THE GALOIS ACTION ON THE TORSOR OF HOMOTOOPY
CLASSES OF PATHS ON A PROJECTIVE LINE MINUS A
FINITE NUMBER OF POINTS

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0. Introduction.

0.1. Deligne on a conference in Schloss Ringberg considered the mixed
Hodge structure on the fundamental group of \( P^1 \setminus \{0, 1, -1, \infty\} \). He showed
that the motivic Galois Lie algebra associated to this mixed Hodge structure
contains a free Lie subalgebra on generators in degree 1, 3, 5, \ldots, \( 2n + 1 \), \ldots,
 corresponding to \( \log 2 \), \( \zeta(3) \), \( \zeta(5) \), \ldots, \( \zeta(2n + 1) \), \ldots.

In [W1] and [DW] we were studying actions of Galois groups on funda-
mental groups. In this note we are studying the action of the Galois group
\( \text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \) on the torsor of (\( \ell \)-adic) paths from 01 to -1 on \( P^1_{\mathbb{Q}} \setminus \{0, 1, \infty\} \). We
show that the associated graded Lie algebra of the image of \( \text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}(\mu_{\infty})) \)
contains a free Lie subalgebra over \( \mathbb{Q}_\ell \) on generators in degree 1, 3, 5, \ldots, \( 2n + 1 \), \ldots. We use the idea working modulo 2 from Deligne’s talk in Schloss
Ringberg.

In [W1] section 5 we were studying some general aspects of actions of
Galois groups on torsors of paths. To make this paper self contained we
recall some definitions and results from [W1] in sections 1 and 2.

1. Torsors of paths.

1.1. Let \( K \) be a number field and let \( a_1, \ldots, a_{n+1} \) be \( K \)-points of a projective
line \( P^1_K \). Let \( V = P^1_K \setminus \{a_1, \ldots, a_n, a_{n+1}\} \). For simplicity we assume that
\( a_{n+1} = \infty \).

We denote by \( \hat{V}(K) \) the set of \( K \)-points of \( V \) and of tangential base
points defined over \( K \). Let \( z, v \in \hat{V}(K) \). Let \( \pi_1(V^\ell_K, v) \) be the \( \ell \)-completion
of the etale fundamental group of \( V^\ell_K \) and let \( \pi(V^\ell_K, z, v) \) be the set of \( \ell \)-adic
paths from $v$ to $z$ on $V_K$. The set $\pi(V_K, z, v)$ is a $\pi_1(V_K, v)$-torsor. The Galois group $G_K := \text{Gal}(\overline{K}/K)$ acts on $\pi_1(V_K, v)$ and on $\pi(V_K, z, v)$ in a compatible way, i.e., $\sigma(p \cdot S) = \sigma(p) \cdot \sigma(S)$, where $\sigma \in G_K$, $p \in \pi(V_K, z, v)$ and $S \in \pi_1(V_K, v)$.

Let us fix a path $p \in \pi(V_K, z, v)$. We define a bijection of sets

$$t_p : \pi(V_K, z, v) \to \pi_1(V_K, v)$$

setting $t_p(q) := p^{-1} \cdot q$ (the composition of paths is from right to left). The bijection $t_p$ is not $G_K$-equivariant. Using the bijection $t_p$ we transport the action of $G_K$ on $\pi(V_K, z, v)$ into the action of $G_K$ on $\pi_1(V_K, v)$.

Let $\sigma \in G_K$. We set

$$f_p(\sigma) := p^{-1} \cdot \sigma(p).$$

The element $f_p(\sigma) \in \pi_1(V_K, v)$. Let us define a new action of $G_K$ on $\pi_1(V_K, v)$ setting

$$\sigma_p(S) := f_p(\sigma) \cdot \sigma(S).$$

Observe that

$$(\tau \cdot \sigma)_p = \tau_p \cdot \sigma_p,$$

i.e., we have an action of $G_K$ on $\pi_1(V_K, v)$. We have

$$t_p(\sigma(q)) = \sigma_p(t_p(q)),$$

i.e., the bijection $t_p$ is $G_K$-equivariant if we equip $\pi_1(V_K, v)$ with the new action of $G_K$.

