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ON SIMPLE-INJECTIVE MODULES

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Throughout this paper, rings are associative with identity and modules are unitary. For terminologies and notations we shall follows [1].

Let R be a ring and L_R a right R-module. Then a right R-module M_R is said to be L-*simple-injective* (resp. L-*FI-injective*) if for any submodule K of L_R, any homomorphism $\theta : K \to M$ with image simple (resp. finitely generated) can be extended to a homomorphism $\eta : L \to M$. The definition of "simple-injective modules" was introduced by Harada [8]. Trivially, any L-injective module is L-FI-injective and any L-FI-injective module is L-simple-injective. In case R is a semiprimary ring, any finitely cogenerated L-simple-injective right R-module is L-injective (see e.g. [3, Proposition 2] or [10, Lemma 2.1]). In this paper, we shall give other conditions for an R-simple-injective module to be injective (or R-FI-injective). A module M_R is called *semicompact* if any finitely solvable system $(x_i, X_i)_{i\in I}$ of M with $X_i = l_M(A_i)$ for some $A_i \subseteq R$ is solvable, where $l_M(A_i) = \{x \in M \mid xA_i = 0\}$. For a module M_R with $P = \mathrm{End}M_R$, if ${}_PM$ is linearly compact, then M_R is trivially semicompact.

In this paper, for an R-simple-injective module M_R with essential socle, we shall show that M_R is R-FI-injective if ${}_PM$ is AB-5*, where $P = \mathrm{End}M_R$ (Theorem 4), and show that M_R is injective if and only if M_R is semicopmact (Theorem 9). These results are obtained as special cases of certain results using bilinear maps, which are generalizations of Theorem 3.2 and Proposition 4.1 in Ánh, Helbera and Menini [2].

Let P and Q be rings, and ${}_PM$, N_Q and ${}_PU_Q$ a left P-module, a right Q-module and a P-Q-bimodule, respectively, and let $\varphi : M \times N \to U$ be a P-Q-bilinear map. Then we say that $({}_PM, N_Q)$ is a *pair with respect to* U (or φ) or simply a *pair* (see [10]). For elements $x \in M$, $y \in N$ and subsets $X \subseteq M$, $Y \subseteq N$, by xy we denote the element $\varphi(x, y)$, and by $r_N(X)$ (resp. $l_M(Y)$) we denote the right (resp. left) annihilator module $\{y \in N \mid Xy = 0\}$ $(\le N_Q)$ (resp. $\{x \in M \mid xY = 0\}$ $(\le {}_PM)$). Moreover for an element $x \in M$, submodules $Z \le Y \le N_Q$ and a homomorphism $\theta : Y \to U$ by $\hat{x} : N \to U$ we denote the left multiplication map by x and by $\theta|_Z$ we denote the restriction map of θ to Z.

Let $({}_PM, N_Q)$ be a pair with respect to U. Then U_Q is said to be (M, N)-*injective* if the following condition $(*)$ holds for any submodule K of N_Q and any homomorphism $\theta : K \to U$.

$(*)$ $\theta : K \to U$ is given by left multiplication by an element of M.

Moreover U_Q is said to be (M,N)-*FI-injective* (resp. (M,N)-*CI-injective* or (M,N)-*simple-injective*) if the condition $(*)$ holds for any $K(\leq N_Q)$ and any homomorphism $\theta : K \to U$ whose image is finitely generated (resp. cyclic or simple).

Let ${}_PM_R$ and L_R be a P-R-bimodule and a right R-module, respectively, and let $({}_PL^*, L_R)$ be a pair with respect to a natural map $\psi : L^* \times L \to M$, where ${}_PL^* = \mathrm{Hom}_R(L,M)$. Then (L^*,L)-injectivity of M_R implies L-injectivity of M_R and in particular (in case $L = R$) (M,R)-injectivity of M_R implies injectivity of M_R.

