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SOME HOMOTOPY GROUPS OF THE HOMOGENEOUS SPACE E_6/F_4

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1. Introduction

Let F_4 and E_6 be the compact, connected, simply connected, simple, exceptional Lie groups of rank 4 and 6 respectively. We consider the homogeneous space E_6/F_4. Cohen and Selick constructed in [3] a map $\lambda : \Omega^2 S^{17} \to \Omega S^9$ which is $\mathrm{ad}(\sigma_9)$ on the bottom cell where $\mathrm{ad} : \pi_{16}(S^9) \to \pi_{15}(\Omega S^9)$ is an adjoint isomorphism and σ_9 a generator of $\pi_{16}(S^9)$. They also showed that there does not exist a spherical fibration $S^9 \to E \to S^{17}$ giving rise to the λ. Thus the homotopy fibre of λ is expected to be homotopy equivalent to $\Omega^2(E_6/F_4)$.

Conlon has determined $\pi_i(E_6/F_4)$ for $i \leq 23$ in [2]. In this paper we calculate $\pi_i(E_6/F_4 : 2)$ for $i \leq 39$ where we denote by $\pi_i(Y : p)$ the p-primary component of $\pi_i(Y)$. The calculation will be done by making use of the fibration

$$X \xrightarrow{i} S^9 \xrightarrow{p} E_6/F_4$$

where X is the homotopy fibre of the natural inclusion of S^9 in E_6/F_4. Our results are stated as follows.

i	$i \leq 8$	9	10	11	12	13	14	15	16	17	18
$\pi_i(E_6/F_4 : 2)$	0	∞	2	2	8	0	0	2	0	$\infty + (2)^2$	$(2)^3$

19	20	21	22	23	24	25	26	27	28	29	30	31
2	$8+2$	0	0	4	$16+2$	2	$(2)^3$	2	$8+2$	8	2	$(2)^2$

32	33	34	35	36	37	38	39
$32+8+2$	$(2)^2$	$(2)^2$	$4+(2)^3$	$8+2$	$(2)^2$	2	0

Here an integer 'n' indicates a cyclic group $\mathbf{Z}_n$ of order n, the symbol '∞' an infinite cyclic group, the symbol '+' the direct sum of groups and '$(n)^k$' indicates the direct sum of k-copies of $\mathbf{Z}_n$. These results are stated in Theorem 4.4 in which we also give their generators.

The results on $\pi_i(E_6/F_4 : 2)$ are expected to determine the homotopy type of homotopy fibre of λ.

We use freely the notation in [14].

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2. Preliminaries

We denote by $\mathbb{O}\mathrm{P}$ the octonionic projective plane. As is well known (see Conlon [2]), we have

$$E_6/F_4 \simeq \Sigma(\mathbb{O}\mathrm{P}) \cup e^{26} \quad \text{and} \quad \Sigma(\mathbb{O}\mathrm{P}) \simeq S^9 \cup_{h_9} e^{17}$$

where $h_9 = \sigma_9 + \alpha_2(9) + \alpha_1(9) \in \pi_{16}(S^9) = \mathbf{Z}_{16}\{\sigma_9\} \oplus \mathbf{Z}_3\{\alpha_2(9)\} \oplus \mathbf{Z}_5\{\alpha_1(9)\}$.

We denote by $i_{\mathbb{O}} \colon S^9 \to S^9 \cup_{h_9} e^{17}$ the inclusion and by $\omega \in \pi_{17}(\Sigma(\mathbb{O}\mathrm{P}), S^9)$ the homotopy class of the characteristic map of 17-dimensional cell of $\Sigma(\mathbb{O}\mathrm{P})$. For the boundary homomorphism $\partial : \pi_{17}(\Sigma(\mathbb{O}\mathrm{P}), S^9) \to \pi_{16}(S^9)$, we have

$$\partial(\omega) = h_9.$$

Then by Theorem 1.4 of [5], we have

$$\pi_{25}(\Sigma(\mathbb{O}\mathrm{P}), S^9) = \mathbf{Z}\{[\omega, \iota_9]\} \oplus \mathbf{Z}_2\{\omega\widehat{\bar{\nu}_{16}}\} \oplus \mathbf{Z}_2\{\omega\widehat{\varepsilon_{16}}\}$$

and

$$\pi_{26}(\Sigma(\mathbb{O}\mathrm{P}), S^9) = \mathbf{Z}_2\{[\omega, \eta_9]\} \oplus \mathbf{Z}_2\{\omega\widehat{\nu_{16}^3}\} \oplus \mathbf{Z}_2\{\omega\widehat{\mu_{16}}\} \oplus \mathbf{Z}_2\{\omega\widehat{\eta_{16}\varepsilon_{17}}\}.$$

