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A NOTE ON COMMUTATIVE GELFAND THEORY FOR REAL BANACH ALGEBRAS

Dedicated to Professor Saburou Saitoh on his 60th birthday (Kanreki)

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Abstract. Pfaffenberger and Phillips [2] consider a real and unital case of the classical commutative Gelfand theorem and obtain two representation theorems. One is to represent a unital real commutative Banach algebra A as an algebra of continuous functions on the unital homomorphism space Φ_A. The other is to represent A as an algebra of continuous sections on the maximal ideal space M_A. In this note, we point out that similar theorems for non-unital case hold and show that two representation theorems are essentially identical.

1. Preliminary and results

Let B be a real commutative Banach algebra and Φ_B the set of non-zero $\mathbb{R}$-algebra homomorphism $\varphi\colon B \to \mathbb{C}$. Then we have $\|\varphi\| \stackrel{\text{def}}{=} \sup_{\|a\|\leq 1} |\varphi(a)| \leq 1$ for each $\varphi \in \Phi_B$. Actually, suppose that there exists an $a \in B$ such that $\|a\| < 1$ and $|\varphi(a)| = 1$. Set $\theta = -\arg\varphi(a)$ and then $e^{i\theta}\varphi(a) = 1$. Also set

$$b = \sum_{n=1}^{\infty} a^n \cos n\theta \quad \text{and} \quad c = \sum_{n=1}^{\infty} a^n \sin n\theta.$$

Elementary trigonometric identities lead to

$$b = a\cos\theta + ab\cos\theta - ac\sin\theta \quad \text{and} \quad 0 = a\sin\theta + ab\sin\theta - c + ac\cos\theta.$$

Apply φ to these equations, multiply the resulting second equation by i and add it to the resulting first equation then we obtain

$$\varphi(b) = \varphi(a)e^{i\theta} + \varphi(a)\varphi(b)e^{i\theta} - i\varphi(c) + i\varphi(a)\varphi(c)e^{i\theta},$$

so that $1 = 0$, a contradiction (we referred the proof of [2, Proposition 1.1, (b)]). Let B_c be the complexification of B. Then Φ_B is a subset of the closed unit ball of the dual space ${B_c}^*$ and hence we can give Φ_B the relative topology of ${B_c}^*$ with the weak*-topology. Therefore, as well-known, Φ_B is a locally compact Hausdorff space. We denote by $C(\Phi_B)$ the algebra of continuous complex-valued functions on Φ_B and set $C_0(\Phi_B) = \{f \in C(\Phi_B) : f \text{ vanishes at infinity}\}$. Then $C_0(\Phi_B)$ is a real commutative C^*-algebra with

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supremum norm and a standard method leads to the following representation theorem (cf. [2, Theorem 1.4]).

Theorem 1. *Let B be a real commutative Banach algebra.*

(1) *The mapping $\Lambda_\Phi\colon B \to C_0(\Phi_B)$ given by $\Lambda_\Phi(b)(\varphi) = \varphi(b)$ $(\stackrel{\text{def}}{=} \hat{b}(\varphi))$ for each $\varphi \in \Phi_B$ and each $b \in B$ is a norm-decreasing real algebra homomorphism which is one-to-one if and only if B is semisimple.*

(2) *If B is unital, then* $\mathrm{Sp}_B(b) = \hat{b}(\Phi_B)$ *for each $b \in B$. Also, if B is non-unital, then* $\mathrm{Sp}_B(b) = \hat{b}(\Phi_B) \cup \{0\}$ *for each $b \in B$.*

We next provide a Gelfand theorem for real commutative Banach algebras using the maximal regular ideal space M_B as a common domain of an algebra of functions.

In the next section, we see that $\operatorname{Ker}\varphi$ is a maximal regular ideal of B for each $\varphi \in \Phi_B$. Following [2], we give M_B the quotient topology arising from the map $\epsilon\colon \Phi_B \to M_B$ defined by $\epsilon(\varphi) = \operatorname{Ker}\varphi$, $\varphi \in \Phi_B$. That is, M_B has the strongest topology which makes the map ϵ continuous. Let $\sigma\colon \Phi_B \to \Phi_B$ be the homeomorphism $\sigma(\varphi) = \bar{\varphi}$. Then $\sigma^2 = e$, the identity, and we have an action of $\mathbb{Z}_2 = \{e, \sigma\}$ on Φ_B. Let $\beta\colon \Phi_B/\mathbb{Z}_2 \to M_B$ be the bijection $\beta(\mathbb{Z}_2\varphi) = \operatorname{Ker}\varphi$ and then $\epsilon = \beta \circ \pi$, where $\pi\colon \Phi_B \to \Phi_B/\mathbb{Z}_2$ is the natural map. Then we have the following result from Lemmas 1 and 2 in the next section.