Let $({}_PM, N_Q)$ be a pair, and $y \in N$ and $K \leq N_Q$. Then by $y^{-1}K$ we denote the following right ideal of Q; $y^{-1}K = \{a \in Q \mid ya \in K\}$.

Lemma 1. *Let $({}_PM, N_Q)$ be a pair with respect to U and $y \in N$ and $K \leq N_Q$. Then the following hold.*

(1) $l_M(K)y \leq l_U(y^{-1}K) = \{\theta(y) \mid \theta \in \mathrm{Hom}_Q(yQ+K,U)$ *and* $K \leq \mathrm{Ker}\theta\}$.
(2) $l_M(K)y = l_U(y^{-1}K)$ *if and only if any homomorphism* $\theta : yQ+K \to U$ *with* $K \leq \mathrm{Ker}\theta$ *is given by left multiplication by an element of* M.

Proof. It is clear that $l_M(K)y \leq l_U(y^{-1}K)$ and $\{\theta(y) \mid \theta \in Hom_Q(yQ+K,U)$ and $K \leq \mathrm{Ker}\theta\} \leq l_U(y^{-1}K)$. For any element $u \in l_U(y^{-1}K)$, a map $\theta : yQ+K \to U$ via $\theta(ya+z) = ua$ ($a \in Q$, $z \in K$) is well-defined and a Q-homomorphism with $\theta(y) = u$ and $K \leq \mathrm{Ker}\theta$ since $ya+z = 0$ implies $ua \in u(y^{-1}K) = 0$. Hence (1) is obtained. Moreover (2) is an immediate consequence of (1). □

Lemma 2. *Let $({}_PM, N_Q)$ be a pair with respect to U. Then the following are equivalent.*

(1) U_Q *is* (M,N)-*FI-injective.*
(2) U_Q *is* (M,N)-*CI-injective.*
(3) $l_U(y^{-1}K) = l_M(K)y$ *for any element* $y \in N$ *and any submodule* K *of* N_Q.

Proof. (2) $\Rightarrow$ (1). Assume (2). Let Y and K be submodules of N_Q with $Y = \Sigma_{i=1}^n y_iQ$ and let $\theta : Y+K \to U$ be a homomorphism with $K \leq \mathrm{Ker}\theta$. By induction on n, we show that θ is given by left multiplication by an element of M. Put $Y_1 = \Sigma_{i=1}^{n-1} y_iQ$ and $Y_2 = y_nQ$. By the assumption (2), $\theta|_{Y_2+K} = \hat{z}$ for some $z \in M$. Since $(\theta - \hat{z})(Y_2+K) = 0$, by inductional assumption $\theta - \hat{z} : Y_1 + (Y_2+K) \to U$ is given by left multiplication $\hat{w}$ by some element w of M. Hence we have $\theta = \hat{x}$ with $x = z+w$.

The converse (1) $\Rightarrow$ (2) is trivial and the equivalence (2) $\Leftrightarrow$ (3) follows from Lemma 1. □

Let ${}_PM$ be a left P-module. Then a family $\{L_i\}_{i\in I}$ of submodules of M is called an *inverse system* of M if for any indices $i, j \in I$, there exists an index $k \in I$ such that $L_k \le L_i \cap L_j$. A module ${}_PM$ is said to be $AB5^*$ if for any submodule K of M and any inverse system $\{L_i\}_{i\in I}$ of M, $\cap_{i\in I}(K + L_i) = K + \cap_{i\in I}L_i$ holds. By [4, Theorem 6] (or [5, Lemma 2.2]) a module ${}_PM$ is AB5* if and only if there exists a pair $({}_PM, N_Q)$ with some ring Q and some right Q-module N_Q such that $l_M r_N(X) = X$ and $r_N l_M(Y) = Y$ hold for any submodules $X \le {}_PM$ and $Y \le N_Q$. In Theorem 3 below, we consider a condition which is weaker than AB5*. The following theorem is a generalization of [2, Theorem 3.2].

