A NOTE ON COMMUTATIVE GELFAND THEORY FOR REAL BANACH ALGEBRAS

Dedicated to Professor Saburou Saitoh on his 60th birthday (Kanreki)

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Abstract. Pfaffenberger and Phillips [2] consider a real and unital case of the classical commutative Gelfand theorem and obtain two representation theorems. One is to represent a unital real commutative Banach algebra \( A \) as an algebra of continuous functions on the unital homomorphism space \( \Phi_A \). The other is to represent \( A \) as an algebra of continuous sections on the maximal ideal space \( M_A \). In this note, we point out that similar theorems for non-unital case hold and show that two representation theorems are essentially identical.

1. Preliminary and results

Let \( B \) be a real commutative Banach algebra and \( \Phi_B \) the set of non-zero \( \mathbb{R} \)-algebra homomorphism \( \varphi : B \to \mathbb{C} \). Then we have \( \|\varphi\| \overset{\text{def}}{=} \sup_{\|a\| \leq 1} |\varphi(a)| \leq 1 \) for each \( \varphi \in \Phi_B \). Actually, suppose that there exists an \( a \in B \) such that \( \|a\| < 1 \) and \( |\varphi(a)| = 1 \). Set \( \theta = -\arg \varphi(a) \) and then \( e^{i\theta} \varphi(a) = 1 \). Also set

\[
b = \sum_{n=1}^{\infty} a^n \cos n\theta \quad \text{and} \quad c = \sum_{n=1}^{\infty} a^n \sin n\theta.
\]

Elementary trigonometric identities lead to

\[
b = a \cos \theta + ab \cos \theta - ac \sin \theta \quad \text{and} \quad 0 = a \sin \theta + ab \sin \theta - c + ac \cos \theta.
\]

Apply \( \varphi \) to these equations, multiply the resulting second equation by \( i \) and add it to the resulting first equation then we obtain

\[
\varphi(b) = \varphi(a)e^{i\theta} + \varphi(a)\varphi(b)e^{i\theta} - i\varphi(c) + i\varphi(a)\varphi(c)e^{i\theta},
\]

so that \( 1 = 0 \), a contradiction (we referred the proof of [2, Proposition 1.1, (b)]). Let \( B_c \) be the complexification of \( B \). Then \( \Phi_B \) is a subset of the closed unit ball of the dual space \( B_c^* \) and hence we can give \( \Phi_B \) the relative topology of \( B_c^* \) with the weak* topology. Therefore, as well-known, \( \Phi_B \) is a locally compact Hausdorff space. We denote by \( C(\Phi_B) \) the algebra of continuous complex-valued functions on \( \Phi_B \) and set \( C_0(\Phi_B) = \{ f \in C(\Phi_B) : f \text{ vanishes at infinity} \} \). Then \( C_0(\Phi_B) \) is a real commutative \( C^* \)-algebra with

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Theorem 1. Let $B$ be a real commutative Banach algebra.

1. The mapping $\Lambda_\varphi: B \to C_0(\Phi_B)$ given by $\Lambda_\varphi(b)(\varphi) = \varphi(b)$ \text{(def) $\hat{b}(\varphi)$) for each $\varphi \in \Phi_B$ and each $b \in B$ is a norm-decreasing real algebra homomorphism which is one-to-one if and only if $B$ is semisimple.

2. If $B$ is unital, then $\text{Sp}_B(b) = \hat{b}(\Phi_B)$ for each $b \in B$. Also, if $B$ is non-unital, then $\text{Sp}_B(b) = \hat{b}(\Phi_B) \cup \{0\}$ for each $b \in B$.

We next provide a Gelfand theorem for real commutative Banach algebras using the maximal regular ideal space $M_B$ as a common domain of an algebra of functions.

In the next section, we see that $\text{Ker} \varphi$ is a maximal regular ideal of $B$ for each $\varphi \in \Phi_B$. Following [2], we give $M_B$ the quotient topology arising from the map $\varphi: \Phi_B \to M_B$ defined by $\varphi(\varphi) = \text{Ker} \varphi$, $\varphi \in \Phi_B$. That is, $M_B$ has the strongest topology which makes the map $\varphi$ continuous. Let $\sigma: \Phi_B \to \Phi_B$ be the homeomorphism $\sigma(\varphi) = \tilde{\varphi}$. Then $\sigma^2 = \epsilon$, the identity, and we have an action of $\mathbb{Z}_2 = \{e, \sigma\}$ on $\Phi_B$. Let $\beta: \Phi_B/\mathbb{Z}_2 \to M_B$ be the bijection $\beta(\mathbb{Z}_2 \varphi) = \text{Ker} \varphi$ and then $\varphi = \beta \circ \sigma$, where $\pi: \Phi_B \to \Phi_B/\mathbb{Z}_2$ is the natural map. Then we have the following result from Lemmas 1 and 2 in the next section.

