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CONNECTIVE COVERINGS OF A FEW CELL COMPLEXES

Kohhei YAMAGUCHI

Abstract. We shall determine the 2-connective coverings of a few cell complexes of the form $S^2 \cup_f e^n$ for $n \geq 4$ and $0 \neq f \in \pi_{n-1}(S^2)$.

1. Introduction.

The principal motivation of this paper comes from the work due to J. Wu [7], who showed that the 2-connective covering of $L_m = S^2 \cup_{m\eta_2} e^4$ is homotopy equivalent to $\mathrm{P}^4(m) \vee S^5$, where $\eta_2 \in \pi_3(S^2)$ is the Hopf map map and $\mathrm{P}^{k+1}(m)$ denotes the Moore space of type $(k, \mathbb{Z}/m)$ given by $\mathrm{P}^{k+1}(m) = S^k \cup_{m\iota_k} e^{k+1}$. We would like to generalize his result for all 2-cell complexes X of the form $X = S^2 \cup_f e^n$ ($n \geq 4$, $0 \neq f \in \pi_{n-1}(S^2)$). Since the induced homomorphism $\eta_{2*} : \pi_k(S^3) \xrightarrow{\cong} \pi_k(S^2)$ is an isomorphism for any $k \geq 2$, there is a unique element $g \in \pi_{n-1}(S^3)$ such that $\eta_2 \circ g = f$. Then the main purpose of this note is to show the following result.

Theorem 1.1. *Let $n \geq 4$ be an integer and let X be a 2-cell complex of the form $X = S^2 \cup_f e^n$ $(0 \neq f \in \pi_{n-1}(S^2))$. Then if $\tilde{X}$ denotes the 2-connective covering of X, there is a homotopy equivalence*

$$\tilde{X} \simeq S^3 \cup_g e^n \vee S^{n+1}, \tag{1.1}$$

where the map $g \in \pi_{n-1}(S^3)$ satisfies the condition $\eta_2 \circ g = f$.

Corollary 1.2. *Under the same assumptions as Theorem 1.1, we have:*

(1) *If $X = S^2 \cup_{m\eta_2} e^4$, $\tilde{X} \simeq \mathrm{P}^4(m) \vee S^5$.*
(2) *If $X = S^2 \cup_{\eta_2^2} e^5$, $\tilde{X} \simeq S^3 \cup_{\eta_3} e^5 \vee S^6$.*
(3) *If $X = S^2 \cup_{\eta_2^3} e^6$, $\tilde{X} \simeq S^3 \cup_{\eta_3^2} e^6 \vee S^7$.*
(4) *If $X = S^2 \cup_{\eta_2 \circ \omega} e^7$, $\tilde{X} \simeq S^3 \cup_\omega e^7 \vee S^8$, where $\omega \in \pi_6(S^3) \cong \mathbb{Z}/12$ denotes Blackers-Massey element.*

Remark. (1) Let $q : S^2 \cup_f e^n \to S^n$ be the pinch map and F_f be its homotopy fiber. It is known that the $(n+2)$-skeleton of F_f is homotopy equivalent to $S^2 \vee S^{n+1}$ ([2]). This fact may be closely related to the statement of

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Theorem 1.1 although we cannot explain it clearly. It is also known that $[f_1, f_2] = 0$ for any $f_1 \in \pi_k(S^2)$, $f_2 \in \pi_l(S^2)$ if $(k,l) \neq (2,2)$ ([3]), and this fact is a crucial point for our proof of Theorem 1.1.

(2) This result will be used for studying the problem of homotopy type classifications of m-twisted complex projective spaces in [5]. In fact, if we use this result, we can extend the dimension that James excision isomorphism holds (cf. [4]) and it may be useful for computing higher homotopy groups$\pi_*(S^2 \cup_f e^n)$ without using Gray's method [2].

