A LOWERBOUND FOR THE LS CATEGORY OF A FORMAL ELLIPTIC SPACE

Dedicated to the memory of Professor Akie TAMAMURA

YASUSUKE KOTANI AND TOSHIHIRO YAMAGUCHI

Abstract. We give a lower bound for the LS category of a formal elliptic space in terms of its rational cohomology.

1. Introduction

The Lusternik-Schnirelmann (LS) category, \( \text{cat} \ X \), for a space \( X \) is the least integer \( m \) such that \( X \) can be covered by \( m + 1 \) open sets, each contractible in \( X \). The rational LS category, \( \text{cat}_0(X) \), is the least integer \( n \) such that \( X \simeq_0 Y \) and \( \text{cat} Y = n \). A simply connected CW complex \( X \) is called (rationally) elliptic if \( \dim H^*(X; \mathbb{Q}) < 1 \) and \( \dim \pi_*(X) \otimes \mathbb{Q} < 1 \). An elliptic space \( X \) has a positive Euler characteristic, i.e.,

\[
\sum_i (-1)^i \dim H^i(X; \mathbb{Q}) > 0,
\]

if and only if it satisfies \( \chi(X) = \sum_i (-1)^i \dim \pi_i(X) \otimes \mathbb{Q} = 0 \) [6]. Then it is called often an \( "F_0\)-space". For example, a homogeneous space \( G/H \), where \( G \) is a compact, connected Lie group and \( H \) is a closed subgroup of maximal rank, is an \( F_0 \)-space. The rational cup length of a space \( Z \), \( \text{cup}_0(Z) \), is the greatest integer \( n \) such that the \( n \)-product \( H^+ \cdots H^+(Z; \mathbb{Q}) \neq 0 \).

Also the rational Toomer invariant of \( Z \), \( e_0(Z) \), is given by using the Sullivan minimal model [4] \( M(Z) = (\vee V, d) \) as \( \sup \{ n \mid \text{there is an element } \alpha \in \wedge^n V \text{ such that } [\alpha] \neq 0 \text{ in } H^*(Z; \mathbb{Q}) \} \).

We remark that

\[
\text{cup}_0(Y) \leq e_0(Y) = \text{cat}_0(Y) \leq \text{cat} Y
\]

for an elliptic space \( Y \) (see [3]).

There is a problem [9]: \textit{If } \( Y \text{ is elliptic, then } \dim H^*(Y; \mathbb{Q}) \leq 2^{\text{cat} Y}. \)

It is true if \( \dim \pi_{\text{even}}(Y) \otimes \mathbb{Q} \leq 1 \) [9].

Lemma. For an \( F_0 \)-space \( X \), \( \dim H^*(X; \mathbb{Q}) \leq 2^{\text{cup}_0(X)} \).

Especially, if \( X \) has the homotopy type of the \( r \)-product of even dimensional spheres, then \( \dim H^*(X; \mathbb{Q}) = 2^r = 2^{\text{cup}_0 X} = 2^{\text{cat} X} \).

If an elliptic space \( Y \) is formal [4, p.156], roughly speaking, if the Sullivan minimal model [4] is a formal consequence of its rational cohomology, it has the rational homotopy type of the total space of a fibration \( X \to E \to S \) in which \( X \) is an \( F_0 \)-space and \( S \) is the product of odd dimensional spheres or the one point space (see [2]). Then we have
Theorem. If an elliptic space $Y$ is formal, then $\dim H^*(Y; \mathbb{Q}) \leq 2^{\text{cup}_0(Y)}$.

From the above remark (*), we deduce a partial answer for our problem.

Corollary. If an elliptic space $Y$ is formal, then $\dim H^*(Y; \mathbb{Q}) \leq 2^{\text{cat} Y}$.

In the next section, we give the proofs. In the third section, we compare our results with the toral rank conjecture [4, p.516] of Halperin for certain spaces.

The authors thank our referee for his suitable advices.

2. Proofs

Let $X$ be an $F_0$-space. Then there is an isomorphism as algebras

$$H^*(X; \mathbb{Q}) \cong \mathbb{Q}[x_1, \ldots, x_p]/(f_1, \ldots, f_p)$$

and there is an equation

$$\dim H^*(X; \mathbb{Q}) = |f_1| \cdots |f_p|/|x_1| \cdots |x_p|$$

where $\{|x_i|\}_i$ are all even and $f_1, \ldots, f_p$ is a regular sequence [6]. Here $|*|$ is the degree of $*$ as an element in a graded algebra and $f_i$ are homogeneous polynomials with no linear terms.