Proposition 1. The space $M_B$ is locally compact and Hausdorff, and the map $\varphi$ is both open and closed. Moreover, $\beta: \Phi_B/\mathbb{Z}_2 \to M_B$ is a homeomorphism.

From the above proposition, we know that $M_B$ is just the quotient of $\Phi_B$ under the (not necessarily free) action of $\mathbb{Z}_2$ on $\Phi_B$. Let

$$\Phi^R_B = \{\varphi \in \Phi_B : \varphi(B) = \mathbb{R}\}, \quad \Phi^C_B = \{\varphi \in \Phi_B : \varphi(B) = \mathbb{C}\},$$

$$M^R_B = \{I \in M_B : B/I \cong \mathbb{R}\} \text{ and } M^C_B = \{I \in M_B : B/I \cong \mathbb{C}\}.$$ 

Clearly $\Phi^R_B$ is a closed subset of $\Phi_B$ and so $\Phi^C_B$ is an open subset of $\Phi_B$. Also, in the next section, we see that

$$M^R_B = \epsilon(\Phi^R_B), \quad M^C_B = \epsilon(\Phi^C_B), \quad \Phi_B = \Phi^R_B \cup \Phi^C_B \text{ and } M_B = M^R_B \cup M^C_B.$$ 

Then $M^R_B$ is closed and $M^C_B$ is open. Also $B$ is called almost complex if $\Phi^C_B = \Phi_B$ \text{(cf. [1, 2])}.

Now as usual in representing algebras as sections we form the set

$$E_B = \bigcup_{I \in M_B} B/I,$$
where \( \cup \) denotes disjoint union. Of course, each \( B/I \) is a field and an algebra over \( \mathbb{R} \), and we have an obvious map \( p: E_B \to M_B \). That is, \( p(I, b + I) = I, (I \in M_B, b \in B) \), where \( E_B \cong \{ (I, b + I) : I \in M_B, b \in B \} \).

The problem is to topologize \( E_B \) in a reasonable way such that \( I \to b + I \) is a continuous section for each \( b \in B \). Following [2], let \( (\mathbb{C} \times \Phi_B)' = (\mathbb{C} \times \Phi_B^C) \cup (\mathbb{R} \times \Phi_B^R) \) endowed with the relative topology in \( \mathbb{C} \times \Phi_B \). We consider the map \( g: (\mathbb{C} \times \Phi_B)' \to E_B \) defined by

\[
g(z, \varphi) = (\text{Ker} \varphi, b + \text{Ker} \varphi),
\]

where \( b \) is chosen such that \( \varphi(b) = z \). Of course, this map is well-defined and surjective. We give \( E_B \) the quotient topology induced by the map \( g \) and denote by \( \Gamma(E_B) \) the set of all continuous sections on \( M_B \). Moreover, we set

\[
\Gamma^b(E_B) = \{ s \in \Gamma(E_B) : \|s\| = \sup_{I \in M_B} |s(I)| < \infty \}
\]

and

\[
\Gamma_0(E_B) = \{ s \in \Gamma(E_B) : s \text{ vanishes at infinity, that is } \lim_{I \to \infty} |s(I)| = 0 \},
\]

where \( |s(I)| = |\varphi(b)|, s(I) = (I, b + I), b \in B \) and \( I = \text{Ker} \varphi = \text{Ker} \tilde{\varphi} \). Then we have the following representation theorem in a way similar to the proof of [2, Theorem 3.5].

**Theorem 2.** Let \( B \) be a real commutative Banach algebra and let \( p: E_B \to M_B \) be the associated bundle of real fields. Then

1. \( \Gamma^b(E_B) \) is a real commutative Banach algebra given the supremum norm and \( \Gamma_0(E_B) \) is a closed subalgebra of it.
2. \( \Lambda_M: B \to \Gamma_0(E_B) \) defined by \( \Lambda_M(b)(I) = (I, b + I), I \in M_B \) is a norm-decreasing algebra homomorphism with kernel, \( \text{Rad} B \).
3. For \( b \in B \), \( \|\Lambda_M(b)\| = \lim_{n \to \infty} \|b^n\|^{1/n} \).

**Remark 1.** Theorems 1 and 2 are non-unital versions of [2, Theorem 1.4 and 3.5].

We next see that these representation theorems are essentially identical. To do this, set

\[
C_h(\Phi_B) = \{ f \in C(\Phi_B) : f(\varphi) = \overline{f(\varphi)} \text{ for all } \varphi \in \Phi_B \},
\]

\[
C^b_h(\Phi_B) = \{ f \in C_h(\Phi_B) : \| f \| < \infty \},
\]

and

\[
C_{h,0}(\Phi_B) = \{ f \in C_h(\Phi_B) : f \text{ vanishes at infinity } \}.
\]