2. The case $n \geq 5$.

Let $n \geq 4$ be an integer and consider the space $X = S^2 \cup_f e^n$ $(0 \neq f \in \pi_{n-1}(S^2))$. Let $\iota_f \in [X, \mathbb{C}\mathrm{P}^\infty] \cong H^2(X,\mathbb{Z}) \cong \mathbb{Z}$ be the map which represents the generator and let $\tilde{X}$ be the homotopy fiber of the map ι_f. It is easy to see that $\tilde{X}$ is a 2-connective covering of X and there is a fibration sequence

$$S^1 \to \tilde{X} \xrightarrow{\varphi} X. \tag{2.1}$$

First, we treat the case $n \geq 5$. (The case $n = 4$ will be considered in the next section.) If we consider the Serre spectral sequence associated to (2.1), we have

$$H^k(\tilde{X},\mathbb{Z}) \cong \begin{cases} \mathbb{Z} & \text{if } k = 0, 3, n, n+1 \\ 0 & \text{otherwise} \end{cases}$$

and we obtain a homotopy equivalence

$$\tilde{X} \simeq S^3 \cup_g e^n \cup_\theta e^{n+1} = K \cup_\theta e^{n+1} \qquad (g \in \pi_{n-1}(S^3), \theta \in \pi_n(K)), \tag{2.2}$$

where we write $K = S^3 \cup_g e^n$. In this case, without loss of generalities, we may identify $\tilde{X} = S^3 \cup_g e^n \cup_\theta e^{n+1} = K \cup_\theta e^{n+1}$ and we may also suppose that φ is a cellular map. Then because $\varphi(K) \subset X$, there is a commutative diagram

$$\begin{array}{ccc} K & \xrightarrow{j} & \tilde{X} \\ \| & & \downarrow{\scriptstyle \varphi} \\ K & \xrightarrow{\varphi_1} & X \end{array}$$

where $j : K = S^3 \cup_g e^n \to \tilde{X}$ denotes the inclusion. Furthermore, since the 3-skeleton of X is S^2, $\varphi(S^3) \subset S^2$. Hence, the map φ_1 also defines the map $\overline{\varphi} : (K, S^3) \to (X, S^2)$.

Lemma 2.1. *$\varphi_{1*} : \pi_n(K) \to \pi_n(X)$ is a surjective homomorphism.*

Proof. Since $n \geq 5$, $(\tilde{X}, S^3)$ and (X, S^2) are at least 4-connected. Hence, if we consider the commutative diagram

$$\begin{array}{ccc}
\pi_3(S^3) & \xrightarrow{\cong} & \pi_3(\tilde{X}) \\
{\scriptstyle (\varphi|S^3)_*}\big\downarrow & & {\scriptstyle \varphi_*}\big\downarrow{\scriptstyle \cong} \\
\pi_3(S^2) & \xrightarrow{\cong} & \pi_3(X)
\end{array}$$

we have that $(\varphi|S^3)_* : \pi_3(S^3) \xrightarrow{\cong} \pi_3(S^2)$ is an isomorphism. Hence, without loss of generalities, we may assume that

$$\varphi|S^3 = \eta_2 \qquad \text{(up to homotopy equivalence).} \tag{2.3}$$

Consider the commutative diagram

$$\begin{array}{ccccc}
\pi_n(K) & \xrightarrow{j_*} & \pi_n(\tilde{X}) & \longrightarrow & 0 \\
\| & & {\scriptstyle \varphi_*}\big\downarrow{\scriptstyle \cong} & & \\
\pi_n(K) & \xrightarrow{\varphi_{1*}} & \pi_n(X) & &
\end{array}$$

where the upper horizontal sequence is exact. Since j_* is surjective, $\varphi_{1*} : \pi_n(K) \to \pi_n(X)$ is also surjective. □

Lemma 2.2. *The attaching map g satisfies the condition $\eta_2 \circ g = f$.*

Proof. Consider the commutative diagram

$$\begin{array}{ccccccccc}
\pi_n(K) & \longrightarrow & \pi_n(K, S^3) & \longrightarrow & \pi_{n-1}(S^3) & \longrightarrow & \pi_{n-1}(K) & \longrightarrow & 0 \\
{\scriptstyle \varphi_{1*}}\big\downarrow & & {\scriptstyle \overline{\varphi}_*}\big\downarrow & & {\scriptstyle \eta_{2*}}\big\downarrow{\scriptstyle \cong} & & {\scriptstyle \varphi_{1*}'}\big\downarrow{\scriptstyle \cong} & & \\
\pi_n(X) & \longrightarrow & \pi_n(X, S^2) & \longrightarrow & \pi_{n-1}(S^2) & \longrightarrow & \pi_{n-1}(X) & \longrightarrow & 0
\end{array}$$

where horizontal sequences are exact.