Proposition 1. *The space M_B is locally compact and Hausdorff, and the map ϵ is both open and closed. Moreover, $\beta\colon \Phi_B/\mathbb{Z}_2 \to M_B$ is a homeomorphism.*

From the above proposition, we know that M_B is just the quotient of Φ_B under the (not necessarily free) action of $\mathbb{Z}_2$ on Φ_B. Let

$$\Phi_B^{\mathbb{R}} = \{\varphi \in \Phi_B : \varphi(B) = \mathbb{R}\},\ \Phi_B^{\mathbb{C}} = \{\varphi \in \Phi_B : \varphi(B) = \mathbb{C}\},$$
$$M_B^{\mathbb{R}} = \{I \in M_B : B/I \cong \mathbb{R}\} \text{ and } M_B^{\mathbb{C}} = \{I \in M_B : B/I \cong \mathbb{C}\}.$$

Clearly $\Phi_B^{\mathbb{R}}$ is a closed subset of Φ_B and so $\Phi_B^{\mathbb{C}}$ is an open subset of Φ_B. Also, in the next section, we see that

$$M_B^{\mathbb{R}} = \epsilon(\Phi_B^{\mathbb{R}}),\ M_B^{\mathbb{C}} = \epsilon(\Phi_B^{\mathbb{C}}),\ \Phi_B = \Phi_B^{\mathbb{R}} \cup \Phi_B^{\mathbb{C}} \text{ and } M_B = M_B^{\mathbb{R}} \cup M_B^{\mathbb{C}}.$$

Then $M_B^{\mathbb{R}}$ is closed and $M_B^{\mathbb{C}}$ is open. Also B is called almost complex if $\Phi_B^{\mathbb{C}} = \Phi_B$ (cf. [1, 2]).

Now as usual in representing algebras as sections we form the set

$$E_B = \dot{\bigcup_{I \in M_B}} B/I,$$

where $\uplus$ denotes disjoint union. Of course, each B/I is a field and an algebra over $\mathbb{R}$, and we have an obvious map $p\colon E_B \to M_B$. That is, $p(I, b+I) = I$, $(I \in M_B, b \in B)$, where $E_B \cong \{(I, b+I) : I \in M_B, b \in B\}$.

The problem is to topologize E_B in a reasonable way such that $I \mapsto b+I$ is a continuous section for each $b \in B$. Following [2], let $(\mathbb{C} \times \Phi_B)' = (\mathbb{C} \times \Phi_B^{\mathbb{C}}) \cup (\mathbb{R} \times \Phi_B^{\mathbb{R}})$ endowed with the relative topology in $\mathbb{C} \times \Phi_B$. We consider the map $g\colon (\mathbb{C} \times \Phi_B)' \to E_B$ defined by

$$g(z, \varphi) = (\operatorname{Ker}\varphi, b + \operatorname{Ker}\varphi),$$

where b is chosen such that $\varphi(b) = z$. Of course, this map is well-defined and surjective. We give E_B the quotient topology induced by the map g and denote by $\Gamma(E_B)$ the set of all continuous sections on M_B. Moreover, we set

$$\Gamma^b(E_B) = \{s \in \Gamma(E_B) : \|s\| = \sup_{I \in M_B} |s(I)| < \infty\}$$

and

$$\Gamma_0(E_B) = \{s \in \Gamma(E_B) : s \text{ vanishes at infinity, that is } \lim_{I \to \infty} |s(I)| = 0\},$$

where $|s(I)| = |\varphi(b)|, s(I) = (I, b+I), b \in B$ and $I = \operatorname{Ker}\varphi = \operatorname{Ker}\bar{\varphi}$. Then we have the following representation theorem in a way similar to the proof of [2, Theorem 3.5].

Theorem 2. *Let B be a real commutative Banach algebra and let $p\colon E_B \to M_B$ be the associated bundle of real fields. Then*

(1) *$\Gamma^b(E_B)$ is a real commutative Banach algebra given the supremum norm and $\Gamma_0(E_B)$ is a closed subalgebra of it.*
(2) *$\Lambda_M\colon B \to \Gamma_0(E_B)$ defined by $\Lambda_M(b)(I) = (I, b+I)$, $I \in M_B$ is a norm-decreasing algebra homomorphism with kernel,* Rad B.
(3) *For $b \in B, \|\Lambda_M(b)\| = \lim_{n\to\infty} \|b^n\|^{1/n}$.*

Remark 1. *Theorems 1 and 2 are non-unital versions of* [2, Theorem 1.4 and 3.5].

