Math. J. Okayama Univ. **47** (2005), 55–63

DERIVATIONS AND AUTOMORPHISMS ON THE ALGEBRA OF NON-COMMUTATIVE POWER SERIES

Kentaro IHARA

Abstract. Motivated by the study of multiple zeta values, we discuss several linear operators on the algebra of non-commutative formal power series $k\langle\langle x, y\rangle\rangle$ over a field k of characteristic zero. Especially a family of derivations whose elements commute with each other is defined and automorphisms which correspond to these derivations via exponential map are described explicitly.

1. Introduction

Let $\mathfrak{H} = k\langle x, y\rangle$ be the non-commutative polynomial algebra in two indeterminates x and y over a field k of characteristic zero. For any positive integer n, define the derivations $D'_n, \overline{D'}_n$ and ∂_n on $\mathfrak{H}$ by

$$D'_n(x) = 0,\quad D'_n(y) = x^n y, \qquad \overline{D'}_n(x) = xy^n,\quad \overline{D'}_n(y) = 0,$$
$$\partial_n(x) = x(x+y)^{n-1}y, \qquad \partial_n(y) = -x(x+y)^{n-1}y.$$

Note that any derivation on $\mathfrak{H}$ is uniquely determined by the values on the generators x and y. Define the derivations on $\widehat{\mathfrak{H}} := k\langle\langle x, y\rangle\rangle$, the non-commutative power series algebra, by $D' = \sum_{n\geq 1} \frac{D'_n}{n}$, $\overline{D'} = \sum_{n\geq 1} \frac{\overline{D'}_n}{n}$, and $\partial = \sum_{n\geq 1} \frac{\partial_n}{n}$.

In [1], Kaneko, Zagier and the author have established a relation among their exponentials:

$$\exp(\partial) = \exp(\overline{D'}) \exp(-D'), \tag{1.1}$$

where, as usual, $\exp(d) = e^d = \sum_{m\geq 0} \frac{d^m}{m!}$ for $d = D', \overline{D'}$ and ∂, which are the automorphisms of $\widehat{\mathfrak{H}}$. To show the identity, we need to compute the images of the automorphisms $e^{D'}, e^{\overline{D'}}, e^{\partial}$ on the generators x and y. They are given by

$$e^{D'}(x) = x,\quad e^{D'}(y) = (1-x)^{-1}y, \qquad e^{\overline{D'}}(x) = x(1-y)^{-1},\quad e^{\overline{D'}}(y) = y,$$
$$e^{\partial}(x) = x(1-y)^{-1}, \qquad e^{\partial}(y) = (1-x(1-y)^{-1})^{-1}y,$$

Mathematics Subject Classification. 16W25.

Key words and phrases. automorphism, non-commutaive power series.

This is a part of the author's doctoral dissertation submitted to Kyushu university, in March 2005.

where $(1-w)^{-1} = 1 + w + w^2 + \cdots$ defines an element in $\widehat{\mathfrak{H}}$ for any element w of degree ≥ 1. In general, it is a hard task to compute the exponential of a given derivation. Using this formula, one can compute the composition of the automorphisms and easily check the identity. In the recent study of *multiple zeta values* these derivations and the relation (1.1) were used effectively. (For more information, see below.)

In this paper, we shall consider closely a series of derivations

$$D_n := D_n(\alpha, \beta, \gamma, \delta) \qquad (\alpha, \beta, \gamma, \delta \in k)$$

(introduced in Section 7 in [1], see below for the precise definition), which generalizes the above $D'_n, \overline{D'_n}$ and ∂_n. First, in Section 2, Proposition 1, we shall prove their mutual commutativity, i.e., $D_m D_n = D_n D_m$ for any fixed parameters $\alpha, \beta, \gamma, \delta$, and then, in Section 3, Theorem 1, we shall give an explicit formula that computes how the exponential of the derivation $\sum_n c_n D_n(\alpha, \beta, \gamma, \delta)$, (where $\{c_1, c_2, \ldots\}$ is an arbitrary sequence of elements of k) acts on the generators of $\widehat{\mathfrak{H}}$.