By the dimensional reason, φ_{1*}' is bijective. Then because φ_{1*} is surjective, the Five Lemma indicates that $\overline{\varphi}_* : \pi_n(K, S^3) \to \pi_n(X, S^2)$ is surjecive. However, because $\pi_n(K, S^3) \cong \mathbb{Z} \cong \pi_n(X, S^2)$, in fact,

(2.4) $\overline{\varphi}_* : \pi_n(K, S^3) \xrightarrow{\cong} \pi_n(X, S^2)$ is bijective.

Let $\overline{g} \in \pi_n(K, S^3) \cong \mathbb{Z}$ (resp. $\overline{f} \in \pi_n(X, S^2)$) denote the characteristic maps of the top cells e^n of K (resp. of X), and consider the commutative diagram

$$\begin{array}{ccc}
\mathbb{Z} \cdot \overline{g} = \pi_n(K, S^3) & \xrightarrow{\partial_n'} & \pi_{n-1}(S^3) \\
{\scriptstyle \overline{\varphi}_*}\big\downarrow{\scriptstyle \cong} & & {\scriptstyle \eta_{2*}}\big\downarrow{\scriptstyle \cong} \\
\mathbb{Z} \cdot \overline{f} = \pi_n(X, S^2) & \xrightarrow{\partial_n} & \pi_{n-1}(S^2)
\end{array} \tag{2.5}$$

Since $\overline{\varphi}_*$ is bijective, $\overline{\varphi}_*(\overline{g}) = \pm\overline{f}$. Hence,

$$\eta_2 \circ g = \eta_{2*}(g) = \eta_{2*} \circ \partial'_n(\overline{g}) = \partial_n \circ \overline{\varphi}_*((\overline{g}) = \partial_n(\pm\overline{f}) = \pm f.$$

Because there is a homotopy equivalence $S^3 \cup_g e^n \simeq S^3 \cup_{-g} e^n$, we may assume $\eta_2 \circ g = f$ and this completes the proof. $\square$

Since $0 \neq f \in \pi_{n-1}(S^2)$ and $n \geq 5$, the order of f is finite. Let $m \geq 2$ be the order of the map $f \in \pi_{n-1}(S^2)$. Since $\eta_2 \circ g = f$, the order of g is also m. If we consider the homotopy exact sequences of the pairs (K, S^3) and (X, S^2), we have isomorphisms

$$\text{Ker } \partial'_n = \langle m \cdot \overline{g} \rangle \cong \mathbb{Z}, \quad \text{Ker } \partial_n = \langle m \cdot \overline{f} \rangle \cong \mathbb{Z}, \tag{2.6}$$

where $\partial'_n : \pi_n(K, S^3) \to \pi_{n-1}(S^3)$ and $\partial_n : \pi_n(X, S^2) \to \pi_{n-1}(S^2)$ denote the corresponding boundary operators.

Lemma 2.3. $\varphi_{1*} : \pi_n(K) \xrightarrow{\cong} \pi_n(X)$ *is an isomorphism.*

Proof. Since φ_{1*} is surjective (by Lemma 2.1), it suffices to show that there is an isomorphism $\pi_n(K) \cong \pi_n(X)$ as abelian groups. If we consider the homotopy exact sequence $\pi_n(S^3) \xrightarrow{i'_*} \pi_n(K) \to \text{Ker } \partial'_n \to 0$, we have an isomorphism $\pi_n(K) \cong \mathbb{Z} \oplus i'_*(\pi_n(S^3))$, where $i' : S^3 \to K$ denotes the inclusion. Similarly, if we denote by $i : S^2 \to X$ the inclusion, we have an isomorphism $\pi_n(X) \cong \mathbb{Z} \oplus i_*(\pi_n(S^2))$. Hence, it is sufficient to show that there is an isomorphism