Proof of Lemma. Suppose that $|x_1| \leq \cdots \leq |x_p|$. Let $\Phi_i$ denote the set of all monomials occurring in $f_i$, i.e., $f_i = \sum_j c_{ij} \sigma_{ij}$ for some $c_{ij} \neq 0 \in \mathbb{Q}$ and $\sigma_{ij} \in \Phi_i$. Since $\dim \mathbb{Q}[x_1, \ldots, x_p]/(f_1, \ldots, f_p) < \infty$, sets $\Phi_1, \ldots, \Phi_p$ must satisfy the “polynomial condition” P.C. [5, p.119] due to Friedlander and Halperin: for each $s$ and for each set of $s$ variables $x_{i_1}, \ldots, x_{i_s}$, there are at least $s$ sets $\Phi_{j_1}, \ldots, \Phi_{j_s}$ containing a monomial in $\mathbb{Q}[x_{i_1}, \ldots, x_{i_s}]$ [5, Theorem 3]. By changing the indexes of $\Phi_i$’s, we can regard $\Phi_i$ as an element of $\mathbb{Q}[x_1, \ldots, x_i]$ for any $i$. Thus we may assume that each $f_i$ contains a term of the form $x_1^{k_{i_1}} \cdots x_i^{k_{i_i}}$ where $k_{i_1}, \ldots, k_{i_i}$ are non-negative integers with $k_{i_1} + \cdots + k_{i_i} \geq 2$. Then, for each $1 \leq i \leq p$, we have

$$|f_i|/|x_i| = (k_{i_1}|x_1| + \cdots + k_{i_i}|x_i|)/|x_i|$$

$$\leq k_{i_1} + \cdots + k_{i_i}$$

$$\leq 2^{k_{i_1} + \cdots + k_{i_i} - 1}$$

$$\leq 2^{\deg f_i - 1},$$

where $\deg f_i$ is the degree of $f_i$ as a polynomial. Thus

$$\dim H^*(X; \mathbb{Q}) = |f_1| \cdots |f_p|/|x_1| \cdots |x_p| \leq 2^{\deg f_1 + \cdots + \deg f_p - p}.$$ 

Since the Jacobian $\det(\partial f_i/\partial x_j)$ is the fundamental class of the Poincaré duality algebra [8, Proposition 3], we have $\deg f_1 + \cdots + \deg f_p - p \leq \text{cup}_0(X)$. 

□
Recall the following Lemma in [9] for the proof of Theorem.

**Lemma 2.1** ([9, Lemma 2.1]). Let \( E \) be the total space of the rational fibration \( F \to E \to S^{2n+1} \) with an elliptic space \( F \) satisfying \( \dim H^*(F; Q) \leq 2^{\epsilon_0(F)} \). Then \( \dim H^*(E; Q) \leq 2^{\epsilon_0(E)} \).

**Proof of Theorem.** Since a formal elliptic space \( Y \) is hyperformal, there is an isomorphism as algebras

\[
H^*(Y; Q) \cong Q[x_1, \ldots, x_p] \otimes \wedge(y_1, \ldots, y_q)/(h_1, \ldots, h_p) \quad (p, q \geq 0),
\]

where the elements \( h_i \) \((i = 1, \ldots, p)\) are written as \( h_i = f_i + g_i \) with a regular sequence \( f_1, \ldots, f_p \) in \( Q[x_1, \ldots, x_p] \) and elements \( g_i \) in the ideal generated by \( y_1, \ldots, y_q \) [2, p.576–577]. We regard the algebra \( H^*(Y; Q) \) as the exterior algebra \( \wedge(y_1, \ldots, y_q) \) if \( p = 0 \). Thus \( Y \) has the rational homotopy type of the total space of a fibration

\[
X \to E_q \to S^{[y_1]} \times \cdots \times S^{[y_q]}.
\]

where \( X \) is the \( F_0 \)-space with \( H^*(X; Q) \cong Q[x_1, \ldots, x_p]/(f_1, \ldots, f_p) \) and \( S^{[y_i]} \) is the \( [y_i] \)-dimensional sphere. If \( q > 0 \), by using Lemma 2.1 with [4, p.388, Example 4] inductively for fibrations

\[
X \to E_1 \to S^{[y_1]}
\]

\[
E_i \to E_{i+1} \to S^{[y_{i+1}]}
\]

\[
E_{q-1} \to E_q \to S^{[y_q]},
\]

we have \( \dim H^*(E_i; Q) \leq 2^{\cup_{0}(E_i)} \) for \( i = 1, \ldots, q \). Here each \( E_i \) is an elliptic space with

\[
H^*(E_i; Q) \cong Q[x_1, \ldots, x_p] \otimes \wedge(y_1, \ldots, y_i)/(h_1, \ldots, h_p),
\]

where \( h_j \) is the obvious projection of \( h_j \). \( \square \)