As mentioned above, these results are motivated by a recent study of the multiple zeta values. Before closing this introduction, we shall explain more about our background problem and related results from the viewpoint of studies of the multiple zeta values. The *multiple zeta values* (MZV's) are defined by the series

$$\zeta(k_1, k_2, \ldots, k_n) = \sum_{m_1 > m_2 > \cdots > m_n > 0} \frac{1}{m_1^{k_1} m_2^{k_2} \cdots m_n^{k_n}} \in \mathbf{R},$$

where the exponents k_i's are positive integers and $k_1 > 1$. They satisfy many linear relations over $\mathbf{Q}$, the simplest of which is $\zeta(3) = \zeta(2,1)$ found by Euler. In [1], Kaneko, Zagier and the author found certain relations among MZV's called *derivation relations*. They also showed that the derivation relations are equivalent in some sense to the other relations which had been found by Ohno.

To review this, we use the algebraic setup introduced by Hoffman [2]. In the rest of this introduction, we take the rationals $\mathbf{Q}$ as the base field k. Let $\mathfrak{H}^0 := \mathbf{Q} + x\mathfrak{H}y$ be a subalgebra of $\mathfrak{H} = \mathbf{Q}\langle x, y\rangle$, and $Z : \mathfrak{H}^0 \longrightarrow \mathbf{R}$ be the $\mathbf{Q}$-linear map which maps each monomial (word) $x^{k_1-1}yx^{k_2-1}y \cdots x^{k_n-1}y$ in $\mathfrak{H}^0$ to the value $\zeta(k_1, k_2, \ldots, k_n)$. Having defined the map Z, one of the main problems in the theory of MZV's is to find the structure of Ker Z. Producing elements of Ker Z amounts to finding linear relations among MZV's. For example $x^2y - xy^2$ is in Ker Z, which corresponds to $\zeta(3) = \zeta(2,1)$. Then the derivation relations are stated as

$$\partial_n(\mathfrak{H}^0) \subset \text{Ker } Z$$

for any $n \geq 1$. The D'_n and $\overline{D'}_n$ are used to describe the other relations which were proved by Y. Ohno [3], and the identity (1.1) showes the equivalence between the derivation relations and Ohno's relations. From these point of view, we think that it is important to study the class of derivations $D_n(\alpha, \beta, \gamma, \delta)$ and their exponential automorphisms.

Acknowledgements. The author is grateful to Prof. Masanobu Kaneko, Jyun Kajikawa and the referee who read the manuscript and suggested several important improvements of the exposition.