Then we can easily see that \( C^b_h(\Phi_B) \) is a unital real commutative Banach algebra given the supremum norm and \( C_{h,0}(\Phi_B) \) is a closed subalgebra of it. Also \( \Lambda_\Phi(B) \subset C_{h,0}(\Phi_B) \) clearly holds.
Now for each $f \in C_h(\Phi_B)$ we define $f^*(\varphi) = \overline{f(\varphi)}$, $(\varphi \in \Phi_B)$. Then $C^b_h(\Phi_B)$ becomes a unital real commutative $C^*$-algebra under this involution and $C_{h,0}(\Phi_B)$ is a $C^*$-subalgebra of $C^b_h(\Phi_B)$. Also let $s \in \Gamma(E_B)$ and $I \in M_B$. Then $s(I) = (I, b+I)$ for some $b \in B$. Choose $\varphi \in \Phi_B$ with $I = \text{Ker} \varphi$. Then there exists an element $\hat{b} \in B$ such that $\varphi(\hat{b}) = \overline{\varphi(b)}$. Set $s^*(I) = (I, \hat{b}+I)$. This is clearly well-defined and we see later that $s^* \in \Gamma(E_B)$ (Lemma 5). Therefore $\Gamma^b(E_B)$ becomes a unital real commutative $C^*$-algebra under this involution and $\Gamma_0(E_B)$ is a $C^*$-subalgebra of $\Gamma^b(E_B)$.

In this setting, we have the following:

**Theorem 3.** There is an isometric real algebra $*$-isomorphism $\rho$ of $\Gamma^b(E_B)$ onto $C^b_h(\Phi_B)$ such that $\rho(\Gamma_0(E_B)) = C_{h,0}(\Phi_B)$ and $\rho \circ \Lambda_M = \Lambda_\Phi$.

We know that the Gelfand representation theorems 1 and 2 are essentially identical by the above theorem.

Combining [2, Theorem 5.3] and Theorem 3, we have the following:

**Corollary 1.** If $B$ is a unital commutative almost complex $C^*$-algebra, then $\Lambda_\Phi : B \to C_h(\Phi_B)$ is an isometric $*$-isomorphism.

## 2. Known results and lemmas

We will remind the reader of the following well-known results since they are basic to all that we do.

Let $\varphi \in \Phi_B$. Since range $\varphi$ is a non-zero real subalgebra of $\mathbb{C}$, it must be either $\mathbb{R}$ or $\mathbb{C}$. In fact, let $A = \text{range} \varphi$ and then $A$ is a non-zero linear subspace of $\mathbb{C}$ over $\mathbb{R}$. Then $\dim A = 1$ or 2. If $\dim A = 2$, then $A = \mathbb{C}$. If $\dim A = 1$, then $A = \mathbb{R}a$ for some non-zero complex number $a \in A$. Since $A$ is an algebra, it follows that $a^2 = ra$ for some $r \in \mathbb{R}$ and then $a \in \mathbb{R}$. Hence $A$ must be $\mathbb{R}$. We thus obtain that range $\varphi = \mathbb{R} \Leftrightarrow \text{Ker} \varphi$ has codimension 1 in $B$ and range $\varphi = \mathbb{C} \Leftrightarrow \text{Ker} \varphi$ has codimension 2 in $B$. Moreover, $\text{Ker} \varphi$ is a maximal regular ideal of $B$. Actually, choose $e \in B$ with $\varphi(e) = 1$ and hence $B(1-e) \subset \text{Ker} \varphi$, namely, $\text{Ker} \varphi$ is regular. Now let $I$ be an ideal of $B$ with $\text{Ker} \varphi \subseteq I \subset B$. Then $I/\text{Ker} \varphi$ is a non-zero real subalgebra of $B/\text{Ker} \varphi$. Take $e \in B$ and $u \in I$ with $\varphi(e) = 1$ and $u \notin \text{Ker} \varphi$. Since $B/\text{Ker} \varphi$ is a field and $e+\text{Ker} \varphi$ is the identity of $B/\text{Ker} \varphi$, we can find $v \in B$ with $(u + \text{Ker} \varphi)(v + \text{Ker} \varphi) = e + \text{Ker} \varphi$ and hence $uv - e \in \text{Ker} \varphi$. Then $e = uv + e - uv \in I + \text{Ker} \varphi = I$, so that $b = be + b(1-e) \in I + \text{Ker} \varphi = I$ for each $b \in B$. That is, we have $B = I$ and so $\text{Ker} \varphi$ is maximal. Let $I$ be a maximal regular ideal of $B$. Since $B/I$ is a real commutative normed division algebra, it follows from the Gelfand-Mazur theorem that $B/I \cong \mathbb{R}$ or $\mathbb{C}$ and then $I$ has codimension 1 or 2. In case of $\text{codim} I = 1$, we have that $B/I \cong \mathbb{R}$ as an algebra over $\mathbb{R}$ and this isomorphism is unique since $\mathbb{R}$ has no non-trivial $\mathbb{R}$-algebra automorphisms. Thus the composition $\varphi : B \to B/I \cong \mathbb{R}$
is the unique element of $\Phi_B$ with kernel $I$. Moreover we have range $\varphi = \mathbb{R}$. In case of codim $I = 2$, we have that $B/I \cong \mathbb{C}$ as an algebra over $\mathbb{R}$ and since $\mathbb{C}$ has exactly one non-trivial $\mathbb{R}$-algebra automorphism given by conjugation, we see that there are exactly two elements $\varphi, \tilde{\varphi} \in \Phi_B$ with kernel $I$. Moreover we have range $\varphi = \text{range } \tilde{\varphi} = \mathbb{C}$.