Theorem 3. *Let $({}_PM, N_Q)$ be a pair with respect to U such that U_Q has essential socle. Then the following are equivalent.*

(1) *U_Q is (M, N)-FI-injective.*

(2) (i) *U_Q is (M, N)-simple-injective.*

(ii) *$\cap_{i\in I}(l_M(K)y + L_i) = l_M(K)y + \cap_{i\in I}L_i$ holds for any element y of N, any submodule K of N_Q and any inverse system $\{L_i\}_{i\in I}$ of ${}_PU$ with $L_i = l_U r_Q(L_i) \le l_U(y^{-1}K)$ $(i \in I)$ (see (1) of Lemma 1).*

Proof. (1) $\Rightarrow$ (2). This follows immediately from Lemma 2.

(2) $\Rightarrow$ (1). By Lemma 2, it suffices to show that U_Q is (M, N)-CI-injective. The proof is a modification of [2, Theorem 3.2]. Let $\theta : yQ + K \to U$ be a homomorphism with $K \le \mathrm{Ker}\theta$, where $y \in N$ and $K \le N_Q$, and put $W = \{L \le {}_PU \mid \theta(y) \in l_M(K)y + L \text{ and } L = l_U r_Q(L) \le l_U(y^{-1}K)\}$. Then W is non-empty since $l_U r_Q(\theta(y)) \in W$. For any non-empty chain $\{L_i\}_{i\in I}$ in W, by (ii) we have $\theta(y) \in \cap_{i\in I}(l_M(K)y + L_i) = l_M(K)y + \cap_{i\in I}L_i$ and $\cap_{i\in I}L_i = l_U(\Sigma_{i\in I} r_Q(L_i)) \le l_U(y^{-1}K)$. Therefore by Zorn's lemma there exists a minimal element L in W. Hence for some elements $x \in l_M(K)$ and $u \in L$, we have $\theta(y) = xy + u$ i.e. $u = \theta(y) - xy$, and by minimality of L, $l_U r_Q(u) = L$ holds. Put $A = r_Q(u)$. We show that $u = 0$. Assume $u \ne 0$. Since uQ has a non-zero socle, for some element $a \in Q$, uaQ is simple and in particular $a \notin A$. Put $\eta = (\theta - \hat{x})|_{(yaQ+yA+K)}$. Since $\theta(y) - xy = u$ and $\eta(K) = 0$, $\mathrm{Im}\eta = uaQ + uA = uaQ$ is simple. Hence by (i) there exists an element $w \in l_M(K)$ such that $\eta = \hat{w}$. Put $z = x + w$. Then $z \in l_M(K)$ and $\theta|_{(yaQ+yA+K)} = \hat{z}$ and in particular $(\theta(y) - zy)(aQ + A) = 0$. Hence putting $v = \theta(y) - zy$, we have $\theta(y) = zy + v \in l_M(K)y + l_U r_Q(v)$. But $r_Q(u) = A < aQ + A \le r_Q(v)$, so $l_U r_Q(v) < l_U r_Q(u) = L$, which contradicts the minimality of L. Thus we have that $u = 0$, so θ is given by left multiplication by $x \in M$. □

As an immediate consequence of Theorem 3, we have;

Theorem 4. *Let M_R be a module with $P = \mathrm{End} M_R$ and assume that $_PM$ is AB-5*. If M_R is an R-simple-injective module with essential socle, then M_R is R-FI-injective.*

A module M_R is called *quasi-simple-injective* (resp. *quasi-FI-injective*) if M_R is M-simple-injective (resp. M-FI-injective). Let M_R be a module with $P = \mathrm{End} M_R$ and consider a pair $({}_PP, M_R)$ with respect to a map $\varphi : P \times M \to M$ via $\varphi(a, x) = ax$. Then by Theorem 3, we have;

Proposition 5. *Let M_R be a module with $P = \mathrm{End} M_R$ and assume that $_PM$ is AB5*. If M_R is a quasi-simple-injective module with essential socle, then M_R is quasi-FI-injective.*