2. Derivations

Let $\widehat{\mathfrak{H}} = k\langle\langle x, y\rangle\rangle$ be the algebra of non-commutative formal power series in two indeterminates over a commutative field k of characteristic zero. The algebra $\widehat{\mathfrak{H}}$ is complete with respect to the grading defined by $\deg x = \deg y = 1$. The space of all derivations of $\widehat{\mathfrak{H}}$ over k form a Lie algebra, denoted by $\mathrm{Der}(\widehat{\mathfrak{H}})$, with usual commutater bracket: $[d, d'] := dd' - d'd$. On the other hand, the set of all automorphisms of $\widehat{\mathfrak{H}}$ over k form a group, denoted by $\mathrm{Aut}(\widehat{\mathfrak{H}})$. Note that both derivations and automorphisms on $\widehat{\mathfrak{H}}$ are determined by the values on generators of $\widehat{\mathfrak{H}}$. Let $\mathrm{Der}^+(\widehat{\mathfrak{H}})$ be the Lie subalgebra consisting of derivations which increase the degree, or equivalently which induce the zero derivation on the associated graded algebra $\mathrm{gr}(\widehat{\mathfrak{H}}) = \bigoplus \widehat{\mathfrak{H}}_n/\widehat{\mathfrak{H}}_{n+1}$, where $\widehat{\mathfrak{H}}_n$ is the subspace of $\widehat{\mathfrak{H}}$ generated by the words of degree $\geq n$. Let $\mathrm{Aut}^1(\widehat{\mathfrak{H}})$ be the subgroup of $\mathrm{Aut}(\widehat{\mathfrak{H}})$ consisting of automorphisms ϕ such that $\phi(x) - x$ and $\phi(y) - y$ belong to $\widehat{\mathfrak{H}}_2$, or equivalently which induce the identity automorphism on $\mathrm{gr}(\widehat{\mathfrak{H}})$. There is a one to one correspondence between the Lie subalgebra $\mathrm{Der}^+(\widehat{\mathfrak{H}})$ and the subgroup $\mathrm{Aut}^1(\widehat{\mathfrak{H}})$ via the exponential and the logarithm maps; $\exp(d) = e^d = \sum_{m\geq 0} \frac{d^m}{m!}$ for $d \in \mathrm{Der}^+(\widehat{\mathfrak{H}})$ and $\log \phi = -\sum_{m\geq 1} \frac{(1-\phi)^m}{m}$ for $\phi \in \mathrm{Aut}^1(\widehat{\mathfrak{H}})$. We have used the condition that the characteristic of k is zero to make these definitions well defined.

Let $\{a, b\}$ be an arbitrary set of (topological) generators of $\widehat{\mathfrak{H}}$, for example a and b are both linear combinations of x and y which are not proportional. In this paper we will fix such $\{a, b\}$ once and for all.

In Proposition 14 of [1], the following derivations were defined, which generalize $\{D_n\}, \{\overline{D}_n\}$ and $\{\partial_n\}$ in the introduction. Here we consider them in a slightly generalized setting that the generators a and b need not be of degree 1 homogeneous elements. The proof of Proposition 1 below was omitted in [1] and so we give it here.

Definition 1. For all $n > 0$ and elements $\alpha, \beta, \gamma, \delta$ in k, define the derivations $D_n := D_n(\alpha, \beta, \gamma, \delta) = D_n(\{a, b\}; \alpha, \beta, \gamma, \delta)$ by

$$D_n(a) = 0, \qquad D_n(b) = \alpha a^{n+1} + \beta a^n b + \gamma b a^n + \delta b a^{n-1} b,$$

which are clearly in $\mathrm{Der}^+(\widehat{\mathfrak{H}})$.

The derivations $D'_n, \overline{D'}_n$ and ∂_n in the introduction are special cases of D_n:

$$\begin{aligned} D'_n &= D_n(\{x, y\}; 0, 1, 0, 0), \qquad \overline{D'}_n = D_n(\{y, x\}; 0, 0, 1, 0), \\ \partial_n &= D_n(\{x + y, x\}; 0, 0, 1, -1) = D_n(\{x + y, y\}; 0, -1, 0, 1). \end{aligned}$$

Proposition 1. *The derivations $\{D_n\}$ commute with each other:*

$$[D_m, D_n] = 0 \text{ for all } m, n \geq 1.$$

Proof. Clearly the $[D_m, D_n]$ is also a derivation on $\widehat{\mathfrak{H}}$, so it is enough to check the images of a and b are both 0. $[D_m, D_n](a) = 0$ is trivial. Put $u = \alpha a + \beta b$ and $v = \gamma a + \delta b$, then we have $D_n(b) = a^n u + b a^{n-1} v$. First we have

$$\begin{aligned} D_m(u) &= \beta(a^m u + b a^{m-1} v) = a^m(\beta u - \alpha v) + u a^{m-1} v, \\ D_m(v) &= \delta(a^m u + b a^{m-1} v) = a^m(\delta u - \gamma v) + v a^{m-1} v. \end{aligned}$$