Here we denote by $\widehat{\alpha}$ an element of $\pi_n(CS^{16}, S^{16})$ such that $\partial(\widehat{\alpha}) = \alpha$ for the boundary homomorphism $\partial : \pi_n(CS^{16}, S^{16}) \to \pi_{n-1}(S^{16})$.

We calculate $\partial : \pi_{25}(\Sigma(\mathbb{O}\mathrm{P}), S^9) \to \pi_{24}(S^9)$ and $\partial : \pi_{26}(\Sigma(\mathbb{O}\mathrm{P}), S^9) \to \pi_{25}(S^9)$ where

$$\pi_{24}(S^9) = \mathbf{Z}_{16}\{\rho'\} \oplus \mathbf{Z}_2\{\sigma_9\bar{\nu}_{16}\} \oplus \mathbf{Z}_2\{\sigma_9\varepsilon_{16}\} \oplus \mathbf{Z}_2\{\bar{\varepsilon}_9\} \oplus \mathbf{Z}_3\{\alpha_4(9)\} \oplus \mathbf{Z}_5\{\alpha_2(9)\}$$

and

$$\pi_{25}(S^9) = \mathbf{Z}_2\{\sigma_9\nu_{16}^3\} \oplus \mathbf{Z}_2\{\sigma_9\mu_{16}\} \oplus \mathbf{Z}_2\{\sigma_9\eta_{16}\varepsilon_{17}\} \oplus \mathbf{Z}_2\{\mu_9\sigma_{18}\}.$$

By Theorem 8.18 in Chapter X of [18] and by the fact that the odd primary component of $\pi_{17}(S^9)$ is trivial, we have $[\alpha_2(9), \iota_9] = [\iota_9, \iota_9]\alpha_2(17) = 0$ and $[\alpha_1(9), \iota_9] = [\iota_9, \iota_9]\alpha_1(17) = 0$. Thus we have

$$\begin{aligned}
\partial[\omega, \iota_9] &= -[\partial\omega, \iota_9] && \text{by (2.1) of [6]} \\
&= -[\sigma_9, \iota_9] - [\alpha_2(9), \iota_9] - [\alpha_1(9), \iota_9] \\
&= -[\iota_9, \iota_9]\sigma_{17} \\
&= (\sigma_9\eta_{16} + \bar{\nu}_9 + \varepsilon_9)\sigma_{17} && \text{by (7.1) of [14]} \\
&= \sigma_9\bar{\nu}_{16} + \sigma_9\varepsilon_{16} && \text{by Lemma 6.4 and 10.7 of [14].}
\end{aligned}$$

So we have

$$\begin{aligned}\partial[\omega,\eta_9] &= -[\partial\omega,\iota_9]\eta_{24}\\ &= \sigma_9\bar{\nu}_{16}\eta_{24}+\sigma_9\varepsilon_{16}\eta_{24}\\ &= \sigma_9\nu_{16}^3+\sigma_9\eta_{16}\varepsilon_{17} \qquad \text{by Lemma 6.3 of [14].}\end{aligned}$$

By the naturality of the boundary homomorphism, we have

$$\partial(\omega\widehat{\bar{\nu}_{16}}) = h_9\bar{\nu}_{16} = \sigma_9\bar{\nu}_{16}, \quad \partial(\omega\widehat{\varepsilon_{16}}) = \sigma_9\varepsilon_{16},$$

$$\partial(\omega\widehat{\nu_{16}^3}) = \sigma_9\nu_{16}^3, \quad \partial(\omega\widehat{\mu_{16}}) = \sigma_9\mu_{16}, \quad \partial(\omega\widehat{\eta_{16}\varepsilon_{17}}) = \sigma_9\eta_{16}\varepsilon_{17}.$$