We next see that these representation theorems are essentially identical. To do this, set

$$C_h(\Phi_B) = \{f \in C(\Phi_B) : f(\bar{\varphi}) = \overline{f(\varphi)} \text{ for all } \varphi \in \Phi_B\},$$
$$C_h^b(\Phi_B) = \{f \in C_h(\Phi_B) : \|f\| < \infty\},$$

and

$$C_{h,0}(\Phi_B) = \{f \in C_h(\Phi_B) : f \text{ vanishes at infinity }\}.$$

Then we can easily see that $C_h^b(\Phi_B)$ is a unital real commutative Banach algebra given the supremum norm and $C_{h,0}(\Phi_B)$ is a closed subalgebra of it. Also $\Lambda_\Phi(B) \subset C_{h,0}(\Phi_B)$ clearly holds.

Now for each $f \in C_h(\Phi_B)$ we define $f^*(\varphi) = \overline{f(\varphi)}$, $(\varphi \in \Phi_B)$. Then $C_h^b(\Phi_B)$ becomes a unital real commutative C^*-algebra under this involution and $C_{h,0}(\Phi_B)$ is a C^*-subalgebra of $C_h^b(\Phi_B)$. Also let $s \in \Gamma(E_B)$ and $I \in M_B$. Then $s(I) = (I, b+I)$ for some $b \in B$. Choose $\varphi \in \Phi_B$ with $I = \operatorname{Ker}\varphi$. Then there exists an element $\bar{b} \in B$ such that $\varphi(\bar{b}) = \overline{\varphi(b)}$. Set $s^*(I) = (I, \bar{b}+I)$. This is clearly well-defined and we see later that $s^* \in \Gamma(E_B)$ (Lemma 5). Therefore $\Gamma^b(E_B)$ becomes a unital real commutative C^*-algebra under this involution and $\Gamma_0(E_B)$ is a C^*-subalgebra of $\Gamma^b(E_B)$.

In this setting, we have the following:

Theorem 3. *There is an isometric real algebra $*$-isomorphism ρ of $\Gamma^b(E_B)$ onto $C_h^b(\Phi_B)$ such that $\rho(\Gamma_0(E_B)) = C_{h,0}(\Phi_B)$ and $\rho \circ \Lambda_M = \Lambda_\Phi$.*

We know that the Gelfand representation theorems 1 and 2 are essentially identical by the above theorem.

Combining [2, Theorem 5.3] and Theorem 3, we have the following:

Corollary 1. *If B is a unital commutative almost complex C^*-algebra, then $\Lambda_\Phi \colon B \to C_h(\Phi_B)$ is an isometric $*$-isomorphism.*

2. Known results and lemmas

We will remind the reader of the following well-known results since they are basic to all that we do.

Let $\varphi \in \Phi_B$. Since $\operatorname{range}\varphi$ is a non-zero real subalgebra of $\mathbb{C}$, it must be either $\mathbb{R}$ or $\mathbb{C}$. In fact, let $A = \operatorname{range}\varphi$ and then A is a non-zero linear subspace of $\mathbb{C}$ over $\mathbb{R}$. Then $\dim A = 1$ or 2. If $\dim A = 2$, then $A = \mathbb{C}$. If $\dim A = 1$, then $A = \mathbb{R}a$ for some non-zero complex number $a \in A$. Since A is an algebra, it follows that $a^2 = ra$ for some $r \in \mathbb{R}$ and then $a \in \mathbb{R}$. Hence A must be $\mathbb{R}$. We thus obtain that $\operatorname{range}\varphi = \mathbb{R} \Leftrightarrow \operatorname{Ker}\varphi$ has codimension 1 in B and $\operatorname{range}\varphi = \mathbb{C} \Leftrightarrow \operatorname{Ker}\varphi$ has codimension 2 in B. Moreover, $\operatorname{Ker}\varphi$ is a maximal regular ideal of B. Actually, choose $e \in B$ with $\varphi(e) = 1$ and hence $B(1-e) \subset \operatorname{Ker}\varphi$, namely, $\operatorname{Ker}\varphi$ is regular. Now let I be an ideal of B with $\operatorname{Ker}\varphi \subsetneq I \subset B$. Then $I/\operatorname{Ker}\varphi$ is a non-zero real subalgebra of $B/\operatorname{Ker}\varphi$. Take $e \in B$ and $u \in I$ with $\varphi(e) = 1$ and $u \notin \operatorname{Ker}\varphi$. Since $B/\operatorname{Ker}\varphi$ is a field and $e + \operatorname{Ker}\varphi$ is the identity of $B/\operatorname{Ker}\varphi$, we can find $v \in B$ with $(u + \operatorname{Ker}\varphi)(v + \operatorname{Ker}\varphi) = e + \operatorname{Ker}\varphi$ and hence $uv - e \in \operatorname{Ker}\varphi$. Then $e = uv + e - uv \in I + \operatorname{Ker}\varphi = I$, so that $b = be + b(1-e) \in I + \operatorname{Ker}\varphi = I$ for each $b \in B$. That is, we have $B = I$ and so $\operatorname{Ker}\varphi$ is maximal. Let I be a maximal regular ideal of B. Since B/I is a real commutative normed division algebra, it follows from the Gelfand-Mazur theorem that $B/I \cong \mathbb{R}$ or $\mathbb{C}$ and then I has codimension 1 or 2. In case of $\operatorname{codim} I = 1$, we have that $B/I \cong \mathbb{R}$ as an algebra over $\mathbb{R}$ and this isomorphism is unique since $\mathbb{R}$ has no non-trivial $\mathbb{R}$-algebra automorphisms. Thus the composition $\varphi \colon B \to B/I \cong \mathbb{R}$