From this, we have

$$\begin{aligned} D_m D_n(b) &= D_m(a^n u + b a^{n-1} v) \\ &= a^n D_m(u) + D_m(b) a^{n-1} v + b a^{n-1} D_m(v) \\ &= a^n(a^m(\beta u - \alpha v) + u a^{m-1} v) + (a^m u + b a^{m-1} v) a^{n-1} v \\ &\quad + b a^{n-1}(a^m(\delta u - \gamma v) + v a^{m-1} v) \\ &= a^{m+n}(\beta u - \alpha v) + (a^n u a^{m-1} v + a^m u a^{n-1} v) \\ &\quad + b a^{m+n-1}(\delta u - \gamma v) + (b a^{m-1} v a^{n-1} v + b a^{n-1} v a^{m-1} v). \end{aligned}$$

Since the last expression is symmetric in m and n, we have $[D_m, D_n](b) = 0$. □

To consider any linear combination of D_n's, we use the notation D_f which was introduced in [1]:

Definition 2. Let $f(X) = \sum_{n \geq 1} c_n X^n \in Xk[[X]]$ be a formal power series in one indeterminate X without constant term. We define the derivation $D_f := D_f(\alpha, \beta, \gamma, \delta) \in \mathrm{Der}^+(\widehat{\mathfrak{H}})$ by $D_f(\alpha, \beta, \gamma, \delta) = \sum_{n \geq 1} c_n D_n(\alpha, \beta, \gamma, \delta)$.

Note that the action of D_f on generators $\{a, b\}$ is given by

$$\begin{aligned} D_f(a) &= 0, \\ D_f(b) &= \alpha f(a)a + \beta f(a)b + \gamma b f(a) + \delta b \frac{f(a)}{a} b \\ &= f(a)u + b\frac{f(a)}{a} v \end{aligned}$$

where $u = \alpha a + \beta b$ and $v = \gamma a + \delta b$. The element $\frac{f(a)}{a} \in \widehat{\mathfrak{H}}$ is given by substituting a for X in the power series $\frac{f(X)}{X} \in k[[X]]$.

It clearly holds that $tD_f = D_{tf}$ and $D_f + D_g = D_{f+g}$ for any elements $t \in k$ and $f(X), g(X)$ in $Xk[[X]]$. As a consequence of Proposition 1, we have $[D_f, D_g] = 0$ for any $f, g \in Xk[[X]]$. In other words, the space of derivations $\{D_f \mid f \in Xk[[X]]\}$ forms a commutative Lie subalgebra in $\mathrm{Der}^+(\widehat{\mathfrak{H}})$.

3. Automorphisms

In the previous section we defined the derivations D_f depending on the fixed basis $\{a, b\}$, coefficients $\alpha, \beta, \gamma, \delta$, and the series $f(X) \in Xk[[X]]$. In this section we give an explicit description of the corresponding exponential automorphisms.

Definition 3. Let $h(X) \in 1 + Xk[[X]]$ be a power series with constant term 1. We define an automorphism Δ_h as follows: Denote by ε and ε' the two roots of the quadratic equation $T^2 + (\beta + \gamma)T + \alpha\delta = 0$ and put $\omega = \varepsilon - \varepsilon'$. The elements $\varepsilon, \varepsilon'$ and ω belong to a quadratic extension K of k, and the elements $\varepsilon + \varepsilon' = -(\beta + \gamma)$ and $\varepsilon\varepsilon' = \alpha\delta$ are in k.