By the argument above, we have

$$\operatorname{Ker}\{\partial : \pi_{25}(\Sigma(\mathbb{O}\mathrm{P}),S^9)\to\pi_{24}(S^9)\} = \mathbf{Z}\{[\omega,\iota_9]+\omega\widehat{\bar{\nu}_{16}}+\omega\widehat{\varepsilon_{16}}\}$$

and

$$\operatorname{Coker}\{\partial : \pi_{26}(\Sigma(\mathbb{O}\mathrm{P}),S^9)\to\pi_{25}(S^9)\} = \mathbf{Z}_2\{\mu_9\sigma_{18}\}.$$

Therefore we have

$$\pi_{25}(\Sigma(\mathbb{O}\mathrm{P})) = \mathbf{Z}\{\theta\}\oplus\mathbf{Z}_2\{i_{\mathbb{O}*}(\mu_9\sigma_{18})\}$$

where θ satisfies

$$j_{\mathbb{O}*}(\theta) = [\omega,\iota_9]+\omega\widehat{\bar{\nu}_{16}}+\omega\widehat{\varepsilon_{16}}$$

for the homomorphism $j_{\mathbb{O}*} : \pi_{25}(\Sigma(\mathbb{O}\mathrm{P}))\to\pi_{25}(\Sigma(\mathbb{O}\mathrm{P}),S^9)$.

Araki has determined integral cohomology of E_6/F_4 in [1] as follows:

$$H^*(E_6/F_4;\mathbf{Z})\cong\wedge(x_9,x_{17}).$$

Then by Theorem 3.3 of [6], we have the following.

Proposition 2.1.

$$E_6/F_4\simeq\Sigma(\mathbb{O}\mathrm{P})\cup_\beta e^{26},$$

where $\beta\equiv\theta \bmod i_{\mathbb{O}*}(\mu_9\sigma_{18})$.

Let X denote the homotopy fibre of the natural inclusion of the S^9 in E_6/F_4. Thus the Serre spectral sequence implies that the integral cohomology ring satisfies

$$H^*(X;\mathbf{Z})\cong\Gamma(y_{16}),$$

where $\Gamma(y_{16})$ denotes the divided polynomial algebra on a generator y_{16} of degree 16. Hence we have

$$X\simeq S^{16}\cup_g e^{32}\cup e^{48}\cup\dots,$$

where $g\in\pi_{31}(S^{16})\cong\mathbf{Z}\{[\iota_{16},\iota_{16}]\}\oplus\mathbf{Z}_{32}\{\rho_{16}\}\oplus\mathbf{Z}_2\{\bar{\varepsilon}_{16}\}\oplus\mathbf{Z}_3\{\alpha_4(16)\}\oplus\mathbf{Z}_5\{\alpha_2(16)\}$. By the equality $y_{16}^2=2y_{32}$, we have

$$g=\pm[\iota_{16},\iota_{16}]+a_1\rho_{16}+a_2\bar{\varepsilon}_{16}+a_3\alpha_4(16)+a_4\alpha_2(16),$$

where a_i are integers. We consider the homotopy exact sequence associated with the fibration

$$X \xrightarrow{i} S^9 \xrightarrow{p} E_6/F_4.$$

Then for the inclusion $j : S^{16} \to X$, we have

$$i_*j_*(\iota_{16}) = ah_9,$$

where a is an integer prime to $2, 3$ and 5.

We consider

$$\begin{aligned} h_9 \circ g &= h_9(\pm[\iota_{16}, \iota_{16}] + a_1\rho_{16} + a_2\bar{\varepsilon}_{16} + a_3\alpha_4(16) + a_4\alpha_2(16)) \\ &= \pm h_9[\iota_{16}, \iota_{16}] + a_1\sigma_9\rho_{16} + a_2\sigma_9\bar{\varepsilon}_{16} + a_3\alpha_2(9)\alpha_4(16) + a_4\alpha_1(9)\alpha_2(16). \end{aligned}$$