**Lemma 1.** Let $G$ be a finite group acting on a topological space $Y$ and let $X = Y/G$ endowed with the quotient topology. Let $p: Y \to X$ be the natural map. Then

1. The map $p$ is both open and closed.
2. For each compact subset $K$ in $X$, $p^{-1}(K)$ is also compact in $Y$.
3. If $Y$ is locally compact, so is $X$.
4. If $Y$ is Hausdorff, so is $X$.

**Proof.**

1. Let $U$ be an open (closed) subset of $Y$. Then the saturation $GU = \cup_{g \in G} g(U)$ of $U$ is clearly open (closed). Also since $GU = p^{-1}(p(U))$, it follows that $p$ is open (closed).

2. Let $K$ be a compact subset of $X$ and $\{y_\lambda\}$ a net in $p^{-1}(K)$. Then $\{p(y_\lambda)\}$ is a net in $K$ and so there is a subnet $\{y_{\lambda'}\}$ of $\{y_\lambda\}$ such that $\{p(y_{\lambda'})\}$ converges to some point of $K$, say $Gy$. Let $Gy = \{y, g_1(y), \ldots, g_{n-1}(y)\}$, where $G = \{e, g_1, \ldots, g_{n-1}\}$. Then we have $Gy \subset p^{-1}(K)$. Now, we assert that a certain subnet of $\{y_{\lambda'}\}$ converges to one of $y, g_1(y), \ldots, g_{n-1}(y)$. Suppose contrary. Then we can easily find an open neighborhood $U$ of $y$ and a subnet $\{y_{\lambda''}\}$ of $\{y_{\lambda'}\}$ such that every $y_{\lambda''}$ does not belong to $U \cup g_1(U) \cup \cdots \cup g_{n-1}(U)$. Set $V = U \cup g_1(U) \cup \cdots \cup g_{n-1}(U)$ and so $V = GV$. Also since $V$ is an open neighborhood of $y$ and $p$ is open, $p(V)$ must be an open neighborhood of $Gy$. Hence there exists a point $p(y_{\lambda''})$ which belongs to $p(V)$. Therefore $y_{\lambda''}$ must be in the saturation of $V$ namely $GV$. However since $V = GV$ and every $y_{\lambda''}$ does not belong to $GV$, this is a contradiction. We thus obtain that any net in $p^{-1}(K)$ has a subnet which converges to some point in $p^{-1}(K)$, that is $p^{-1}(K)$ is compact.

3. Since $p$ is both open and closed by (1), $X$ must be locally compact from a standard topological argument.

4. Suppose that $Y$ is Hausdorff and let $x_1, x_2 \in X$ with $x_1 \neq x_2$. Let $y_1 \in p^{-1}(x_1)$ and $y_2 \in p^{-1}(x_2)$, and then $y_1 \neq y_2$. Since $G$ is finite, we can find an open neighborhood $U_1$ of $y_1$ and an open neighborhood $U_2$ of $y_2$ such that $g(U_1) \cap h(U_2) = \emptyset$ for all $g, h \in G$. Since $p$ is open, $p(U_1)$ and $p(U_2)$ are disjoint open neighborhoods of $x_1$ and $x_2$, respectively. Consequently $X$ is also Hausdorff.

**Lemma 2.** Let $X$ be a topological space and let $Y$ and $Z$ be two sets with surjections $\pi_Y: X \to Y$ and $\pi_Z: X \to Z$. We give $Y$ and $Z$ the quotient topologies induced by $\pi_Y$ and $\pi_Z$, respectively. Moreover, assume that $\pi_Y$
is both open and closed, and there is a bijection \( \theta: Y \rightarrow Z \) such that \( \pi_Z = \theta \circ \pi_Y \). Then \( \pi_Z \) is also both open and closed, and \( \theta \) is a homeomorphism.