Let $\Delta_h \in \mathrm{Aut}^1(\widehat{\mathfrak{H}})$ be the automorphism defined by the following action on generators: $\Delta_h(a) = a$ and

$$\begin{aligned} \Delta_h(b) &= h(a)^{\beta+\varepsilon}\Big[b + \frac{h(a)^{-\omega} - 1}{-\omega}(\alpha a - \varepsilon b)\Big] \qquad (3.1) \\ &\quad \times \Big[1 + \frac{h(a)^{\omega} - 1}{\omega a}(\varepsilon a - \delta b)\Big]^{-1} h(a)^{\gamma+\varepsilon} \\ &= h(a)^{\beta}\Big[(h(a)^{\varepsilon} - h(a)^{\varepsilon'})\alpha a - (\varepsilon' h(a)^{\varepsilon} - \varepsilon h(a)^{\varepsilon'})b\Big] \qquad (3.2) \\ &\quad \times \Big[(\varepsilon h(a)^{\varepsilon} - \varepsilon' h(a)^{\varepsilon'}) - \frac{h(a)^{\varepsilon} - h(a)^{\varepsilon'}}{a}\delta b\Big]^{-1} h(a)^{-\beta} \end{aligned}$$

where $h(a)^{\lambda} = \exp(\lambda \log h(a))$ for any $\lambda \in K$, and the quotients $(h(a)^{\omega} - 1)/\omega a$ and $(h(a)^{\varepsilon} - h(a)^{\varepsilon'})/a$ define the elements of $K\langle\langle x, y\rangle\rangle$, since each numerator has no constant term, one can divide it by a. In the case $\omega = 0$, we regard the elements $(h(a)^{-\omega} - 1)/(-\omega)$ and $(h(a)^{\omega} - 1)/\omega a$ as $\log h(a)$

and $(\log h(a))/a$ respectively. Since the expression (3.2) is symmetric in ε and ε', it defines an element of $\widehat{\mathfrak{H}}$.

First we check the expression (3.1) equals (3.2):

$$\begin{aligned}
A_h := {}& h(a)^{\beta+\varepsilon}\Big[b + \frac{h(a)^{-\omega}-1}{-\omega}(\alpha a - \varepsilon b)\Big] \\
={}& h(a)^{\beta}\Big[\frac{h(a)^{\varepsilon}-h(a)^{\varepsilon'}}{\omega}\alpha a + \big(h(a)^{\varepsilon} - \frac{\varepsilon(h(a)^{\varepsilon}-h(a)^{\varepsilon'})}{\omega}\big)b\Big] \\
={}& h(a)^{\beta}\Big[\frac{h(a)^{\varepsilon}-h(a)^{\varepsilon'}}{\omega}\alpha a - \frac{\varepsilon' h(a)^{\varepsilon}-\varepsilon h(a)^{\varepsilon'}}{\omega}b\Big]. \qquad (3.3)
\end{aligned}$$

On the other hand, we have

$$\begin{aligned}
B_h^{-1} := {}& \Big[1 + \frac{h(a)^{\omega}-1}{\omega a}(\varepsilon a - \delta b)\Big]^{-1} h(a)^{\gamma+\varepsilon} \\
={}& \Big[1 + \frac{h(a)^{\omega}-1}{\omega a}(\varepsilon a - \delta b)\Big]^{-1} h(a)^{-(\beta+\varepsilon')} \\
={}& \Big[h(a)^{\varepsilon'} + \frac{\varepsilon(h(a)^{\varepsilon}-h(a)^{\varepsilon'})}{\omega} - \frac{h(a)^{\varepsilon}-h(a)^{\varepsilon'}}{\omega a}\delta b\Big]^{-1} h(a)^{-\beta} \\
={}& \Big[\frac{\varepsilon h(a)^{\varepsilon}-\varepsilon' h(a)^{\varepsilon'}}{\omega} - \frac{h(a)^{\varepsilon}-h(a)^{\varepsilon'}}{\omega a}\delta b\Big]^{-1} h(a)^{-\beta}. \qquad (3.4)
\end{aligned}$$

Thus we have showed that (3.1)=(3.2). Since the equation (3.4) defines an invertible element of $\widehat{\mathfrak{H}}$, we denote the inverse by B_h. Hence we have $\Delta_h(b) = A_h B_h^{-1}$.