By (7.2), Corollary 7.12 and Theorem 8.18 in Chapter X of [18], we have

$$\begin{aligned} h_9[\iota_{16}, \iota_{16}] &= [h_9, h_9] \\ &= [\sigma_9, \sigma_9] + [\alpha_2(9), \alpha_2(9)] + [\alpha_1(9), \alpha_1(9)] \\ &= [\iota_9, \iota_9]\sigma_{17}^2 + [\iota_9, \iota_9]\alpha_2(17)^2 + [\iota_9, \iota_9]\alpha_1(17)^2. \end{aligned}$$

By (7.1), Lemma 6.4 and 10.7 of [14] and by the fact that odd primary component of $\pi_{17}(S^9)$ is trivial, we have

$$\begin{aligned} h_9[\iota_{16}, \iota_{16}] &= [h_9, h_9] \\ &= [\iota_9, \iota_9]\sigma_{17}^2 \\ &= (\sigma_9\eta_{16} + \bar{\nu}_9 + \varepsilon_9)\sigma_{17}^2 \\ &= 0. \end{aligned}$$

By Theorem 7.6 of [15], the homomorphism $E^6 : \pi_{25}(S^3 : 3) \to \pi_{31}(S^9 : 3)$ is trivial. So we have $\alpha_2(9)\alpha_4(16) = E^6(\alpha_2(3)\alpha_4(10)) = 0$. By the fact $\pi_{31}(S^9 : 5) = 0$, we have $\alpha_1(9)\alpha_2(16) = 0$. Hence we have

$$h_9 \circ g = a_1\sigma_9\rho_{16} + a_2\sigma_9\bar{\varepsilon}_{16}.$$

On the other hand, since $j_*(g) = 0 \in \pi_{31}(X)$, we have

$$h_9 \circ g = i_*j_*(g) = 0.$$

Then by the fact that $\sigma_9\rho_{16}$ and $\sigma_9\bar{\varepsilon}_{16}$ are generators of order 16 and 2 respectively, we have

$$g \equiv \pm[\iota_{16}, \iota_{16}] \bmod 16\rho_{16}, \alpha_4(16), \alpha_2(16).$$

3. Homotopy groups of the X

From now on, we restrict our attention to the 2-primary component and so omit for simplicity the notation '2' indicating 2-primary component of homotopy group.

By the argument in the previous section, we have

$$E_6/F_4 \underset{2}{\simeq} S^9 \cup_{\sigma_9} e^{17} \cup e^{26}$$

and

$$X \underset{2}{\simeq} S^{16} \cup_g e^{32} \cup e^{48} \cup \dots ,$$

where

$$g \equiv [\iota_{16}, \iota_{16}] \bmod 16\rho_{16}.$$

For dimensional reasons, we have

$$\pi_i(X) \cong \pi_i(S^{16} \cup_g e^{32})$$

for $i \leq 46$. We consider the homotopy exact sequence

$$\cdots \to \pi_i(S^{16}) \xrightarrow{j_*} \pi_i(S^{16} \cup_g e^{32}) \to \pi_i(S^{16} \cup_g e^{32}, S^{16}) \xrightarrow{\partial} \pi_{i-1}(S^{16}) \xrightarrow{j_*} \cdots$$

associated with the pair $(S^{16} \cup_g e^{32}, S^{16})$. The collapsing map $\pi : (S^{16} \cup_g e^{32}, S^{16}) \to (S^{32}, *)$ induces a homomorphism

$$\pi_* : \pi_i(S^{16} \cup_g e^{32}, S^{16}) \to \pi_i(S^{32})$$

which is an isomorphism for $i \leq 45$ by the Blakers-Massey theorem. Therefore the following sequence is exact for $i \leq 45$:

$$\cdots \to \pi_i(S^{16}) \xrightarrow{j_*} \pi_i(S^{16} \cup_g e^{32}) \xrightarrow{\pi_*} \pi_i(S^{32}) \xrightarrow{\Delta'} \pi_{i-1}(S^{16}) \xrightarrow{j_*} \cdots ,$$

where

$$\Delta' = \partial \circ \pi_*^{-1} : \pi_i(S^{32}) \xleftarrow{\pi_*} \pi_i(S^{16} \cup_g e^{32}, S^{16}) \xrightarrow{\partial} \pi_{i-1}(S^{16}).$$

For any element $\alpha \in \pi_n(S^{31})$, we have

$$\Delta'(\Sigma\alpha) \equiv [\iota_{16}, \iota_{16}]\alpha \bmod (16\rho_{16})\alpha. \tag{3.1}$$

By use of this formula, we calculate $\Delta' : \pi_{n+1}(S^{32}) \to \pi_n(S^{16})$ for $n \leq 39$.