**Proof.** Let \( U \) be an open subset of \( X \). Then

\[
\pi_Z^{-1}(\pi_Z(U)) = \pi_Y^{-1}(\theta^{-1}(\pi_Z(U))) = \pi_Y^{-1}(\theta^{-1}(\pi_Y(U))) = \pi_Y^{-1}(\pi_Y(U))
\]

and hence \( \pi_Z^{-1}(\pi_Z(U)) \) is open since \( \pi_Y \) is continuous and open. Therefore \( \pi_Z \) is open by definition of the quotient topology. Similarly, we can prove the closedness of \( \pi_Z \). Now note that \( \pi_Z(\pi_Y^{-1}(K)) = \theta(K) \) for each subset \( K \) of \( Y \). This implies that \( \theta \) is a homeomorphism. \( \square \)

Now, following [2], we give \( E_B \) the quotient topology induced by the map \( g \). There is also a natural action of \( \mathbb{Z}_2 = \{ e, \sigma \} \) on \( (\mathbb{C} \times \Phi_B)' \), namely: \( \sigma(z, \varphi) = (\tilde{z}, \tilde{\varphi}) \). The following is a just [2, Lemma 3.2].

**Lemma 3.** There is a natural homeomorphism \( \theta: (\mathbb{C} \times \Phi_B)'/\mathbb{Z}_2 \rightarrow E_B \) such that \( g = \theta \circ \pi_{\mathbb{Z}_2} \), where \( \pi_{\mathbb{Z}_2} \) is the canonical map of \( (\mathbb{C} \times \Phi_B)' \) onto \( (\mathbb{C} \times \Phi_B)'/\mathbb{Z}_2 \). Thus \( g \) is both open and closed.

**Lemma 4.** Let \( Y \) be a topological space and \( X \) a set. Let \( p: Y \rightarrow X \) be a surjection. We give \( X \) the quotient topology induced by \( p \). Let \( x \in X \) and \( y \in p^{-1}(x) \). If \( p \) is open and \( \{ V_\alpha \} \) is a base of neighborhoods of \( y \), then \( \{ p(V_\alpha) \} \) is a base of neighborhoods of \( x \).

**Proof.** Let \( U \) be any neighborhood of \( x \). Then \( p^{-1}(U) \) is a neighborhood of \( y \) and hence \( V_\alpha \subset p^{-1}(U) \) for some \( V_\alpha \). Therefore we have \( p(V_\alpha) \subset p(p^{-1}(U)) = U \) and so \( \{ p(V_\alpha) \} \) is a base of neighborhoods of \( x \). \( \square \)

**Lemma 5.** If \( s \in \Gamma(E_B) \), then \( s^* \in \Gamma(E_B) \), where \( s^* \) is a section for \( p: E_B \rightarrow M_B \) defined in the preceding section.

**Proof.** Let \( s \in \Gamma(E_B) \). We show that \( s^* \) is continuous. To do this, let \( I_0 \in M_B \) be arbitrary and take an element \( \varphi_0 \in \Phi_B \) with \( I_0 = \text{Ker} \, \varphi_0 \). Also take an element \( b_0 \in B \) with \( s(I_0) = (I_0, b_0 + I_0) \) and set \( z_0 = \varphi_0(b_0) \). By definition of \( s^* \), we have that \( s^*(I_0) = (I_0, \overline{b_0} + I_0) \) and \( \overline{z_0} = \overline{\varphi_0(b_0)} = \varphi_0(\overline{b_0}) \). Then \( g(\overline{z_0}, \varphi_0) = s^*(I_0) \in E_B \). By Lemma 4, a basic neighborhood of \( s^*(I_0) \) in \( E_B \) is of the form \( g((O_\varepsilon(\overline{z_0}) \times N)' \), where \( O_\varepsilon(\overline{z_0}) \) is an \( \varepsilon \)-neighborhood of \( z_0 \) in \( \mathbb{C} \), \( N \) is a neighborhood of \( \varphi_0 \) in \( \Phi_B \) and \( (O_\varepsilon(\overline{z_0}) \times N)' = (O_\varepsilon(z_0) \times N) \cap (\mathbb{C} \times \Phi_B)' \). Also we have

\[
g((O_\varepsilon(\overline{z_0}) \times N)') = \{ (\text{Ker} \, \varphi, b + \text{Ker} \, \varphi) : \varphi \in N \text{ and } \varphi(b) \in O_\varepsilon(\overline{z_0}) \}.
\]

Let \( O_\varepsilon(z_0) \) be an \( \varepsilon \)-neighborhood of \( z_0 \) in \( \mathbb{C} \) and then