1.2. We fix generators of $\pi_1(V_K, v)$ in the following way. At each missing point $a_i$ we choose a tangential base point $v_i$ defined over $K$. Let $\gamma_i$ be a path from $v$ to $v_i$. Then $x_i$ is the composition of the path $\gamma_i + a_i$ small loop around $a_i$ in the opposite clockwise direction + the path $\gamma_i^{-1}$. We can assume that $x_{n+1} \cdot x_n \cdot \ldots \cdot \gamma_1 = 1$.

To study the action of $G_K$ on the torsor $\pi(V_K, z, v)$, i.e., the action

$$( )_p : G_K \to \text{Aut}_{set}(\pi_1(V_K, v))$$

it is very convenient to embed $\pi_1(V_K, v)$ into the ring of formal power series in non-commuting variables.

Let $\mathbb{Q}_\ell\{\{X_1, \ldots, X_n\}\}$ (resp. $\mathbb{Q}_\ell\{X_1, \ldots, X_n\}$) be a $\mathbb{Q}_\ell$-algebra of formal power series (resp. of polynomials) in non-commuting variables $X_1, \ldots, X_n$. Let

$$k : \pi_1(V_K, v) \to \mathbb{Q}_\ell\{\{X_1, \ldots, X_n\}\}$$

be a continuous multiplicative embedding given by $k(x_i) = e^{X_i}$ for $i = i, \ldots, n$.

Let us set

$$\Lambda_p(\sigma) := k(f_p(\sigma)).$$
The action of $G_K$ on $\pi_1(V_K, v)$ induces a homomorphism

$$G_K \rightarrow \text{Aut}_{\mathbb{Q}_\ell\text{-algebra}}(\mathbb{Q}_\ell\{\{X_1, \ldots, X_n\}\}).$$

The action of $G_K$ on $\pi(V_K, z, v)$, i.e., the action $(\ )_p$ induces a homomorphism

$$\varphi_p : G_K \rightarrow \text{Aut}_{\mathbb{Q}_\ell\text{-linear}}(\mathbb{Q}_\ell\{\{X_1, \ldots, X_n\}\}).$$

Let $\omega \in \mathbb{Q}_\ell\{\{X_1, \ldots, X_n\}\}$ and let $\sigma \in G_K$. Then

$$\varphi_p(\sigma)(\omega) = \Lambda_p(\sigma) \cdot \sigma(\omega).$$

1.3. We shall study the Lie algebras of derivations of free Lie algebras.

Let $\text{Lie}(V)$ be a free Lie algebra over $\mathbb{Q}_\ell$ on free generators $X_1, \ldots, X_n$. Let $L(V) := \lim_{\leftarrow n} \text{Lie}(V)/\Gamma^n\text{Lie}(V)$. We identify $\text{Lie}(V)$ (resp. $L(V)$) with the Lie algebra of Lie elements of $\mathbb{Q}_\ell\{X_1, \ldots, X_n\}$ (resp. of $\mathbb{Q}_\ell\{\{X_1, \ldots, X_n\}\}$).

If $L$ is a Lie algebra then we denote by $\text{Der} L$ the Lie algebra of derivations of $L$.

Let $n := \{1, \ldots, n\}$. We set

$$\text{Der}^*\text{Lie}(V) := \{D \in \text{Der} \text{Lie}(V) \mid \forall i \in n \exists A_i \in \text{Lie}(V), D(X_i) = [X_i, A_i]\}$$

and

$$\text{Der}^* L(V) := \{D \in \text{Der} L(V) \mid \forall i \in n \exists A_i \in L(V), D(X_i) = [X_i, A_i]\}.$$