We recall here some necessary results on $\pi_{n+i}(S^n)$ for $i \leq 30$ determined by Toda [14], Mimura-Toda [10], Mimura [8], Mimura-Mori-Oda [9], and Oda [12].

Table 1.

i	9	10	11	12	13	14	15	16	17
$\pi_{i+1}(S^{32})$	0	0	0	0	0	0	0	0	0
generator									
$\pi_i(S^{16})$	0	0	0	0	0	0	0	$\mathbf{Z}$	$\mathbf{Z}_2$
generator								ι_{16}	η_{16}
$\pi_i(S^9)$	$\mathbf{Z}$	$\mathbf{Z}_2$	$\mathbf{Z}_2$	$\mathbf{Z}_8$	0	0	$\mathbf{Z}_2$	$\mathbf{Z}_{16}$	$\mathbf{Z}_2 \oplus \mathbf{Z}_2 \oplus \mathbf{Z}_2$
generator	ι_9	η_9	η_9^2	ν_9	0	0	ν_9^2	σ_9	$\sigma_9\eta_{16}, \bar{\nu}_9, \varepsilon_9$

18	19	20	21	22	23
0	0	0	0	0	0
$\mathbf{Z}_2$	$\mathbf{Z}_8$	0	0	$\mathbf{Z}_2$	$\mathbf{Z}_{16}$
η_{16}^2	ν_{16}			ν_{16}^2	σ_{16}
$\mathbf{Z}_2 \oplus \mathbf{Z}_2 \oplus \mathbf{Z}_2 \oplus \mathbf{Z}_2$	$\mathbf{Z}_8 \oplus \mathbf{Z}_2$	$\mathbf{Z}_8 \oplus \mathbf{Z}_2$	0	$\mathbf{Z}_2$	$\mathbf{Z}_{16} \oplus \mathbf{Z}_4$
$\sigma_9\eta_{16}^2, \nu_9^3, \mu_9, \eta_9\varepsilon_{10}$	$\sigma_9\nu_{16}, \eta_9\mu_{10}$	$\zeta_9, \bar{\nu}_9\nu_{17}$		$\sigma_9\nu_{16}^2$	σ_9^2, κ_9

24	25
0	0
$\mathbf{Z}_2 \oplus \mathbf{Z}_2$	$\mathbf{Z}_2 \oplus \mathbf{Z}_2 \oplus \mathbf{Z}_2$
$\bar{\nu}_{16}, \varepsilon_{16}$	$\nu_{16}^3, \mu_{16}, \eta_{16}\varepsilon_{17}$
$\mathbf{Z}_{16} \oplus \mathbf{Z}_2 \oplus \mathbf{Z}_2 \oplus \mathbf{Z}_2$	$\mathbf{Z}_2 \oplus \mathbf{Z}_2 \oplus \mathbf{Z}_2 \oplus \mathbf{Z}_2$
$\rho', \sigma_9\bar{\nu}_{16}, \sigma_9\varepsilon_{16}, \bar{\varepsilon}_9$	$\sigma_9\nu_{16}^3, \sigma_9\mu_{16}, \sigma_9\eta_{16}\varepsilon_{17}, \mu_9\sigma_{18}$

26	27	28	29
0	0	0	0
$\mathbf{Z}_2$	$\mathbf{Z}_8$	0	0
$\eta_{16}\mu_{17}$	ζ_{16}		
$\mathbf{Z}_2 \oplus \mathbf{Z}_2 \oplus \mathbf{Z}_2 \oplus \mathbf{Z}_2$	$\mathbf{Z}_8 \oplus \mathbf{Z}_2$	$\mathbf{Z}_8 \oplus \mathbf{Z}_2$	$\mathbf{Z}_8$
$\sigma_9\eta_{16}\mu_{17}, \nu_9\kappa_{12}, \bar{\mu}_9, \eta_9\mu_{10}\sigma_{19}$	$\sigma_9\zeta_{16}, \eta_9\bar{\mu}_{10}$	$\bar{\zeta}_9, \bar{\sigma}_9$	$\bar{\kappa}_9$