Theorem 3.1. *For any $f(X) \in Xk[[X]]$, set $h(X) = e^{f(X)} \in 1 + Xk[[X]]$. Then we have*

$$\Delta_h = \exp(D_f). \qquad (3.5)$$

Proof. For the derivation D_f, we can consider the 1-dimensional commutative Lie subalgebra $\{tD_f = D_{tf}\}$ spanned by D_f. Then the image of the Lie algebra under the exponential map forms a 1-parameter subgroup $\{e^{tD_f} = e^{D_{tf}}\}$ of $\mathrm{Aut}^1(\widehat{\mathfrak{H}})$. The tangent vector along the path at the unit (identity automorphism on $\widehat{\mathfrak{H}}$) corresponds to $\log(e^{D_f}) = D_f$. Therefore it is enough to show that (i) $\frac{d}{dt}\Delta_{h^t}|_{t=0} = D_f$, and (ii) $\Delta_{gh} = \Delta_g\Delta_h$ for $g, h \in 1 + Xk[[X]]$, i.e., the map $h \mapsto \Delta_h$ is a group homomorphism.

For (i), from the definition of D_f and Δ_h it is clear that $\frac{d}{dt}\Delta_{h^t}(a)|_{t=0} = D_f(a) = 0$. Next we have from (3.1)

$$\Delta_{h^t}(b) = h^{(\beta+\varepsilon)t}\Big[b + \frac{h^{-\omega t}-1}{-\omega}(\alpha a - \varepsilon b)\Big]\Big[1 + \frac{h^{\omega t}-1}{\omega a}(\varepsilon a - \delta b)\Big]^{-1} h^{(\gamma+\varepsilon)t},$$

where we write h for $h(a)$ for simplicity. By using the formula $\frac{d}{dt}h^{\lambda t}|_{t=0} = \frac{d}{dt}e^{\lambda t f(a)}|_{t=0} = \lambda f(a)$ for $\lambda \in K$, we have

$$\begin{aligned}\frac{d}{dt}\Delta_{h^t}(b)\big|_{t=0} &= (\beta+\varepsilon)f(a)b + f(a)(\alpha a - \varepsilon b) - b\frac{f(a)}{a}(\varepsilon a - \delta b)\\ &\qquad\qquad\qquad\qquad\qquad\qquad\qquad + b(\gamma+\varepsilon)f(a)\\ &= \alpha f(a)a + \beta f(a)b + \gamma b f(a) + \delta b\frac{f(a)}{a}b\\ &= f(a)u + b\frac{f(a)}{a}v.\end{aligned}$$

This coincides with the expression in Definition 2. For the proof of (ii) we need the following lemma which will be proved later:

Lemma 3.2. *For any $g, h \in 1 + Xk[[X]]$, we obtain*

$$\Delta_g(A_h)B_g = A_{gh}, \qquad \Delta_g(B_h)B_g = B_{gh} \tag{3.6}$$

where A_h, B_h are the elements defined above.

Using this lemma we can prove (ii):

$$\begin{aligned}\Delta_g(\Delta_h(a)) &= a = \Delta_{gh}(a),\\ \Delta_g(\Delta_h(b)) &= \Delta_g(A_h B_h^{-1}) = \Delta_g(A_h)\Delta_g(B_h)^{-1}\\ &= (A_{gh}B_g^{-1})(B_{gh}B_g^{-1})^{-1} = A_{gh}B_{gh}^{-1} = \Delta_{gh}(b).\end{aligned}$$