\[
g((O_\varepsilon(z_0) \times N)') = \{ (\text{Ker} \, \varphi, b + \text{Ker} \, \varphi) : \varphi \in N \text{ and } \varphi(b) \in O_\varepsilon(z_0) \}.
\]
is a neighborhood of \(s(I_0)\) because \(g(z_0, \varphi_0) = s(I_0) = E_B\). Also since \(s\) is continuous, we can find a neighborhood \(V_0\) of \(I_0\) in \(M_B\) such that \(s(V_0) \subset g((O_\varepsilon(z_0) \times N)^')\). In this case, we have \(s^*(V_0) \subset g((O_\varepsilon(z_0) \times N)^')\). In fact, let \(I \in V_0\) and take an element \(\varphi \in \Phi_B\) with \(I = \text{Ker} \varphi\). Since \(s(I) = (I, b + I)\) for some \(b \in B\), it follows from definition of \(s^*\) that \(s^*(I) = (I, \bar{b} + I)\) and \(\varphi(\bar{b}) = \overline{\varphi(b)}\) for some \(\bar{b} \in B\). Also since \((\text{Ker} \varphi, b + \text{Ker} \varphi) \in g((O_\varepsilon(z_0) \times N)^')\), it follows that \(\varphi \in N\) and \(|\varphi(b) - z_0| < \varepsilon\), and hence \(|\varphi(\bar{b}) - \overline{z_0}| = |\varphi(b) - z_0| < \varepsilon\). Consequently, \(s^*(I) \in g((O_\varepsilon(\overline{z_0}) \times N)^')\). We thus see that \(s^*\) is continuous at each point in \(M_B\).

\[\square\]

3. Proof of Theorem 3

Let \(s \in \Gamma(E_B)\) be arbitrary. For any \(\varphi \in \Phi_B\), set \(I = \text{Ker} \varphi\). Then \(s(I) = (I, b + I)\) for some \(b \in B\). We define \(f_s(\varphi) = \varphi(b)\). This is, of course, well-defined. Then we have that \(f_s(\varphi) = f_s(\psi)\) for all \(\varphi \in \Phi_B\). In fact, let \(\varphi \in \Phi_B\) and put \(I = \text{Ker} \varphi\) and then \(I = \text{Ker} \psi\). Since \(s(I) = (I, b + I)\) for some \(b \in B\), it follows from definition of \(f_s\) that \(f_s(\varphi) = \varphi(b)\) and \(f_s(\psi) = \psi(b)\). Therefore we have \(f_s(\varphi) = \varphi(b) = \overline{\varphi(b)} = f_s(\psi)\). We next claim that \(f_s\) is continuous on \(\Phi_B\). Let \(\varphi_0 \in \Phi_B\) and set \(I_0 = \text{Ker} \varphi_0\). Then \(s(I_0) = (I_0, b_0 + I_0)\) for some \(b_0 \in B\). Put \(z_0 = f_s(\varphi_0)(\varphi_0(b_0))\). Let \(U_{\varepsilon}\) be any \(\varepsilon\)-neighborhood of \(z_0\). If \(\varphi_0 \in \Phi^R_B\), we set \(N_0 = \Phi_B\). If \(\varphi_0 \in \Phi^C_B\), then \(\varphi_0 \neq \overline{\varphi_0}\) and hence there is \(b_1 \in B\) such that \(\varphi_0(b_1) \neq \overline{\varphi_0(b_1)}\), so we set \(N_0 = \{\varphi \in \Phi_B : |\varphi_0(b_1) - \varphi(b_1)| < \delta / 2\}\), where \(\delta = |\varphi_0(b_1) - \overline{\varphi_0(b_1)}| > 0\). Then \(N_0\) is an open neighborhood of \(\varphi_0\). In case of \(\varphi_0 \in \Phi^C_B\), we have that if \(\varphi \in N_0\) then \(\varphi \notin N_0\). In fact, assume that \(\varphi \in N_0\) and \(\bar{\varphi} \in N_0\). Then \(|\varphi_0(b_1) - \varphi(b_1)| < \delta / 2\) and \(|\varphi_0(b_1) - \overline{\varphi(b_1)}| < \delta / 2\), so we have \(|\varphi_0(b_1) - \overline{\varphi_0(b_1)}| < \delta / 2 + \delta / 2 = \delta\), a contradiction. Now note from Lemma 3 that

\[g((U_{\varepsilon} \times N_0)^') = \{(\text{Ker} \varphi, b + \text{Ker} \varphi) : \varphi \in N_0, \varphi(b) \in U_{\varepsilon}\}\]

is an open neighborhood of \(s(I_0)\) in \(E_B\), where \((U_{\varepsilon} \times N_0)^' = (U_{\varepsilon} \times N_0) \cap (\mathbb{C} \times \Phi_B)^'\). Since \(s\) is a continuous section, there exists a neighborhood \(V_0\) of \(I_0\) such that \(s(V_0) \subset g((U_{\varepsilon} \times N_0)^')\). Set \(W_0 = \varepsilon^{-1}(V_0) \cap N_0\) and then it is a neighborhood of \(\varphi_0 \in \Phi_B\). We see that \(f_s(W_0) \subset U_{\varepsilon}\). In fact, let \(\varphi \in W_0\) be arbitrary. Then \(\varphi \in \varepsilon^{-1}(V_0)\) and hence \(\text{Ker} \varphi \in V_0\). Therefore