30	31
0	$\mathbf{Z}$
	ι_{32}
$\mathbf{Z}_2 \oplus \mathbf{Z}_2$	$\mathbf{Z} \oplus \mathbf{Z}_{32} \oplus \mathbf{Z}_2$
$\sigma_{16}^2, \kappa_{16}$	$[\iota_{16}, \iota_{16}], \rho_{16}, \bar{\varepsilon}_{16}$
$\mathbf{Z}_2 \oplus \mathbf{Z}_2 \oplus \mathbf{Z}_2$	$\mathbf{Z}_{16} \oplus \mathbf{Z}_2 \oplus \mathbf{Z}_2 \oplus \mathbf{Z}_2$
$\eta_9\bar{\kappa}_{10}, \sigma_9\kappa_{16}, \sigma_9^3$	$\sigma_9\rho_{16}, \varepsilon_9\kappa_{17}, \nu_9\bar{\sigma}_{12}, \sigma_9\bar{\varepsilon}_{16}$

32
$\mathbf{Z}_2$
η_{32}
$\mathbf{Z}_2 \oplus \mathbf{Z}_2 \oplus \mathbf{Z}_2 \oplus \mathbf{Z}_2$
$\eta_{16}^*, \Sigma\eta^{*\prime}, \omega_{16}, \sigma_{16}\mu_{23}$
$\mathbf{Z}_{16} \oplus \mathbf{Z}_8 \oplus \mathbf{Z}_2 \oplus \mathbf{Z}_2 \oplus \mathbf{Z}_2 \oplus \mathbf{Z}_2$
$\bar{\rho}_9, \nu_9\bar{\kappa}_{12}, \phi_9, \bar{\kappa}_9\nu_{29} - \nu_9\bar{\kappa}_{12}, \sigma_9^2\mu_{23}, \sigma_9\omega_{16}$

33
$\mathbf{Z}_2$
η_{32}^2
$\mathbf{Z}_2 \oplus \mathbf{Z}_2 \oplus \mathbf{Z}_2 \oplus \mathbf{Z}_2 \oplus \mathbf{Z}_2 \oplus \mathbf{Z}_2$
$\eta_{16}^*\eta_{32}, (\Sigma\eta^{*\prime})\eta_{32}, \varepsilon_{16}^*, \sigma_{16}\eta_{23}\mu_{24}, \nu_{16}\kappa_{19}, \bar{\mu}_{16}$
$\mathbf{Z}_2 \oplus \mathbf{Z}_2 \oplus \mathbf{Z}_2 \oplus \mathbf{Z}_2 \oplus \mathbf{Z}_2 \oplus \mathbf{Z}_2 \oplus \mathbf{Z}_2$
$\delta_9, \bar{\mu}_9\sigma_{26}, \bar{\sigma}_9', \sigma_9\bar{\mu}_{16}, \sigma_9^2\eta_{23}\mu_{24}, \sigma_9\nu_{16}\kappa_{19}, \sigma_9\varepsilon_{16}^*$

34
$\mathbf{Z}_8$
ν_{32}
$\mathbf{Z}_8 \oplus \mathbf{Z}_8 \oplus \mathbf{Z}_8 \oplus \mathbf{Z}_2$
$\nu_{16}^*, \xi_{16}, \Sigma^3\lambda, \eta_{16}\bar{\mu}_{17}$
$\mathbf{Z}_8 \oplus \mathbf{Z}_4 \oplus \mathbf{Z}_2 \oplus \mathbf{Z}_2 \oplus \mathbf{Z}_2$
$\sigma_9\xi_{16}, \sigma_9\nu_{16}^*, \sigma_9\eta_{16}\bar{\mu}_{17}, \mu_{3,9}, \eta_9\bar{\mu}_{10}\sigma_{27}$