Proof of Lemma. We continue to write h instead of $h(a)$ and denote equations (3.3) and (3.4) simply as

$$\begin{aligned}A_h &= h^\beta\big[(h^\varepsilon - h^{\varepsilon'})\alpha a - (\varepsilon' h^\varepsilon - \varepsilon h^{\varepsilon'})b\big]/\omega,\\ B_h &= h^\beta\big[(\varepsilon h^\varepsilon - \varepsilon' h^{\varepsilon'}) - \frac{h^\varepsilon - h^{\varepsilon'}}{a}\delta b\big]/\omega,\end{aligned}$$

and $\Delta_h(b) = A_h B_h^{-1}$. We obtain

$$\begin{aligned}\Delta_g(A_h)B_g &= h^\beta\big[(h^\varepsilon - h^{\varepsilon'})\alpha a - (\varepsilon' h^\varepsilon - \varepsilon h^{\varepsilon'})\Delta_g(b)\big]B_g/\omega\\ &= h^\beta\big[(h^\varepsilon - h^{\varepsilon'})\alpha a B_g - (\varepsilon' h^\varepsilon - \varepsilon h^{\varepsilon'})A_g\big]/\omega\\ &= (gh)^\beta\Big[(h^\varepsilon - h^{\varepsilon'})\alpha a\big\{(\varepsilon g^\varepsilon - \varepsilon' g^{\varepsilon'}) - \frac{g^\varepsilon - g^{\varepsilon'}}{a}\delta b\big\}\\ &\qquad - (\varepsilon' h^\varepsilon - \varepsilon h^{\varepsilon'})\big\{(g^\varepsilon - g^{\varepsilon'})\alpha a - (\varepsilon' g^\varepsilon - \varepsilon g^{\varepsilon'})b\big\}\Big]\Big/\omega^2\\ &= (gh)^\beta\Big[\big\{(h^\varepsilon - h^{\varepsilon'})(\varepsilon g^\varepsilon - \varepsilon' g^{\varepsilon'}) - (\varepsilon' h^\varepsilon - \varepsilon h^{\varepsilon'})(g^\varepsilon - g^{\varepsilon'})\big\}\alpha a\\ &\qquad - \big\{(h^\varepsilon - h^{\varepsilon'})(g^\varepsilon - g^{\varepsilon'})\alpha\delta - (\varepsilon' h^\varepsilon - \varepsilon h^{\varepsilon'})(\varepsilon' g^\varepsilon - \varepsilon g^{\varepsilon'})\big\}b\Big]\Big/\omega^2\end{aligned}$$

$$= (gh)^\beta\big[\{(gh)^\varepsilon - (gh)^{\varepsilon'}\}\alpha a - \{\varepsilon'(gh)^\varepsilon - \varepsilon(gh)^{\varepsilon'}\}b\big]/\omega = A_{gh}.$$

Similarly we obtain

$$\begin{aligned}
\Delta_g(B_h)B_g &= h^\beta\Big[(\varepsilon h^\varepsilon - \varepsilon' h^{\varepsilon'}) - \frac{h^\varepsilon - h^{\varepsilon'}}{a}\delta\Delta_g(b)\Big]B_g\Big/\omega \\
&= h^\beta\Big[(\varepsilon h^\varepsilon - \varepsilon' h^{\varepsilon'})B_g - \frac{h^\varepsilon - h^{\varepsilon'}}{a}\delta A_g\Big]\Big/\omega \\
&= (gh)^\beta\Big[(\varepsilon h^\varepsilon - \varepsilon' h^{\varepsilon'})\{(\varepsilon g^\varepsilon - \varepsilon' g^{\varepsilon'}) - \frac{g^\varepsilon - g^{\varepsilon'}}{a}\delta b\} \\
&\quad - \frac{h^\varepsilon - h^{\varepsilon'}}{a}\delta\{(g^\varepsilon - g^{\varepsilon'})\alpha a - (\varepsilon' g^\varepsilon - \varepsilon g^{\varepsilon'})b\}\Big]\Big/\omega^2 \\
&= (gh)^\beta\Big[\{(\varepsilon h^\varepsilon - \varepsilon' h^{\varepsilon'})(\varepsilon g^\varepsilon - \varepsilon' g^{\varepsilon'}) - (h^\varepsilon - h\varepsilon')(g^\varepsilon - g\varepsilon')\alpha\delta\} \\
&\quad - \Big\{\frac{(\varepsilon h^\varepsilon - \varepsilon' h^{\varepsilon'})(g^\varepsilon - g^{\varepsilon'})}{a} - \frac{(h^\varepsilon - h^{\varepsilon'})(\varepsilon' g^\varepsilon - \varepsilon g^{\varepsilon'})}{a}\Big\}\delta b\Big]\Big/\omega^2 \\
&= (gh)^\beta\Big[\{\varepsilon(gh)^\varepsilon - \varepsilon'(gh)^{\varepsilon'}\} - \frac{(gh)^\varepsilon - (gh)^{\varepsilon'}}{a}\delta b\Big]\Big/\omega = B_{gh}.
\end{aligned}$$