is the unique element of Φ_B with kernel I. Moreover we have range $\varphi = \mathbb{R}$. In case of codim $I = 2$, we have that $B/I \cong \mathbb{C}$ as an algebra over $\mathbb{R}$ and since $\mathbb{C}$ has exactly one non-trivial $\mathbb{R}$-algebra automorphism given by conjugation, we see that there are exactly two elements $\varphi, \bar{\varphi} \in \Phi_B$ with kernel I. Moreover we have range $\varphi =$ range $\bar{\varphi} = \mathbb{C}$.

Lemma 1. *Let G be a finite group acting on a topological space Y and let $X = Y/G$ endowed with the quotient topology. Let $p\colon Y \to X$ be the natural map. Then*

(1) *The map p is both open and closed.*
(2) *For each compact subset K in X, $p^{-1}(K)$ is also compact in Y.*
(3) *If Y is locally compact, so is X.*
(4) *If Y is Hausdorff, so is X.*

Proof. (1) Let U be an open (closed) subset of Y. Then the saturation $GU = \cup_{g\in G}\, g(U)$ of U is clearly open (closed). Also since $GU = p^{-1}(p(U))$, it follows that p is open (closed).

(2) Let K be a compact subset of X and $\{y_\lambda\}$ a net in $p^{-1}(K)$. Then $\{p(y_\lambda)\}$ is a net in K and so there is a subnet $\{y_{\lambda'}\}$ of $\{y_\lambda\}$ such that $\{p(y_{\lambda'})\}$ converges to some point of K, say Gy. Let $Gy = \{y, g_1(y), \cdots, g_{n-1}(y)\}$, where $G = \{e, g_1, \cdots, g_{n-1}\}$. Then we have $Gy \subset p^{-1}(K)$. Now, we assert that a certain subnet of $\{y_{\lambda'}\}$ converges to one of $y, g_1(y), \cdots, g_{n-1}(y)$. Suppose contrary. Then we can easily find an open neighborhood U of y and a subnet $\{y_{\lambda''}\}$ of $\{y_{\lambda'}\}$ such that every $y_{\lambda''}$ does not belong to $U \cup g_1(U) \cup \cdots \cup g_{n-1}(U)$. Set $V = U \cup g_1(U) \cup \cdots \cup g_{n-1}(U)$ and so $V = GV$. Also since V is an open neighborhood of y and p is open, $p(V)$ must be an open neighborhood of Gy. Hence there exists a point $p(y_{\lambda_0''})$ which belongs to $p(V)$. Therefore $y_{\lambda_0''}$ must be in the saturation of V namely GV. However since $V = GV$ and every $y_{\lambda''}$ does not belong to GV, this is a contradiction. We thus obtain that any net in $p^{-1}(K)$ has a subnet which converges to some point in $p^{-1}(K)$, that is $p^{-1}(K)$ is compact.

(3) Since p is both open and closed by (1), X must be locally compact from a standard topological argument.

(4) Suppose that Y is Hausdorff and let $x_1, x_2 \in X$ with $x_1 \neq x_2$. Let $y_1 \in p^{-1}(x_1)$ and $y_2 \in p^{-1}(x_2)$, and then $y_1 \neq y_2$. Since G is finite, we can find an open neighborhood U_1 of y_1 and an open neighborhood U_2 of y_2 such that $g(U_1) \cap h(U_2) = \emptyset$ for all $g, h \in G$. Since p is open, $p(U_1)$ and $p(U_2)$ are disjoint open neighborhoods of x_1 and x_2, respectively. Consequently X is also Hausdorff. □

Lemma 2. *Let X be a topological space and let Y and Z be two sets with surjections $\pi_Y\colon X \to Y$ and $\pi_Z\colon X \to Z$. We give Y and Z the quotient topologies induced by π_Y and π_Z, respectively. Moreover, assume that π_Y*

is both open and closed, and there is a bijection $\theta\colon Y \to Z$ *such that* $\pi_Z = \theta \circ \pi_Y$. *Then* π_Z *is also both open and closed, and* θ *is a homeomorphism.*