### 3. Toral rank vs LS category

Let \( X \) be a simply connected finite cell complex. Recall that the toral rank of a space \( X \), \( \text{rk}(X) \), is the largest integer \( n \) such that an \( n \)-torus can act continuously on \( X \) with all its isotropy subgroups finite. In [7], S. Halperin conjectured that \( 2^{\text{rk}(X)} \leq \dim H^*(X; Q) \), which gives an upper bound for toral rank. We compare the two bounds around formal elliptic spaces in the following table. Refer [4, Part II and Section 32] for the Sullivan minimal model theory.
<table>
<thead>
<tr>
<th>$X$: elliptic</th>
<th>$2^{\text{rk}(X)} \leq \dim H^*(X; \mathbb{Q})$</th>
<th>$\dim H^*(X; \mathbb{Q}) \leq 2^{\text{cat} X}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\pi_{\text{even}}(X) \otimes \mathbb{Q} = 0$</td>
<td>? (a)</td>
<td>yes ([9])</td>
</tr>
<tr>
<td>$X$: formal</td>
<td>$\chi_\pi(X) = 0$</td>
<td>yes (since $\text{rk}(X) = 0$)</td>
</tr>
<tr>
<td></td>
<td>$\chi_\pi(X) &lt; 0$</td>
<td>yes (Lemma)</td>
</tr>
<tr>
<td>$X$: pure</td>
<td>yes ([7, Proposition 1.5])</td>
<td>yes (Corollary)</td>
</tr>
</tbody>
</table>

Here $X$ is called pure [4, p.435] if the differential $d$ satisfies $dV^{\text{even}} = 0$ and $dV^{\text{odd}} \subseteq \wedge V^{\text{even}}$ for the Sullivan minimal model $\mathcal{M}(X) = (\wedge V, d)$ of $X$, where $V = \oplus_{i>1} V^i$ with $\dim V^i = \dim \pi_i(X) \otimes \mathbb{Q}$. For example, a homogeneous space is pure. Note a pure space with $\dim V^{\text{even}} = \dim V^{\text{odd}}$ is an $F_0$-space.

(a) is “yes” if $X$ is a space of two-stage Sullivan minimal model and coformal, i.e., $V$ decomposes as $V \cong U \oplus W$ with $dU = 0$, $dW \subseteq \wedge U$ and $d$ is quadratic, respectively [1, Proposition 3.1].

(b) is obtained from $\text{rk}(X) \leq -\chi_\pi(X) = -(\dim V^{\text{even}} - \dim V^{\text{odd}}) = q$ [7, Theorem 1.1] and $\dim H^*(X; \mathbb{Q}) \geq 2^q$ when $H^*(X; \mathbb{Q})$ is given by $\mathbb{Q}[x_1, \ldots, x_p] \otimes \wedge \langle y_1, \ldots, y_q \rangle/(h_1, \ldots, h_p)$, where $\mathcal{M}(X) = (\wedge(x_1, \ldots, x_p, y_1, \ldots, y_q, v_1, \ldots, v_p), d)$ with $dx_i = dy_i = 0$, $dv_i = h_i$.

(c) is “yes” if $\dim V^{\text{even}} = 1$, because then there is an isomorphism $H^*(X; \mathbb{Q}) \cong \mathbb{Q}[x]/(x^n) \otimes \wedge(y_1, \ldots, y_q)$.

References

A LOWER BOUND FOR THE LS CATEGORY OF A FORMAL ELLIPTIC SPACE

Yasusuke Kotani
Faculty of Science
Kochi University
Kochi 780-8520, Japan
e-mail address: kotani@math.kochi-u.ac.jp

Toshihiro Yamaguchi
Faculty of Education
Kochi University
Kochi 780-8520, Japan
e-mail address: tyamag@cc.kochi-u.ac.jp

(Received September 28, 2004)
(Revised November 28, 2004)