Thus we conclude the proof of Lemma and Theorem 3.1. □

The following theorem is a special case of Theorem 1, but is worth stating separately because of the conciseness of the expression.

Theorem 3.3. *Suppose that $\alpha, \beta, \gamma, \delta \in k$ satisfy $\alpha\delta - \beta\gamma = 0$. Then the derivation D_f is defined by the images $D_f(a) = 0$, $D_f(b) = w\frac{f(a)}{a}w'$ on the generators for some $w, w' \in ka + kb$ and the automorphism $\Delta_h = \exp(D_f)$ for $h = e^f$ sends the generators to*

$$\Delta_h(a) = a,$$

$$\Delta_h(b) = \Big[b + \frac{h(a)^{\beta-\gamma} - 1}{\beta - \gamma}u\Big]\Big[1 - \frac{h(a)^{\beta-\gamma} - 1}{(\beta-\gamma)a}v\Big]^{-1}, \tag{3.7}$$

where $u = \alpha a + \beta b$, $v = \gamma a + \delta b$.

Proof. The condition $\alpha\delta - \beta\gamma = 0$ is equivalent to that one can write $u = \lambda w'$ and $v = \mu w'$ for some $\lambda, \mu \in k$ and $w' \in ka + kb$. If we put $w = \lambda a + \mu b$ then we have

$$\begin{aligned}
D_f(b) &= f(a)u + b\frac{f(a)}{a}v = f(a)\lambda w' + b\frac{f(a)}{a}\mu w' \\
&= (\lambda a + \mu b)\frac{f(a)}{a}w' = w\frac{f(a)}{a}w'.
\end{aligned}$$

In this case one can take $-\beta$ and $-\gamma$ respectively as the roots ε and ε' of the quadratic equation. Then we have $-\omega = \beta - \gamma$. Therefore from (3.3) we obtain

$$A_h = \Big[b + \frac{h(a)^{\beta-\gamma} - 1}{\beta - \gamma}(\alpha a + \beta b)\Big].$$

On the other hand, since the equation (3.4) is symmetric in ε and ε', we have

$$\begin{aligned} B_h^{-1} &= \Big[1 + \frac{h(a)^{-\omega} - 1}{-\omega a}(\varepsilon' a - \delta b)\Big]^{-1} h(a)^{-(\beta+\varepsilon)} \\ &= \Big[1 - \frac{h(a)^{\beta-\gamma} - 1}{(\beta-\gamma)a}(\gamma a + \delta b)\Big]^{-1}. \end{aligned}$$

Hence we conclude the proof. □

References

[1] K. Ihara, M. Kaneko, D. Zagier, *Derivations and double shuffle relations for multiple zeta values*, preprint (2004), Max-Planck-Institut für Mathematik preprint series 2004-100.

[2] M. Hoffman, *The algebra of multiple harmonic series*, J. of Algebra **194** (1997), 477-495.

[3] Y. Ohno, *A generalization of the duality and sum formulas on the multiple zeta values*, J. of Number Th. **74** (1999), 39-43.

Kentaro Ihara
Graduate School of Mathematics
Kyushu University 33 Fukuoka, 812-8581 Japan
e-mail address: ihara@math.kyushu-u.ac.jp

(*Received November 22, 2004*)