Proof. Let U be an open subset of X. Then

$$\begin{aligned} {\pi_Z}^{-1}(\pi_Z(U)) &= {\pi_Y}^{-1}(\theta^{-1}(\pi_Z(U))) \\ &= {\pi_Y}^{-1}(\theta^{-1}(\theta(\pi_Y(U)))) \\ &= {\pi_Y}^{-1}(\pi_Y(U)) \end{aligned}$$

and hence ${\pi_Z}^{-1}(\pi_Z(U))$ is open since π_Y is continuous and open. Therefore π_Z is open by definition of the quotient topology. Similarly, we can prove the closedness of π_Z. Now note that $\pi_Z({\pi_Y}^{-1}(K)) = \theta(K)$ for each subset K of Y. This implies that θ is a homeomorphism. □

Now, following [2], we give E_B the quotient topology induced by the map g. There is also a natural action of $\mathbb{Z}_2 = \{e, \sigma\}$ on $(\mathbb{C} \times \Phi_B)'$, namely: $\sigma(z, \varphi) = (\bar{z}, \bar{\varphi})$. The following is a just [2, Lemma 3.2].

Lemma 3. *There is a natural homeomorphism* $\theta\colon (\mathbb{C} \times \Phi_B)'/\mathbb{Z}_2 \to E_B$ *such that* $g = \theta \circ \pi_{\mathbb{Z}_2}$, *where* $\pi_{\mathbb{Z}_2}$ *is the canonical map of* $(\mathbb{C} \times \Phi_B)'$ *onto* $(\mathbb{C} \times \Phi_B)'/\mathbb{Z}_2$. *Thus* g *is both open and closed.*

Lemma 4. *Let* Y *be a topological space and* X *a set. Let* $p\colon Y \to X$ *be a surjection. We give* X *the quotient topology induced by* p. *Let* $x \in X$ *and* $y \in p^{-1}(x)$. *If* p *is open and* $\{V_\alpha\}$ *is a base of neighborhoods of* y, *then* $\{p(V_\alpha)\}$ *is a base of neighborhoods of* x.

Proof. Let U be any neighborhood of x. Then $p^{-1}(U)$ is a neighborhood of y and hence $V_\alpha \subset p^{-1}(U)$ for some V_α. Therefore we have $p(V_\alpha) \subset p(p^{-1}(U)) = U$ and so $\{p(V_\alpha)\}$ is a base of neighborhoods of x. □

Lemma 5. *If* $s \in \Gamma(E_B)$, *then* $s^* \in \Gamma(E_B)$, *where* s^* *is a section for* $p\colon E_B \to M_B$ *defined in the preceding section.*

Proof. Let $s \in \Gamma(E_B)$. We show that s^* is continuous. To do this, let $I_0 \in M_B$ be arbitrary and take an element $\varphi_0 \in \Phi_B$ with $I_0 = \operatorname{Ker}\varphi_0$. Also take an element $b_0 \in B$ with $s(I_0) = (I_0, b_0 + I_0)$ and set $z_0 = \varphi_0(b_0)$. By definition of s^*, we have that $s^*(I_0) = (I_0, \overline{b_0} + I_0)$ and $\overline{z_0} = \overline{\varphi_0(b_0)} = \varphi_0(\overline{b_0})$. Then $g(\overline{z_0}, \varphi_0) = s^*(I_0) \in E_B$. By Lemma 4, a basic neighborhood of $s^*(I_0)$ in E_B is of the form $g((O_\varepsilon(\overline{z_0}) \times N)')$, where $O_\varepsilon(\overline{z_0})$ is an ε-neighborhood of $\overline{z_0}$ in $\mathbb{C}$, N is a neighborhood of φ_0 in Φ_B and $(O_\varepsilon(\overline{z_0}) \times N)' = (O_\varepsilon(\overline{z_0}) \times N) \cap (\mathbb{C} \times \Phi_B)'$. Also we have

$$g((O_\varepsilon(\overline{z_0}) \times N)') = \{(\operatorname{Ker}\varphi, b + \operatorname{Ker}\varphi) : \varphi \in N \text{ and } \varphi(b) \in O_\varepsilon(\overline{z_0})\}.$$