\[s(\text{Ker} \varphi) \in s(V_0) \subset g((U_{\varepsilon} \times N_0)^') = \{(\text{Ker} \psi, b + \text{Ker} \psi) : \psi \in N_0, \psi(b) \in U_{\varepsilon}\}\]

and so \(s(\text{Ker} \varphi) = (\text{Ker} \psi, b + \text{Ker} \psi)\), for some \(b \in B\) and \(\psi \in N_0\) with \(\psi(b) \in U_{\varepsilon}\). Then \(\text{Ker} \varphi = \text{Ker} \psi\), \(f_s(\varphi) = \varphi(b)\) and \(f_s(\psi) = \psi(b)\). If \(\psi = \varphi\), then \(f_s(\varphi) = \varphi(b) = \overline{\psi(b)} = f_s(\psi)\). If \(\psi \neq \varphi\), then \(\psi = \varphi\) and hence \(f_s(\varphi) = \varphi(b) = \overline{\psi(b)} = f_s(\psi)\). In case of \(\varphi_0 \in \Phi^R_B\), we have \(z_0 \in \mathbb{R}\) and hence \(U_{\varepsilon}\) is conjugate invariant, so \(\overline{\psi(b)} \in U_{\varepsilon}\), that is \(f_s(\varphi) \in U_{\varepsilon}\). In case of \(\varphi_0 \in \Phi^C_B\), since \(\varphi \in N_0\), we have \(\varphi \notin N_0\). However since \(\psi \in N_0\) and \(\psi = \varphi\), we have
Consequently, we have obtained a natural map \( f_s(W_0) \subset U_\varepsilon \). We thus obtain a natural map \( s \mapsto f_s \) of \( \Gamma(E_B) \) into \( C_h(\Phi_B) \).

We next show that this map is surjective. To do this, let \( f \in C_h(\Phi_B) \) be arbitrary. For any \( I \in M_B \), choose \( \varphi \in \Phi_B \) with \( I = \text{Ker} \varphi \). In this case, we can take \( b \in B \) with \( \varphi(b) = f(\varphi) \). In fact, if \( \varphi \in \Phi_B^R \), then \( \varphi = \overline{\varphi} \) and hence \( f(\varphi) = f(\overline{\varphi}) = \overline{f(\varphi)} \), so \( f(\varphi) \in \mathbb{R} \). Moreover, \( \varphi(B) = \mathbb{R} \) and so we can find such \( b \in B \). If \( \varphi \in \Phi_B^C \), then \( \varphi(B) = \mathbb{C} \) and so there exists clearly such \( b \in B \). Now we define \( s_f(I) = (I, b + I) \). This is well-defined. In fact, let \( \psi \in \Phi_B, I = \text{Ker} \psi, c \in B \) and \( s \psi(c) = f(\psi) \). If \( \varphi \neq \psi \), then \( \psi = \overline{\varphi} \) and so

\[
\overline{\varphi(c)} = \overline{\varphi(c)} = \psi(c) = f(\psi) = f(\overline{\varphi}) = \overline{f(\varphi)} = \overline{f(b)},
\]

hence \( c - b \in \text{Ker} \varphi = I \). Similarly, we can treat the case \( \varphi = \psi \). This shows that our definition is well-defined. Now we claim that \( s_f \) is a continuous section for \( p : E_B \to M_B \). It suffices to see that \( s_f \) is continuous. So let \( I_0 \in M_B \) be arbitrary. Choose \( \varphi_0 \in \Phi_B \) with \( \text{Ker} \varphi_0 = I_0 \) and take \( b_0 \in B \) with \( f(\varphi_0) = \varphi_0(b_0) \) (we can of course take such \( b_0 \in B \) as observed above). Then \( s_f(I_0) = (\text{Ker} \varphi_0, b_0 + \text{Ker} \varphi_0) \). Let \( z_0 = \varphi_0(b_0) \). Then we have \( f(\varphi_0) = z_0 \) and \( g(z_0, 0, 0) = s_f(I_0) \in E_B \). By Lemma 4, a basic neighborhood of \( s_f(I_0) \) in \( E_B \) is of the form \( g((O_\varepsilon \times N)' \), where \( O_\varepsilon \) is an \( \varepsilon \)-neighborhood of \( z_0 \) in \( \mathbb{C} \), \( N \) is a neighborhood of \( \varphi_0 \) in \( \Phi_B \) and \( (O_\varepsilon \times N)' = (O_\varepsilon \times N) \cap (\mathbb{C} \times \Phi_B)' \). Also we have

\[
g((O_\varepsilon \times N)') = \{(\text{Ker} \varphi, b + \text{Ker} \varphi) : \varphi \in N \text{ and } \varphi(b) \in O_\varepsilon \}.
\]