The derivation $D \in \text{Der}^*\text{Lie}(V)$ such that $D(X_i) = [X_i, A_i]$ for $i \in n$ we denote by $D_{(A_1, \ldots, A_n)}$ or $D_{(A_i)_{i \in n}}$. Let $< X_i >$ be a vector subspace of $\text{Lie}(V)$ generated by $X_i$. Observe that we have an isomorphism of vector spaces

$$\text{Der}^*\text{Lie}(V) \approx \bigoplus_{i=1}^n (\text{Lie}(V)/< X_i >)$$

which maps $D_{(A_1, \ldots, A_n)}$ onto $(A_1, \ldots, A_n)$. We introduce on $\bigoplus_{i=1}^n (\text{Lie}(V)/< X_i >)$ a new bracket $\{ \}$ defined in the following way

$$\{(A_i)_{i \in n}, (B_i)_{i \in n}\} := ([A_i, B_i] + D_{(A_j)_{j \in n}}(B_i) - D_{(B_j)_{j \in n}}(A_i))_{i \in n}.$$ 

**Lemma 1.3.1.** The vector space $\bigoplus_{i=1}^n (\text{Lie}(V)/< X_i >)$ equip with the bracket $\{ \}$ is a Lie algebra isomorphic to the Lie algebra $\text{Der}^*\text{Lie}(V)$. The isomorphism of Lie algebras maps $(A_i)_{i \in n}$ onto $D_{(A_j)_{j \in n}}$. \hfill \Box

The vector space $\bigoplus_{i=1}^n (\text{Lie}(V)/< X_i >)$ equip with the Lie bracket $\{ \}$ we shall denote by $(\bigoplus_{i=1}^n (\text{Lie}(V)/< X_i >), \{ \})$.

We define a semi-direct product of Lie algebras
\[ \mathrm{Lie}(V) \times \mathrm{Der}^* \mathrm{Lie}(V) \]
defining a Lie bracket \( \{ \} \) on the product of vector spaces \( \mathrm{Lie}(V) \times \mathrm{Der}^* \mathrm{Lie}(V) \) in the following way
\[
\{(\lambda, D\beta), (\lambda_1, D\beta_1)\} := ([\lambda, \lambda_1] + D\beta(\lambda_1) - D\beta_1(\lambda), [D\beta, D\beta_1]).
\]
Hence the Lie bracket in a semi-direct product of Lie algebras
\[ \mathrm{Lie}(V) \times (\bigoplus_{i=1}^n (\mathrm{Lie}(V)/<X_i>), \{ \}) \]
is given by
\[
\{(\lambda, \beta), (\lambda_1, \beta_1)\} := ([\lambda, \lambda_1] + D\beta(\lambda_1) - D\beta_1(\lambda), \{\beta, \beta_1\}).
\]

We recall that \( \mathbb{Q}_\ell\{X_1, \ldots, X_n\} \) is a \( \mathbb{Q}_\ell \)-algebra of polynomials in non-commuting variables \( X_1, \ldots, X_n \). Observe that any derivation of the Lie algebra \( \mathrm{Lie}(V) \) (resp. \( L(V) \)) induces a derivation of the \( \mathbb{Q}_\ell \)-algebra \( \mathbb{Q}_\ell\{X_1, \ldots, X_n\} \) (resp \( \mathbb{Q}_\ell\{\{X_1, \ldots, X_n\}\} \)). Let \( \omega \in \mathbb{Q}_\ell\{X_1, \ldots, X_n\} \) (resp \( \omega \in \mathbb{Q}_\ell\{\{X_1, \ldots, X_n\}\} \)). We denote by \( L_\omega \) the left multiplication by \( \omega \) in the corresponding \( \mathbb{Q}_\ell \)-algebra. We denote by \( L_{\mathrm{Lie}(V)} \) (resp. \( L_{L(V)} \)) the set of left multiplications by elements of \( \mathrm{Lie}(V) \) (resp. \( L(V) \)). Observe that the semi-direct product
\[ L_{\mathrm{Lie}(V)} \times \mathrm{Der}^* \mathrm{Lie}(V) \subset \mathrm{End}_{\mathbb{Q}_\ell - \text{linear}}(\mathbb{Q}_\ell\{X_1, \ldots, X_n\}). \]
Notice that the Lie algebras \( \mathrm{Lie}(V) \times \mathrm{Der}^* \mathrm{Lie}(V) \) and \( L_{\mathrm{Lie}(V)} \times \mathrm{Der}^* \mathrm{Lie}(V) \) are obviously isomorphic. The same is true if we replace \( \mathrm{Lie}(V) \) by \( L(V) \) and \( \mathbb{Q}_\ell\{X_1, \ldots, X_n\} \) by \( \mathbb{Q}_\ell\{\{X_1, \ldots, X_n\}\} \).