Let $O_\varepsilon(z_0)$ be an ε-neighborhood of z_0 in $\mathbb{C}$ and then

$$g((O_\varepsilon(z_0) \times N)') = \{(\operatorname{Ker}\varphi, b + \operatorname{Ker}\varphi) : \varphi \in N \text{ and } \varphi(b) \in O_\varepsilon(z_0)\}$$

is a neighborhood of $s(I_0)$ because $g(z_0,\varphi_0)=s(I_0)\in E_B$. Also since s is continuous, we can find a neighborhood V_0 of I_0 in M_B such that $s(V_0)\subset g((O_\varepsilon(z_0)\times N)')$. In this case, we have $s^*(V_0)\subset g((O_\varepsilon(z_0)\times N)')$. In fact, let $I\in V_0$ and take an element $\varphi\in\Phi_B$ with $I=\operatorname{Ker}\varphi$. Since $s(I)=(I,b+I)$ for some $b\in B$, it follows from definition of s^* that $s^*(I)=(I,\bar b+I)$ and $\varphi(\bar b)=\overline{\varphi(b)}$ for some $\bar b\in B$. Also since $(\operatorname{Ker}\varphi, b+\operatorname{Ker}\varphi)\in g((O_\varepsilon(z_0)\times N)')$, it follows that $\varphi\in N$ and $|\varphi(b)-z_0|<\varepsilon$, and hence $|\varphi(\bar b)-\overline{z_0}|=|\varphi(b)-z_0|<\varepsilon$. Consequently, $s^*(I)\in g((O_\varepsilon(\overline{z_0})\times N)')$. We thus see that s^* is continuous at each point in M_B. □

3. Proof of Theorem 3

Let $s\in\Gamma(E_B)$ be arbitrary. For any $\varphi\in\Phi_B$, set $I=\operatorname{Ker}\varphi$. Then $s(I)=(I,b+I)$ for some $b\in B$. We define $f_s(\varphi)=\varphi(b)$. This is, of course, well-defined. Then we have that $f_s(\bar\varphi)=\overline{f_s(\varphi)}$ for all $\varphi\in\Phi_B$. In fact, let $\varphi\in\Phi_B$ and put $I=\operatorname{Ker}\varphi$ and then $I=\operatorname{Ker}\bar\varphi$. Since $s(I)=(I,b+I)$ for some $b\in B$, it follows from definition of f_s that $f_s(\varphi)=\varphi(b)$ and $f_s(\bar\varphi)=\bar\varphi(b)$. Therefore we have $f_s(\bar\varphi)=\bar\varphi(b)=\overline{\varphi(b)}=\overline{f_s(\varphi)}$. We next claim that f_s is continuous on Φ_B. Let $\varphi_0\in\Phi_B$ and set $I_0=\operatorname{Ker}\varphi_0$. Then $s(I_0)=(I_0,b_0+I_0)$ for some $b_0\in B$. Put $z_0=f_s(\varphi_0)(=\varphi_0(b_0))$. Let U_ε be any ε-neighborhood of z_0. If $\varphi_0\in\Phi_B^{\mathbb R}$, we set $N_0=\Phi_B$. If $\varphi_0\in\Phi_B^{\mathbb C}$, then $\varphi_0\neq\overline{\varphi_0}$ and hence there is $b_1\in B$ such that $\varphi_0(b_1)\neq\overline{\varphi_0}(b_1)$, so we set $N_0=\{\varphi\in\Phi_B : |\varphi_0(b_1)-\varphi(b_1)|<\delta/2\}$, where $\delta=|\varphi_0(b_1)-\overline{\varphi_0}(b_1)|>0$. Then N_0 is an open neighborhood of φ_0. In case of $\varphi_0\in\Phi_B^{\mathbb C}$, we have that if $\varphi\in N_0$ then $\bar\varphi\notin N_0$. In fact, assume that $\varphi\in N_0$ and $\bar\varphi\in N_0$. Then $|\varphi_0(b_1)-\varphi(b_1)|<\delta/2$ and $|\varphi_0(b_1)-\overline{\varphi(b_1)}|<\delta/2$, so we have $|\varphi_0(b_1)-\overline{\varphi_0}(b_1)|<\delta/2+\delta/2=\delta$, a contradiction. Now note from Lemma 3 that

$$g((U_\varepsilon\times N_0)')=\{(\operatorname{Ker}\varphi, b+\operatorname{Ker}\varphi) : \varphi\in N_0, \varphi(b)\in U_\varepsilon\}$$

is an open neighborhood of $s(I_0)$ in E_B, where $(U_\varepsilon\times N_0)'=(U_\varepsilon\times N_0)\cap(\mathbb C\times\Phi_B)'$. Since s is a continuous section, there exists a neighborhood V_0 of I_0 such that $s(V_0)\subset g((U_\varepsilon\times N_0)')$. Set $W_0=\epsilon^{-1}(V_0)\cap N_0$ and then it is a neighborhood of $\varphi_0\in\Phi_B$. We see that $f_s(W_0)\subset U_\varepsilon$. In fact, let $\varphi\in W_0$ be arbitrary. Then $\varphi\in\epsilon^{-1}(V_0)$ and hence $\operatorname{Ker}\varphi\in V_0$. Therefore