Let $_PM$ be a module. A class $(x_i, X_i)_{i \in I}$ (where $x_i \in M$ and $X_i \leq {}_PM$ for any $i \in I$) is called *solvable* if there exists an $x \in M$ such that $x - x_i \in X_i$ for any $i \in I$, and it is called *finitely solvable* if $(x_i, X_i)_{i \in F}$ is solvable for any finite subset F of I. For a class A of submodules of $_PM$, $_PM$ is said to be A-*linearly compact* if any finitely solvable system $(x_i, X_i)_{i \in I}$ of $_PM$ with $X_i \in A$ is solvable. A module $_PM$ is said to be *linearly compact* if it is C-linearly compact for the class C of submodules of $_PM$. If $_PM$ is linearly compact, then it is clearly A-linearly compact for any class A of submodules of $_PM$.

Let $({}_PM, N_Q)$ be a pair. Then by $A_l(M, N)$ we denote the class $\{X \leq {}_PM \mid X = l_M r_N(X)\}$ of submodules of $_PM$.

Remark 1. Let $({}_PM, N_Q)$ be a pair with respect to a P-Q-bilinear map $\varphi : M \times N \to U$ and X a submodule of $_PM$ with $X = l_M r_N(X)$. Then for a pair $({}_PX, N_Q)$ with respect to the restriction map $\varphi|_{X \times N}$, in case $_PM$ is $A_l(M, N)$-linearly compact, $_PX$ is $A_l(X, N)$-linearly compact.

Let $({}_PM, N_Q)$ be a pair with respect to U. Then U_Q is said to be (M, N)-*F-injective* if for any finitely generated submodule K of N_Q, any homomorphism $\theta : K \to U$ is given by left multiplication by an element of M. Every (M, N)-FI-injective module is clearly (M, N)-F-injective. As a characterization of an (M, N)-injective module, we have the following theorem, which is essentially proved by Matlis [9, Propositions 2 and 3] (also see [11, Proposition 1.1] and [2, Proposition 4.1]). However, for the benefit of reader we provide a proof.

Theorem 6. (see [9], [11] or [2]). *Let $({}_PM, N_Q)$ be a pair with respect to U. Then the following are equivalent.*

(1) *U_Q is (M, N)-injective.*
(2) (i) *U_Q is (M, N)-F-injective.*
  (ii) *$_PM$ is $A_l(M, N)$-linearly compact.*

Proof. (1) $\Rightarrow$ (2). We only show (ii) since (i) is trivial. Let $(x_i, X_i)_{i\in I}$ be a finitely solvable system in ${}_PM$ with $X_i = l_M r_N(X_i)$. Then the map $\theta : \Sigma_{i\in I} r_N(X_i) \to U$ via $\theta(\Sigma y_i) = \Sigma x_i y_i$ ($y_i \in r_N(X_i)$) is well-defined and we have $(\theta - \hat{x}_i)(r_N(X_i)) = 0$ for each $i \in I$. Hence by assumption $(\theta - \hat{x}_0)(\Sigma_{i\in I} r_N(X_i)) = 0$ for some $x_0 \in M$. Therefore $(x_0 - x_i)r_N(X_i) = 0$, so $x_0 - x_i \in l_M r_N(X_i) = X_i$ for each $i \in I$. Thus $(x_i, X_i)_{i\in I}$ is solvable.

(2) $\Rightarrow$ (1). Let $Y \leq N_Q$ and $\theta : Y \to U$ a homomorphism and put $W = \{K \leq Y_Q \mid K \text{ is finitely generated}\}$. By (i), for every $K \in W$ there is an $x_K \in M$ such that $(\theta - \hat{x}_K)(K) = 0$. Since $(x_K, l_M(K))_{K\in W}$ is a finitely solvable system of ${}_PM$, by (ii) there is an $x_0 \in M$ such that $(x_0 - x_K)K = 0$ for every $K \in W$. Hence $(\theta - \hat{x}_0)(K) = 0$ for every $K \in W$, so $\theta = \hat{x}_0$. Thus U_Q is (M, N)-injective. □