1.4. Using the representations
\[ (1.4.1) \quad G_K \to \text{Aut}_{\mathbb{Q}_\ell - \text{algebra}}(\mathbb{Q}_\ell\{\{X_1, \ldots, X_n\}\}) \]
and
\[ \varphi_p : G_K \to \text{Aut}_{\mathbb{Q}_\ell - \text{linear}}(\mathbb{Q}_\ell\{\{X_1, \ldots, X_n\}\}) \]
we shall define filtrations of the Galois group \( G_K \). We set
\[ G_m = G_m(V, v) := \ker(\psi_m : G_K \to \text{Aut}_{\mathbb{Q}_\ell - \text{algebra}}(\mathbb{Q}_\ell\{\{X_1, \ldots, X_n\}\}/I^{m+1})), \]
where \( I \) is the augmentation ideal of the \( \mathbb{Q}_\ell \)-algebra \( \mathbb{Q}_\ell\{\{X_1, \ldots, X_n\}\} \) and \( \psi_m \) is induced by the action (1.4.1) of \( G_K \) on the \( \mathbb{Q}_\ell \)-algebra \( \mathbb{Q}_\ell\{\{X_1, \ldots, X_n\}\} \). We set
\[ H_m = H_m(V, z, v) := \ker(\varphi_{p,m} : G_m \to \text{Aut}_{\mathbb{Q}_\ell - \text{linear}}(\mathbb{Q}_\ell\{\{X_1, \ldots, X_n\}\}/I^m)), \]
where $\varphi_{p,m}$ is induced by $\varphi_p$.

We set

$$G_\infty := \bigcap_{m=1}^{\infty} G_m \text{ and } H_\infty := \bigcap_{m=1}^{\infty} H_m.$$ 

2. Lie algebras of actions of Galois groups on torsors.

2.1. We have seen in section 1 that the action of $G_K$ on the torsor $\pi(V_\bar{K}, z, v)$ leads to the Galois representation

$$\varphi_p : G_K \to \text{Aut}(\mathbb{Q}_\ell \{\{X_1, \ldots, X_n\}\}),$$

where $\varphi_p(\sigma)(\omega) = \Lambda_p(\sigma) \cdot \sigma(\omega)$. It is shown in [W1] Lemma 5.1.7 that for $\sigma \in \text{Gal}(\bar{K}/K(\mu_\ell\infty))$,

$$(2.1.1) \ log \varphi_p(\sigma) = L_{log \varphi_p(\sigma)(1)} + log \sigma.$$ 

Moreover we have

$$(2.1.2) \ (log \sigma)(X_i) = [X_i, log \varphi_{\gamma_i}(\sigma)(1)]$$

for $i = 1, \ldots, n$ (see [W1] Proposition 5.1.8). Passing with the representation $\varphi_p$ to Lie algebras we get a homomorphism of Lie algebras

$$\text{Lie}\varphi_p : \text{Lie}(H_1/H_\infty \otimes \mathbb{Q}) \to \text{End}_{\mathbb{Q}_\ell-\text{linear}}(\mathbb{Q}_\ell \{\{X_1, \ldots, X_n\}\}).$$

It follows from (2.1.1) and (2.1.2) that $\text{Lie}\varphi_p$ factors through

$$\text{Lie}\varphi_p : \text{Lie}(H_1/H_\infty \otimes \mathbb{Q}) \to L_{L(V)} \times \text{Der}^* L(V).$$

We recall that we have a canonical isomorphism

$$L_{L(V)} \times \text{Der}^* L(V) \approx L(V) \times (\bigoplus_{i=1}^{n} (L(V)/ \prec X_i \succ), \{ \}).$$