$$s(\operatorname{Ker}\varphi)\in s(V_0)\subset g((U_\varepsilon\times N_0)')=\{(\operatorname{Ker}\psi, b+\operatorname{Ker}\psi) : \psi\in N_0, \psi(b)\in U_\varepsilon\}$$

and so $s(\operatorname{Ker}\varphi)=(\operatorname{Ker}\psi, b+\operatorname{Ker}\psi)$, for some $b\in B$ and $\psi\in N_0$ with $\psi(b)\in U_\varepsilon$. Then $\operatorname{Ker}\varphi=\operatorname{Ker}\psi$, $f_s(\varphi)=\varphi(b)$ and $f_s(\psi)=\psi(b)$. If $\psi=\varphi$, then $f_s(\varphi)=\varphi(b)=\psi(b)\in U_\varepsilon$. If $\psi\neq\varphi$, then $\psi=\bar\varphi$ and hence $f_s(\varphi)=\varphi(b)=\overline{\psi(b)}$. In case of $\varphi_0\in\Phi_B^{\mathbb R}$, we have $z_0\in\mathbb R$ and hence U_ε is conjugate invariant, so $\overline{\psi(b)}\in U_\varepsilon$, that is $f_s(\varphi)\in U_\varepsilon$. In case of $\varphi_0\in\Phi_B^{\mathbb C}$, since $\varphi\in N_0$, we have $\bar\varphi\notin N_0$. However since $\psi\in N_0$ and $\psi=\bar\varphi$, we have a contradiction. Therefore we must conclude that $f_s(W_0) \subset U_\varepsilon$. We thus obtain a natural map $s \mapsto f_s$ of $\Gamma(E_B)$ into $C_h(\Phi_B)$.

We next show that this map is surjective. To do this, let $f \in C_h(\Phi_B)$ be arbitrary. For any $I \in M_B$, choose $\varphi \in \Phi_B$ with $I = \operatorname{Ker}\varphi$. In this case, we can take $b \in B$ with $\varphi(b) = f(\varphi)$. In fact, if $\varphi \in \Phi_B^{\mathbb{R}}$, then $\varphi = \bar{\varphi}$ and hence $f(\varphi) = f(\bar{\varphi}) = \overline{f(\varphi)}$, so $f(\varphi) \in \mathbb{R}$. Moreover, $\varphi(B) = \mathbb{R}$ and so we can find such $b \in B$. If $\varphi \in \Phi_B^{\mathbb{C}}$, then $\varphi(B) = \mathbb{C}$ and so there exists clearly such $b \in B$. Now we define $s_f(I) = (I, b + I)$. This is well-defined. In fact, let $\psi \in \Phi_B, I = \operatorname{Ker}\psi, c \in B$ and $\psi(c) = f(\psi)$. If $\varphi \neq \psi$, then $\psi = \bar{\varphi}$ and so

$$\overline{\varphi(c)} = \bar{\varphi}(c) = \psi(c) = f(\psi) = f(\bar{\varphi}) = \overline{f(\varphi)} = \overline{\varphi(b)},$$

hence $c - b \in \operatorname{Ker}\varphi = I$. Similarly, we can treat the case $\varphi = \psi$. This shows that our definition is well-defined. Now we claim that s_f is a continuous section for $p\colon E_B \to M_B$. It suffices to see that s_f is continuous. So let $I_0 \in M_B$ be arbitrary. Choose $\varphi_0 \in \Phi_B$ with $\operatorname{Ker}\varphi_0 = I_0$ and take $b_0 \in B$ with $f(\varphi_0) = \varphi_0(b_0)$ (we can of course take such $b_0 \in B$ as observed above). Then $s_f(I_0) = (\operatorname{Ker}\varphi_0, b_0 + \operatorname{Ker}\varphi_0)$. Let $z_0 = \varphi_0(b_0)$. Then we have $f(\varphi_0) = z_0$ and $g(z_0, \varphi_0) = s_f(I_0) \in E_B$. By Lemma 4, a basic neighborhood of $s_f(I_0)$ in E_B is of the form $g((O_\varepsilon \times N)')$, where O_ε is an ε-neighborhood of z_0 in $\mathbb{C}$, N is a neighborhood of φ_0 in Φ_B and $(O_\varepsilon \times N)' = (O_\varepsilon \times N) \cap (\mathbb{C} \times \Phi_B)'$. Also we have

$$g((O_\varepsilon \times N)') = \{(\operatorname{Ker}\varphi, b + \operatorname{Ker}\varphi) : \varphi \in N \text{ and } \varphi(b) \in O_\varepsilon\}.$$