Lemma 7. *Let $({}_PM, N_Q)$ be a pair with respect to U such that ${}_PM$ is $A_l(M, N)$-linearly compact. Then $\cap_{i\in I}(My + L_i) = My + \cap_{i\in I} L_i$ holds for any element y of N and any inverse system $\{L_i\}_{i\in I}$ of ${}_PU$ with $L_i = l_U r_Q(L_i)$ ($i \in I$).*

Proof. It suffices to show that $\cap_{i\in I}(My + L_i) \leq My + \cap_{i\in I} L_i$ since the converse is clear. Let $v \in \cap_{i\in I}(My + L_i)$. Then for each $i \in I$, there is an element $x_i \in M$ such that $v - x_i y \in L_i$. Put $X_i = \{x \in M \mid xy \in L_i\}$. Then $(x_i, X_i)_{i\in I}$ is a finitely solvable system of M since for any finite subset F of I, there is an element $j \in I$ with $L_j \leq L_i$ ($i \in F$), so $(x_j - x_i)y = (x_j y - v) - (x_i y - v) \in L_i$. Hence $(x_i, X_i)_{i\in I}$ is solvable since $X_i = l_M(y r_Q(L_i))$. It follows that there exists an element $x_0 \in M$ such that for each $i \in I$, $(x_0 - x_i)y \in L_i$, so $v - x_0 y = (v - x_i y) - (x_0 - x_i)y \in L_i$. Thus $v = x_0 y + (v - x_0 y) \in My + \cap_{i\in I} L_i$ and we have $\cap_{i\in I}(My + L_i) \leq My + \cap_{i\in I} L_i$. □

The following theorem is a generalization of [2, Proposition 4.1].

Theorem 8. *Let $({}_PM, N_Q)$ be a pair with respect to U such that U_Q has essential socle. Then the following are equivalent.*

(1) *U_Q is (M, N)-injective.*
(2) (i) *U_Q is (M, N)-simple-injective.*
 (ii) *${}_PM$ is $A_l(M, N)$-linearly compact.*

Proof. (1) $\Rightarrow$ (2). (i) Trivial. (ii) By Theorem 6.

(2) $\Rightarrow$ (1). Let K be a submodule of N_Q and consider the pair $({}_P l_M(K), N_Q)$ with respect to U induced from the pair $({}_PM, N_Q)$ with respect to U. Then by Remark 1 and Lemma 7 we have $\cap_{i\in I}(l_M(K)y + L_i) = l_M(K)y + \cap_{i\in I} L_i$ for any element y of N and any inverse system $\{L_i\}_{i\in I}$ of ${}_PU$ with $L_i = l_U r_Q(L_i)$ ($i \in I$). Hence by Theorem 3 U_Q is (M, N)-FI-injective, so by Theorem 6 U_Q is (M, N)-injective. □

Recall that a right R-module M_R is *semicopmact* if ${}_PM$ is A-linearly compact for the class $A = \{X \leq {}_PM \mid X = l_M r_R(X)\}$, where $P = \mathrm{End}M_R$ (see [9] or [11]). By Theorem 8, we have;

Theorem 9. *Let M_R be a module with essential socle. Then the following are equivalent.*

(1) *M_R is injective.*
(2) *M_R is R-simple-injective and semicompact.*

Corollary 10. *Let M_R be a module with $P = \mathrm{End}M_R$ and assume that ${}_PM$ is linearly compact. If M_R is an R-simple-injective module with essential socle, then M_R is injective.*

Applying Theorem 8 to a pair $({}_PP, M_R)$ with respect to a map $\varphi : P \times M \to M$ via $\varphi(a, x) = ax$, we have;

Proposition 11. (cf. Proposition 5). *Let M_R be a module with $P = \mathrm{End}M_R$ and assume that ${}_PP$ is linearly compact. If M_R is a quasi-simple-injective module with essential socle, then M_R is quasi-injective.*