$$i'_*(\pi_n(S^3)) \cong i_*(\pi_n(S^3)). \tag{2.7}$$

Consider the commutative diagram

$$\begin{tikzcd}
\pi_{n+1}(K, S^3) \arrow[r, "\partial'_{n+1}"] \arrow[d, "\overline{\varphi}_*"'] & \pi_n(S^3) \arrow[r, "i'_*"] \arrow[d, "\eta_{2*}"', "\cong"] & \pi_n(K) \arrow[d, "\varphi_{1*}"'] \\
\pi_{n+1}(X, S^2) \arrow[r, "\partial_{n+1}"] & \pi_n(S^2) \arrow[r, "i_*"] & \pi_n(X)
\end{tikzcd}$$

where horizontal sequences are exact. Then we have isomorphisms

$$\begin{cases} i'_*(\pi_n(S^3)) \cong \pi_n(S^3)/\partial'_{n+1}(\pi_{n+1}(K, S^3)), \\ i_*(\pi_n(S^2)) \cong \pi_n(S^2)/\partial_{n+1}(\pi_{n+1}(X, S^2)). \end{cases} \tag{2.8}$$

It follows from the James's isomorphism [4] that we have the isomorphisms

$$\begin{cases} \pi_{n+1}(K, S^3) = \overline{g}_*\pi_{n+1}(D^n, S^{n-1}) = \mathbb{Z}/2 \cdot \overline{g} \circ \eta, \\ \pi_{n+1}(X, S^2) = \mathbb{Z} \cdot [\overline{f}, \iota_2]_r \oplus \overline{f}_*\pi_{n+1}(D^n, S^{n-1}) = \mathbb{Z} \cdot [\overline{f}, \iota_2]_r \oplus \mathbb{Z}/2 \cdot \overline{f} \circ \eta, \end{cases}$$

where $\eta \in \pi_{n+1}(D^n, S^{n-1}) \cong \mathbb{Z}/2$ denotes the generator and $[\ ,\ ]_r$ is a relative Whitehead product. If we recall the commutative diagrams

$$\begin{array}{ccccccc}
\pi_{n+1}(K, S^3) & \xrightarrow{\partial'_{n+1}} & \pi_n(S^3) & & \pi_{n+1}(X, S^2) & \xrightarrow{\partial_{n+1}} & \pi_n(S^2) \\
\overline{g}_* \big\uparrow \cong & & g_* \big\uparrow & & \overline{f}_* \big\uparrow & & f_* \big\uparrow \\
\pi_{n+1}(D^n, S^{n-1}) & \xrightarrow[\cong]{\partial'} & \pi_n(S^{n-1}) & & \pi_{n+1}(D^n, S^{n-1}) & \xrightarrow[\cong]{\partial'} & \pi_n(S^{n-1})
\end{array}$$

then we have

$$\begin{cases} \partial'_{n+1}(\overline{g} \circ \eta) = g \circ \eta_{n-1}, \ \partial_{n+1}(\overline{f} \circ \eta) = f \circ \eta_{n-1}, \\ \partial_{n+1}([\overline{f}, \iota_2]_r) = -[f, \iota_2] = 0. \end{cases} \qquad \text{(by [1] and [3])}$$

Hence, by using (2.8) we have the isomorphisms

$$i'_*(\pi_n(S^3)) \cong \pi_n(S^3)/\langle g \circ \eta_{n-1} \rangle \text{ and } i_*(\pi_n(S^2)) \cong \pi_n(S^2)/\langle f \circ \eta_{n-1} \rangle.$$

However, because $\eta_{2*} : \pi_k(S^3) \xrightarrow{\cong} \pi_k(S^2)$ is an isomorphism for any $k \geq 2$ and $f = \eta_2 \circ g$, the map η_2 also induces an isomorphism

$$\pi_n(S^3)/\langle g \circ \eta_{n-1} \rangle \cong \pi_n(S^2)/\langle f \circ \eta_{n-1} \rangle.$$