Let $\sigma \in \text{Gal}(\bar{K}/K(\mu_\ell\infty))$. We shall calculate coordinates of $(\text{Lie} \varphi_p)(\sigma)$ in $L(V) \times (\bigoplus_{i=1}^{n} (L(V)/ \prec X_i \succ), \{ \}).$

**Lemma 2.1.3.** Let $\sigma \in \text{Gal}(\bar{K}/K(\mu_\ell\infty))$. Then

$$(\text{Lie}\varphi_p)(\sigma) = (log \varphi_p(\sigma)(1), (log \varphi_{\gamma_i}(\sigma)(1))_{i \in \mathbb{N}}).$$

*Proof.* The lemma follows from (2.1.1) and (2.1.2). \qed

We pass with the morphism $\text{Lie}\varphi_p$ to associated graded Lie algebras. Then we get a morphism

$$\text{gr}\text{Lie}\varphi_p : \text{gr}\text{Lie}(H_1/H_\infty \otimes \mathbb{Q}) \to L_{\text{Lie}(V)} \times \text{Der}^* \text{Lie}(V).$$

Let us set

$$\phi_p := \text{gr}\text{Lie}\varphi_p.$$

**Lemma 2.1.4.** Let $\sigma \in H_n$. Then
i) \( \log \varphi_p(\sigma)(1) \equiv \log \Lambda_p(\sigma) \mod \Gamma^{n+1}\text{Lie}(V) \),

ii) the class of \( \log \Lambda_p(\sigma) \mod \Gamma^{n+1}\text{Lie}(V) \) does not depend on a choice of a path \( p \) from \( v \) to \( z \).

**Proof.** The lemma is already proved in [W1]. \( \square \)

Let \( \sigma \in H_n \). We denote by \( \mathcal{L}(z, v)(\sigma) \) the class of \( \log \Lambda_p(\sigma) \mod \Gamma^{n+1}\text{Lie}(V) \).

Now we can calculate coordinates of \( \phi_p(\sigma) \) in \( L_{\text{Lie}(V)} \times \text{Der}^*\text{Lie}(V) \approx \text{Lie}(V) \times (\bigoplus_{i=1}^n (\text{Lie}(V)/<X_i>), \{ \}) \).

**Lemma 2.1.5.** Let \( \sigma \in H_n \). Then

\[
\phi_p(\sigma) = (\mathcal{L}(z, v)(\sigma), (\mathcal{L}(v_i, v)(\sigma))_{i \in \mathbb{N}})
\]

in \( \text{Lie}(V) \times (\bigoplus_{i=1}^n (\text{Lie}(V)/<X_i>), \{ \}) \).

**Proof.** The lemma follows from Lemmas 2.1.3 and 2.1.4. \( \square \)

It follows from Lemma 2.1.5 that the morphism of Lie algebras

\[
\phi_p : \text{grLie}(H_1/H_\infty \otimes \mathbb{Q}) \to L_{\text{Lie}(V)} \times \text{Der}^*\text{Lie}(V).
\]

does not depend on a choice of a path \( p \) from \( v \) to \( z \), hence we shall denote it by \( \phi_{z,v} \).

We set

\[
t_V(z, v) := \text{image}(\phi_{z,v}).
\]

Observe that the Lie algebra \( t_V(v, v) \) is the associated graded Lie algebra of the image of \( \text{Gal}(\overline{K}/K(\mu_{\infty})) \) in \( \text{Aut}(\pi_1(V_{\overline{K}}, v)) \). This Lie algebra was studied in [W1] section 15. To indicate the importance of the Lie algebra \( t_V(v, v) \) we set

\[
\delta_V(v) := t_V(v, v).
\]

3. **Examples.**

Let \( V = P^1_{\mathbb{Q}} \setminus \{0, 1, \infty\} \). In the fundamental group \( \pi_1(V_{\overline{\mathbb{Q}}}, 01) \) we have two generators \( x \) - loop around 0 and \( y \) - loop around 1. We embed \( \pi_1(V_{\overline{\mathbb{Q}}}, 01) \) into \( \mathbb{Q}_\ell\{\{X, Y\}\} \) mapping \( x \) onto \( e^X \) and \( y \) onto \( e^Y \).