Remark 2. A ring R is called a *dual ring* if $l_R r_R(I) = I$ and $r_R l_R(K) = K$ hold for any left ideal I and any right ideal K of R. In [6, Proposition 5.2], Hajarnavis and Norton showed that for any dual ring R, R_R is R-FI-injective and in [6, Example 6.1] they gave an example of a commutative dual ring which is not self-injective. Hence there exists an R-FI-injective right R-module which is not injective. Every right R-module with socle zero is trivially R-simple-injective. Hence for the ring $\mathbf{Z}$ of integers, $\mathbf{Z}_{\mathbf{Z}}$ is a trivial $\mathbf{Z}$-simple-injective module which is not $\mathbf{Z}$-FI-injective. The authors however know no example of an R-simple-injective right R-module M_R with essential socle such that M_R is not R-FI-injective.

Examples. By the example below, we see the following (1) and (2);

(1) There exists a pair $({}_PM, N_Q)$ with respect to U such that U_Q is injective but it is not (M, N)-simple-injective.

(2) There exists a pair $({}_PM, N_Q)$ with respect to U such that U_Q is an (M, N)-injective module with essential socle and ${}_PM$ is $A_l(M, N)$-linearly compact but ${}_PM$ is not linearly compact (cf. Theorem 8).

In [7, Example], Harada constructed a semiprimary left QF-3 ring which is not right QF-3 (also see [12, Example 2]). Let D be a division ring and ${}_DL_D$ a bimodule with $\dim({}_DL) = \infty$. Put $L^* = \mathrm{Hom}_D({}_DL, {}_DD)$ and

$$R = \begin{bmatrix} D & L & D \\ O & D & L^* \\ O & O & D \end{bmatrix}, \ e = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \ f = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Put $P = eRe(\simeq D)$ and $Q = fRf(\simeq D)$ and let $({}_PeR, Rf_Q)$ be a pair with respect to $\varphi : eR \times Rf \to eRf$ via $\varphi(ea, bf) = eabf$. Let $L = \oplus_{i\in I} Dx_i$ $(x_i \neq 0)$ and let y_i $(i \in I)$ and y be elements in L^* such that $x_i y_i = 1$ $(i \in I)$, $x_j y_i = 0$ $(j \neq i)$ and $x_i y = 1$ $(i \in I)$ and put $Y = [DVD]^T \leq Rf_Q$ and $Z = [DWD]^T \leq Rf_Q$, where $V = \oplus_{i\in I} y_i D \leq L^*_D$, $W = V + yD \leq L^*_D$ and $[-]^T$ denotes the transposed matrix of $[-]$. Then as is easily seen $l_{eR}(Y) = 0$. Since by assumption, I is an infinite set, we have $y \notin V$, so $Z/Y_Q \simeq eRf_Q(\simeq Q)$. Let $\theta : Z \to eRf$ be an epimorphism with $\mathrm{Ker}\theta = Y$. If $\theta = \hat{x}$ for some element $x \in eR$, then $xY = \theta(Y) = 0$ hence $x = 0$, a contradiction. Therefore eRf_Q is not $({}_PeR, Rf_Q)$-simple-injective. On the other hand, eRf_Q is injective over the division ring $Q(\simeq D)$. Next consider a pair $({}_RR, Rf_Q)$ with respect to $\psi : R\times Rf \to Rf$ via $\psi(a, bf) = abf$. Since ${}_RRf \simeq \mathrm{Hom}_P(eR, eRf)$, ${}_RRf$ is an injective (i.e. an $({}_RR, Rf_Q)$-injective) module with essential socle, so by Theorem 8 Rf_Q is $A_r(R, Rf)$-linearly compact, where $A_r(R, Rf) = \{Y \leq Rf_Q \mid Y = r_{Rf}l_R(Y)\}$. But Rf_Q is not linearly compact since $\dim(Rf_Q) = \infty$.

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