
\]

except for the case \( n \) is even and \( k = n - 1 \).

**Proof.** Note that \( g_{k+1} = w_{k+1} \) in the above. We proceed by induction. We have an exact sequence

\[
0 \rightarrow A_{k-1} \stackrel{g_k}{\longrightarrow} A_{k-1} \rightarrow A_k \rightarrow 0
\]

and hence the resulting long exact sequence

\[
\cdots \rightarrow H^*(A_{k-1}) \stackrel{[g_k]}{\longrightarrow} H^*(A_{k-1}) \longrightarrow H^*(A_k) \longrightarrow H^*(A_{k-1}) \longrightarrow \cdots.
\]

If \( k \) is odd, \( g_k = w_{k+2} \) is 0 in the cohomology because \( w_{k+2} = Sq^1w_{k+1} \) in \( A_{k-1} \). Thus the long exact sequence splits into short ones

\[
0 \rightarrow H^*(A_{k-1}) \rightarrow H^*(A_k) \rightarrow H^*(A_{k-1}) \rightarrow 0.
\]

It is easy to check that \( H^*(A_k) = H^*(A_{k-1}) \otimes \text{Ann}(w_{k+1}) \), and the inductive step is proved in this case.

If \( k \) is even, \( H^*(A_{k-1}) = H^*(A_{k-2}) \otimes \text{Ann}(g_k) \) and whence the following sequence is exact.

\[
0 \rightarrow H^*(A_{k-2}) \rightarrow H^*(A_k) \rightarrow g_k \cdot H^*(A_{k-2}) \rightarrow 0
\]
Then diagram chasing shows $H^*(A_k) = H^*(A_{k-2}) \otimes \wedge(\alpha_{4m-1})$. Therefore the lemma is proved.

Finally we deal with the case $n$ is even and $k = n - 1$. In this case $g_k (= w_n)$ is a trivial cocycle. To see this, we note $w_i = w'_i + t_n w'_{i-1}$, where $w'_i$ is the $i$-th elementary symmetric polynomial in $t_1, \ldots, t_{n-1}$. Thus in $A_0$, $w_n = w'_{n-1}w'_1 = Sq^1 w'_{n-1}$. Moreover, it is easy to see $w'_i = (w'_1)^i = t_n^i$ and hence $w'_{n-1} = t_n^{n-1}$ in $A_{n-2}$. We can reason similarly if we take $t_i$ instead of $t_n$ for any $i$.

Since the multiplication by $w_n$ induces a null homomorphism on cohomology, we have a short exact sequence

$$0 \rightarrow H^*(A_{n-2}) \rightarrow H^*(A_{n-1}) \rightarrow H^*(A_{n-2}) \rightarrow 0,$$

where $H^*(A_{n-2}) = \wedge(\alpha_3, \alpha_7, \ldots, \alpha_{4m-5})$. Again by diagram chasing, we obtain $H^*(A_{n-1}) = H^*(A_{n-2}) \otimes \wedge(t_n^{n-1})$. Summing up all, we have obtained

**Proposition 2.2.**

$$H^*(H^*(O(n)/V); Sq^1) = \left\{ \begin{array}{ll} \wedge(\alpha_3, \alpha_7, \ldots, \alpha_{4m-1}) & (n = 2m + 1) \\ \wedge(\alpha_3, \alpha_7, \ldots, \alpha_{4m-5}, \beta) & (n = 2m), \end{array} \right.$$ 

where $\beta$ is represented by $t_n^{n-1}$ for arbitrary $i$.

Here in $H^*(BV)$, $Sq^1(\alpha_{4i-1}) = w_{2i}^2$ and $Sq^1(\beta) = t_n^n = w_n$ when $n$ is even, both of which has the image null in $H^*(O(n)/V)$.

In $H^*(H^*(O(n)/V); Sq^1)$ the degree of the generators are as follows: $\deg \alpha_{4i-1} = 4i - 1$ and $\deg \beta = 2k - 1 (= n - 1)$. On the other hand, the rational cohomology of $O(n)/V$ is of the same form. Therefore the Bockstein spectral sequence collapses and $O(n)/V$ does not have higher torsion. It is immediate to see the similar result holds for $SO(n)$.

As in [7], we describe $H^*(O(n)/V; \mathbb{Z})$ as a graded module as follows. Put

$$f(t) = \prod_{i=1}^{n} \frac{1 - t^i}{1 - t};$$

$$g^+(t) = (1 + t^{n-1}) \prod_{i=1}^{k} (1 + t^{4i-1}) = \sum_{i} g^+_i t^i \ (g^+_i \in \mathbb{Z}),$$

$$g^-(t) = \prod_{i=1}^{k} (1 + t^{4i-1}) = \sum_{i} g^-_i t^i \ (g^-_i \in \mathbb{Z}),$$

where $k = \max \left\{ i \in \mathbb{Z} \mid i \leq \frac{n-1}{2} \right\}$. Proposition 3 deduces that these three are the Poincaré polynomials of $H^*(O(n)/V; F_2)$, $H^*(O(n)/V; \mathbb{Q})$ for
even \( n \), and \( H^*(O(n)/V; Q) \) for odd \( n \), respectively. There then exist polynomials 
\[ r^+(t) = \sum_i r^+_i t^i (r^+_i \in \mathbb{Z}) \quad \text{and} \quad r^-(t) = \sum_i r^-_i t^i (r^-_i \in \mathbb{Z}) \]

such that 
\[ f(t) - g^+(t) = \left(1 + \frac{1}{t}\right)r^+(t) \]
for even \( n \) and 
\[ f(t) - g^-(t) = \left(1 + \frac{1}{t}\right)r^-(t) \]
for odd \( n \). (Note that a factorization into monic polynomials in rational coefficients 
can be realized already in integral coefficients since \( \mathbb{Z} \) is integrally closed.) Thus we obtain the next corollary.

**Corollary 2.3.**

\[
H^i(O(n)/V; \mathbb{Z}) = \begin{cases} 
\mathbb{Z}^{g^+_i} \oplus (\mathbb{Z}/2\mathbb{Z})^{r^+_i} & (n \text{: even}), \\
\mathbb{Z}^{g^-_i} \oplus (\mathbb{Z}/2\mathbb{Z})^{r^-_i} & (n \text{: odd}).
\end{cases}
\]

**References**


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