**Proposition 3.1.** The Lie algebras \( \delta_V(01) \) and \( t_V(10, 01) \) are isomorphic.

**Proof.** It follows from Lemma 2.1.5 that

\[
\phi_{01,01}(\sigma) = (0, (0, \mathcal{L}(10, 01)(\sigma))).
\]
and
\[ \phi_{10,01}(\sigma) = (\mathcal{L}(10,01)(\sigma), (0, \mathcal{L}(10,01)(\sigma))) \]
in \( \text{Lie}(V) \tilde{\times} (\langle \text{Lie}(V)/ < X > \rangle \oplus \langle \text{Lie}(V)/ < Y > \rangle, \{ \}) \). It is clear that the map \( \delta_V(01) \to t_V(10,01) \) sending \((0, (0, \mathcal{L})) \) to \((\mathcal{L}, (0, \mathcal{L})) \) is an isomorphism of the corresponding Lie algebras.

\[ \boxed{\text{Proposition 3.2.} \text{ The Lie algebra } t_V(-1,01) \text{ contains a free Lie subalgebra on free generators in degree } 1, 3, 5, \ldots, 2n + 1, \ldots.} \]

**Proof.** The proof is based on Deligne’s ideas indicated in [D]. It follows from Lemma 2.1.5 that
\[ (3.2.1) \quad \phi_{-1,01}(\sigma) = \left( \mathcal{L}(-1,01)(\sigma), (0, \mathcal{L}(10,01)(\sigma)) \right) \]
in \( \text{Lie}(V) \tilde{\times} (\langle \text{Lie}(V)/ < X > \rangle \oplus \langle \text{Lie}(V)/ < Y > \rangle, \{ \}) \). Let \( I_n \) be a vector subspace of \( \text{Lie}(V) \) generated by Lie brackets of the Lie algebra \( \text{Lie}(V) \) which contain at least \( n \) \( Y \)'s. Let us set
\[ I_n := I_n \oplus (I_n \oplus I_n). \]
Observe that \( I_n \) is a Lie ideal of the Lie algebra \( \text{Lie}(V) \tilde{\times} (\langle \text{Lie}(V)/ < X > \rangle \oplus \langle \text{Lie}(V)/ < Y > \rangle, \{ \}) \).

Let \( n > 1 \) and let \( \sigma \in H_n \). It follows from the definition of \( \ell \)-adic polylogarithms in [W1] section 11 and from the definition of the filtration \( \{ H_k \}_{k \in \mathbb{N}} \) of \( G_Q \) that
\[ (3.2.2) \quad \mathcal{L}(10,01)(\sigma) \equiv \ell_n(10)(\sigma)[..[Y,X],X^{n-2}] \mod I_2 + \Gamma^{n+1}L(V) \]
and
\[ (3.2.3) \quad \mathcal{L}(-1,01)(\sigma) \equiv \ell_n(-1)(\sigma)[..[Y,X],X^{n-2}] \mod I_2 + \Gamma^{n+1}L(V). \]
It follows from the work of Soulé (see [S1] and [S2]) and the relation between \( \ell \)-adic polylogarithms and classes of Soulé (see [W1] Corollary 14.3.3 and also [NW] Remark 2 and [W2] Proposition 3.4) that \( \ell_{2n+1}(10) \neq 0 \) and \( \ell_{2n}(10) = 0 \). In [W1] Corollary 11.2.3 and also in [W2] Theorem 2.1 we have proved the identity
\[ 2^{n-1}(\ell_n(-1) + \ell_n(1)) = \ell_n(1) \]
after the restriction to $H_n$. ($\ell_n(1)$ denotes $\ell_n(\overrightarrow{10})$.) Hence we get that

$$\ell_n(1) = \frac{2^{n-1}}{1 - 2^{n-1}}\ell_n(-1)$$

for $n > 1$. This implies that $\ell_{2n}(-1) = 0$.