Since \( f \) is continuous, we can find a neighborhood \( U_0 \) of \( \varphi_0 \) in \( \Phi_B \) so that \( f(U_0) \subset O_\varepsilon \) and \( U_0 \subset N \). Set \( V_0 = \varepsilon(U_0) \) and hence \( V_0 \) is a neighborhood of \( I_0 \) in \( M_B \) since \( \varepsilon : \Phi_B \to M_B \) is open from Proposition 1. We assert that \( s_f(V_0) \subset g((O_\varepsilon \times N)') \). In fact, if \( I \in V_0 \), then there exists \( \varphi \in U_0 \) with \( I = \varepsilon(\varphi) = \text{Ker} \varphi \) and so \( \varphi \in N \) and \( f(\varphi) \in O_\varepsilon \). Now take \( b \in B \) with \( f(\varphi) = \varphi(b) \). Then \( s_f(I) = (I, b + I) = (\text{Ker} \varphi, b + \text{Ker} \varphi) \) and hence \( s_f(I) \in g((O_\varepsilon \times N)') \). Thus we have our assertion \( s_f(V_0) \subset g((O_\varepsilon \times N)') \). Now to see \( f_{s_f} = f \), let \( \varphi \in \Phi_B \) and take \( b \in B \) with \( f(\varphi) = \varphi(b) \). By definition of the natural map, we have \( f_{s_f}(\varphi) = \varphi(b) \) and so \( f_{s_f}(\varphi) = f(\varphi) \). Consequently, \( f_{s_f} = f \) and so we have the natural map is surjective.

Now let \( s \in \Gamma^b(E_B) \). For any \( \varphi \in \Phi_B \), set \( I = \text{Ker} \varphi \). Then \( s(I) = (I, b + I) \) and \( s^*(I) = (I, b + I) \) for some \( b, \bar{b} \in B \) such that \( \overline{\varphi(b)} = \varphi(\bar{b}) \). Hence \( f_s(\varphi) = \varphi(b) \) and so \( |f_s(\varphi)| = |\varphi(b)| = |s(I)| \leq \|s\| \). Therefore \( \|f_s\| \leq \|s\| \) and so \( f_s \in C^*_h(\Phi_B) \). Similarly, \( \|s\| \leq \|f_s\| \) and hence the restriction of this natural map to \( \Gamma^b(E_B) \) is isometric. Moreover, since \( f_{s^*}(\varphi) = \varphi(\bar{b}) = \overline{f_s(\varphi)} \), it follows that \( f_{s^*} = f_s^* \) and so the natural map is *-preserving. Also, we can easily see that this natural map is a real algebra homomorphism.

Finally set \( \rho(s) = f_s \) for each \( s \in \Gamma^b(E_B) \). Then we can easily see that
\[ \rho(\Gamma^b(E_B)) = C_h^b(\Phi_B) \] from the above observation. Moreover, \( \rho(\Gamma_0(E_B)) = C_{h,0}(\Phi_B) \). To see this, let \( f \in C_{h,0}(\Phi_B) \) and \( \delta > 0 \). Set \( K_f = \{ \varphi \in \Phi_B : |f(\varphi)| \geq \delta \} \) and then \( K_f \) is compact. Set \( T_f = \epsilon(K_f) \) and so \( T_f \) is also compact. Then we have \( T_f = \{ I \in M_B : |s_f(I)| \geq \delta \} \). In fact, let \( \varphi \in K_f \) be arbitrary and set \( I = \text{Ker} \, \varphi \). Take \( b \in B \) with \( \varphi(b) = f(\varphi) \) and then \( s_f(I) = (I, b + I) \). By definition of \( |s_f(I)| \), we have \( |s_f(I)| = |\varphi(b)| \) and so \( |s_f(I)| \geq \delta \) since \( \varphi \in K_f \). That is \( T_f \subset \{ I \in M_B : |s_f(I)| \geq \delta \} \). Conversely, let \( I \in M_B \) with \( |s_f(I)| \geq \delta \). Choose \( \psi \in \Phi_B \) with \( I = \text{Ker} \, \psi \) and take \( c \in B \) with \( \psi(c) = f(\psi) \). Then we have \( |f(\psi)| = |\psi(c)| = |s_f(I)| \geq \delta \) and so \( \psi \in K_f \), namely \( I \in T_f \). Therefore we obtain the inverse inclusion. We thus obtain that \( s_f \in \Gamma_0(E_B) \) and hence \( \rho(\Gamma_0(E_B)) \supset C_{h,0}(\Phi_B) \). Now let \( s \in \Gamma_0(E_B) \) and \( \delta > 0 \). Set \( T_s = \{ I \in M_B : |s(I)| \geq \delta \} \) and then \( T_s \) is compact. Set \( K_s = \{ \varphi \in \Phi_B : |s_s(\varphi)| \geq \delta \} \) and then we have \( K_s = \epsilon^{-1}(T_s) \), similarly. Therefore we see from Lemma 1-(2) that \( K_s \) is compact and hence \( \rho(\Gamma_0(E_B)) \subset C_{h,0}(\Phi_B) \). This completes the proof.

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