Since f is continuous, we can find a neighborhood U_0 of φ_0 in Φ_B so that $f(U_0) \subset O_\varepsilon$ and $U_0 \subset N$. Set $V_0 = \epsilon(U_0)$ and hence V_0 is a neighborhood of I_0 in M_B since $\epsilon\colon \Phi_B \to M_B$ is open from Proposition 1. We assert that $s_f(V_0) \subset g((O_\varepsilon \times N)')$. In fact, if $I \in V_0$, then there exists $\varphi \in U_0$ with $I = \epsilon(\varphi) = \operatorname{Ker}\varphi$ and so $\varphi \in N$ and $f(\varphi) \in O_\varepsilon$. Now take $b \in B$ with $f(\varphi) = \varphi(b)$. Then $s_f(I) = (I, b + I) = (\operatorname{Ker}\varphi, b + \operatorname{Ker}\varphi)$ and hence $s_f(I) \in g((O_\varepsilon \times N)')$. Thus we have our assertion $s_f(V_0) \subset g((O_\varepsilon \times N)')$. Now to see $f_{s_f} = f$, let $\varphi \in \Phi_B$ and take $b \in B$ with $f(\varphi) = \varphi(b)$. By definition of the natural map, we have $f_{s_f}(\varphi) = \varphi(b)$ and so $f_{s_f}(\varphi) = f(\varphi)$. Consequently, $f_{s_f} = f$ and so we have the natural map is surjective.

Now let $s \in \Gamma^b(E_B)$. For any $\varphi \in \Phi_B$, set $I = \operatorname{Ker}\varphi$. Then $s(I) = (I, b+I)$ and $s^*(I) = (I, \bar{b}+I)$ for some $b, \bar{b} \in B$ such that $\overline{\varphi(b)} = \varphi(\bar{b})$. Hence $f_s(\varphi) = \varphi(b)$ and so $|f_s(\varphi)| = |\varphi(b)| = |s(I)| \leq \|s\|$. Therefore $\|f_s\| \leq \|s\|$ and so $f_s \in C_h^b(\Phi_B)$. Similarly, $\|s\| \leq \|f_s\|$ and hence the restriction of this natural map to $\Gamma^b(E_B)$ is isometric. Moreover, since $f_{s^*}(\varphi) = \varphi(\bar{b}) = \overline{f_s(\varphi)}$, it follows that $f_{s^*} = {f_s}^*$ and so the natural map is $*$-preserving. Also, we can easily see that this natural map is a real algebra homomorphism.

Finally set $\rho(s) = f_s$ for each $s \in \Gamma^b(E_B)$. Then we can easily see that

$\rho(\Gamma^b(E_B)) = C_h^b(\Phi_B)$ from the above observation. Moreover, $\rho(\Gamma_0(E_B)) = C_{h,0}(\Phi_B)$. To see this, let $f \in C_{h,0}(\Phi_B)$ and $\delta > 0$. Set $K_f = \{\varphi \in \Phi_B : |f(\varphi)| \geq \delta\}$ and then K_f is compact. Set $T_f = \epsilon(K_f)$ and so T_f is also compact. Then we have $T_f = \{I \in M_B : |s_f(I)| \geq \delta\}$. In fact, let $\varphi \in K_f$ be arbitrary and set $I = \operatorname{Ker}\varphi$. Take $b \in B$ with $\varphi(b) = f(\varphi)$ and then $s_f(I) = (I, b + I)$. By definition of $|s_f(I)|$, we have $|s_f(I)| = |\varphi(b)|$ and so $|s_f(I)| \geq \delta$ since $\varphi \in K_f$. That is $T_f \subset \{I \in M_B : |s_f(I)| \geq \delta\}$. Conversely, let $I \in M_B$ with $|s_f(I)| \geq \delta$. Choose $\psi \in \Phi_B$ with $I = \operatorname{Ker}\psi$ and take $c \in B$ with $\psi(c) = f(\psi)$. Then we have $|f(\psi)| = |\psi(c)| = |s_f(I)| \geq \delta$ and so $\psi \in K_f$, namely $I \in T_f$. Therefore we obtain the inverse inclusion. We thus obtain that $s_f \in \Gamma_0(E_B)$ and hence $\rho(\Gamma_0(E_B)) \supset C_{h,0}(\Phi_B)$. Now let $s \in \Gamma_0(E_B)$ and $\delta > 0$. Set $T_s = \{I \in M_B : |s(I)| \geq \delta\}$ and then T_s is compact. Set $K_s = \{\varphi \in \Phi_B : |f_s(\varphi)| \geq \delta\}$ and then we have $K_s = \epsilon^{-1}(T_s)$, similarly. Therefore we see from Lemma 1-(2) that K_s is compact and hence $\rho(\Gamma_0(E_B)) \subset C_{h,0}(\Phi_B)$. This completes the proof.

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