Let $n > 1$ and let $\sigma \in H_n$. It follows from (3.2.1) - (3.2.4) that in the Lie algebra $t_V(-1, \overrightarrow{01})$ there is an element of the form

$$(\ell_n(-1)(\sigma)[..[Y, X], X^{n-2}] + u_n, (0, \frac{2^{n-1}}{1 - 2^{n-1}}\ell_n(-1)(\sigma)[..[Y, X], X^{n-2}] + \omega_n))$$

where $u_n, \omega_n \in I_2$. Let us take $\sigma \in H_{2n+1}$ such that $\ell_{2n+1}(-1)(\sigma) \neq 0$. Multiplying by $(1 - 2^{2n})$ and dividing by $\ell_{2n+1}(-1)(\sigma)$ we get an element of the form

$$z_{2n+1} := ((1 - 2^{2n})[..[Y, X], X^{2n-1}] + u_{2n+1}, (0, 2^{2n}[..[Y, X], X^{2n-1}] + w_{2n+1}))$$

$(u_{2n+1}, w_{2n+1} \in I_2)$ in the Lie algebra $t_V(-1, \overrightarrow{01})$.

Let $n = 1$. It follows from [W1] Proposition 11.0.8 that $\ell_1(-1) = \ell(2)$. The $\ell$-adic logarithm $\ell(2)$ is the Kummer character associated to 2 (see [W1] Proposition 14.1.0.). Hence there is an element $\sigma \in H_1$ such that $\ell(2)(\sigma) \neq 0$. Therefore we get that $\mathcal{L}(\overrightarrow{10}, \overrightarrow{01})(\sigma) = 0$ and $\mathcal{L}(-1, \overrightarrow{01})(\sigma) = \ell(2)(\sigma)Y$. Hence the element

$$z_1 := (Y, (0, 0))$$

belongs to $t_V(-1, \overrightarrow{01})$.

Let us set $t_{2n+1} = ((1 - 2^{2n})[..[Y, X], X^{2n-1}], (0, 2^{2n}[..[Y, X], X^{2n-1}]))$ for $n > 1$ and $t_1 = (Y, (0, 0))$. Observe that for any Lie bracket of length $r$ in the Lie algebra $\text{Lie}(V)\hat{\times}((\text{Lie}(V)/ \langle X \rangle) \oplus (\text{Lie}(V)/ \langle Y \rangle), \{ \})$ we have

$$\{\{z_{i_1}, z_{i_2}\} \ldots z_{i_r}\} \equiv \{\{t_{i_1}, t_{i_2}\} \ldots t_{i_r}\} \mod \mathcal{I}_{r+1}.$$

Let us set $s_{2n+1} = \{[..[Y, X], X^{2n-1}]$ for $n > 0$ and $s_1 = Y$. Notice that the elements $t_1, t_3, \ldots$ and $s_1, s_2, \ldots$ have integer coefficients. Observe that

$$\{\{t_{i_1}, t_{i_2}\} \ldots t_{i_r}\} \equiv \{\{[s_{i_1}, s_{i_2}] \ldots s_{i_r}], (0, 0)\} \mod 2,$$

where $[,]$ is the standard Lie bracket in the free Lie algebra $\text{Lie}(V)$.

The Hall basic Lie elements in $s_1, s_3, \ldots, s_{2n+1}$, ... in the free Lie algebra $\text{Lie}(V)$ are linearly independent. Hence the Hall basic Lie elements $z_1, z_3, \ldots, z_{2n+1}, \ldots$ in the Lie algebra $\text{Lie}(V)\hat{\times}((\text{Lie}(V)/ \langle X \rangle) \oplus (\text{Lie}(V)/ \langle Y \rangle), \{ \})$ are linearly independent. Hence the elements $z_1, z_3, \ldots, z_{2n+1}, \ldots$ are free generators of a free Lie subalgebra of $t_V(-1, \overrightarrow{01})$. $$\square$$
The references:


[DW] J.-C. Douai, Z. Wojtkowiak, On the Galois actions on the fundamental group of $\mathbb{P}^1_{\mathbb{Q}(\mu_n)} \setminus \{0, \mu_n, \infty\}$, Tokyo Journal of Mathematics, Vol. 27, No. 1, 2004, 21–34.


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