Hence, the isomorphism (2.7) is proved. $\square$

Let $\overline{\theta} \in \pi_{n+1}(\tilde{X}, K) \cong \mathbb{Z}$ denote the characteristic map of the top cell e^{n+1} in $\tilde{X}$ and consider the exact sequence of the pair $(\tilde{X}, K)$,

$$\mathbb{Z} \cdot \overline{\theta} = \pi_{n+1}(\tilde{X}, K) \xrightarrow{\partial''_{n+1}} \pi_n(K) \xrightarrow{j_*} \pi_n(\tilde{X}) \longrightarrow 0.$$

Because $j_* : \pi_n(K) \to \pi_n(\tilde{X})$ is surjective and there are isomorphisms

$$\pi_n(K) \xrightarrow[\cong]{\varphi_{1*}} \pi_n(X) \xleftarrow[\cong]{\varphi_*} \pi_n(\tilde{X}),$$

in fact, j_* is an isomorphism. Hence, $\partial''_{n+1} = 0$ and we have $\theta = \partial''_{n+1}(\overline{\theta}) = 0$. So $\tilde{X} \simeq K \vee S^{n+1}$ and we complete the proof for the case $n \geq 5$.

3. The case $n = 4$.

The proof of the case $n = 4$ is essentially due to Jie Wu and the author does not claim its originality. However, for completeness of this paper, we shall give its proof here.

If we assume $n = 4$, without loss of generalities we may assume that $X = L_m = S^2 \cup_{m\eta_2} e^4$ for an integer $m \geq 2$. We note that the equality $x_2 \cdot x_2 = mx_4$ holds, where $x_{2k} \in H^{2k}(X, \mathbb{Z}) \cong \mathbb{Z}$ $(k = 1, 2)$ denote the corresponding generators. Then, if we compute the Serre spectral sequence associated to the fibration (2.1), we have

$$H^k(\tilde{X},\mathbb{Z}) \cong \begin{cases} \mathbb{Z} & \text{if } k=0,5, \\ \mathbb{Z}/m & \text{if } k=4, \\ 0 & \text{otherwise.} \end{cases}$$

So there is a homotopy equivalence

$$\tilde{X} \simeq \mathrm{P}^4(m) \cup_\theta e^5 \quad (\theta \in \pi_4(\mathrm{P}^4(m)). \tag{3.1}$$

It suffices to show that $\theta = 0$. If we use James's isomorphism [4], we have

$$\pi_4(\mathrm{P}^4(m)) = \begin{cases} \mathbb{Z}/2 \cdot i''_*(\eta_3) & \text{if } m \equiv 0 \pmod 2, \\ 0 & \text{if } m \equiv 1 \pmod 2, \end{cases} \tag{3.2}$$

where $i'' : S^3 \to \mathrm{P}^4(m)$ denotes the inclusion. If $m \equiv 1 \pmod 2$, since $\theta \in \pi_4(\mathrm{P}^4(m)) = 0$, $\theta = 0$ and the assertion follows. Next, consider the case $m \equiv 0 \pmod 2$. Because $\theta \in \pi_4(\mathrm{P}^4(m)) = \mathbb{Z}/2 \cdot i''_*(\eta_3)$, $\theta = 0$ or $\theta = i''_*(\eta_3)$. Now we suppose that $\theta = i''_*(\eta_3) \neq 0$. Then let $\overline{\theta} \in \pi_5(\tilde{X}, \mathrm{P}^4(m)) \cong \mathbb{Z}$ denote the characteristic map of the top cell e^5 and consider the exact sequence

$$\mathbb{Z}\cdot\overline{\theta} = \pi_5(\tilde{X}, \mathrm{P}^4(m)) \xrightarrow{\partial_5} \pi_4(\mathrm{P}^4(m)) \longrightarrow \pi_4(\tilde{X}) \longrightarrow 0.$$

Because $\partial_5(\overline{\theta}) = \theta = i''_*(\eta_3)$, we have $\pi_4(\tilde{X}) = 0$. However, since $\pi_4(\tilde{X}) \cong \pi_4(X) \cong \mathbb{Z}/2$ (by [8]), this is a contradiction. Hence $\theta = 0$. □

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Kohhei Yamaguchi
Department of Information Mathematics
University of Electro-Communications
Chofugaoka, Chofu, Tokyo 182-8585 Japan
e-mail address: kohhei@im.uec.ac.jp

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