35	36
0	0
$\mathbf{Z}_8 \oplus \mathbf{Z}_2 \oplus \mathbf{Z}_2$	$\mathbf{Z}_8$
$\bar{\zeta}_{16}, \bar{\sigma}_{16}, \omega_{16}\nu_{32}$	$\bar{\kappa}_{16}$
$\mathbf{Z}_8 \oplus \mathbf{Z}_2 \oplus \mathbf{Z}_2 \oplus \mathbf{Z}_2 \oplus \mathbf{Z}_2 \oplus \mathbf{Z}_2 \oplus \mathbf{Z}_2$	$\mathbf{Z}_8 \oplus \mathbf{Z}_8 \oplus \mathbf{Z}_2$
$\sigma_9\bar{\zeta}_{16}, \sigma_9\omega_{16}\nu_{32}, \sigma_9\bar{\sigma}_{16}, \bar{\kappa}_9\nu_{29}^2, \bar{\sigma}_9\sigma_{28}, \nu_9^2\bar{\kappa}_{15}, \eta_9\mu_{3,10}$	$\sigma_9\bar{\kappa}_{16}, \zeta_{3,9}, \bar{\nu}_9\bar{\sigma}_{17}$

37
$\mathbf{Z}_2$
ν_{32}^2
$\mathbf{Z}_2 \oplus \mathbf{Z}_2 \oplus \mathbf{Z}_2 \oplus \mathbf{Z}_2$
$\eta_{16}\bar{\kappa}_{17}, \sigma_{16}^3, (\Sigma^3\lambda)\nu_{34}, \nu_{16}^*\nu_{34}$
$\mathbf{Z}_2 \oplus \mathbf{Z}_2 \oplus \mathbf{Z}_2 \oplus \mathbf{Z}_2 \oplus \mathbf{Z}_2$
$\sigma_9^4, \sigma_9\eta_{16}\bar{\kappa}_{17}, \sigma_9\nu_{16}^*\nu_{34}, \bar{\nu}_9\bar{\kappa}_{17}, \varepsilon_9\bar{\kappa}_{17}$

38
$\mathbf{Z}_{16}$
σ_{32}
$\mathbf{Z}_{16} \oplus \mathbf{Z}_{16} \oplus \mathbf{Z}_2 \oplus \mathbf{Z}_2 \oplus \mathbf{Z}_2$
$\sigma_{16}^*, \Sigma\sigma^{*\prime}, \omega_{16}\nu_{32}^2, \varepsilon_{16}\kappa_{24}, \nu_{16}\bar{\sigma}_{19}$
$\mathbf{Z}_{16} \oplus \mathbf{Z}_2 \oplus \mathbf{Z}_2 \oplus \mathbf{Z}_2 \oplus \mathbf{Z}_2$
$\sigma_9\sigma_{16}^*, \sigma_9\omega_{16}\nu_{32}^2, \sigma_9\varepsilon_{16}\kappa_{24}, \sigma_9\nu_{16}\bar{\sigma}_{19}, \eta_9\varepsilon_{10}\bar{\kappa}_{18}$

39
$\mathbf{Z}_2 \oplus \mathbf{Z}_2$
$\bar{\nu}_{32}, \varepsilon_{32}$
$\mathbf{Z}_{16} \oplus \mathbf{Z}_8 \oplus \mathbf{Z}_2 \oplus \mathbf{Z}_2 \oplus \mathbf{Z}_2 \oplus \mathbf{Z}_2 \oplus \mathbf{Z}_2 \oplus \mathbf{Z}_2$
$\bar{\rho}_{16}, \nu_{16}\bar{\kappa}_{19}, \phi_{16}, \psi_{16}, \Sigma\bar{\varepsilon}^{*\prime}, \Sigma\bar{\nu}^{*\prime}, \bar{\varepsilon}_{16}^*, \bar{\nu}_{16}^*$
$\mathbf{Z}_{16} \oplus \mathbf{Z}_8 \oplus \mathbf{Z}_2 \oplus \mathbf{Z}_2 \oplus \mathbf{Z}_2$
$\sigma_9\bar{\rho}_{16}, \sigma_9\nu_{16}\bar{\kappa}_{19}, \sigma_9\phi_{16}, \sigma_9\psi_{16}, \phi_9\sigma_{32}$

As for the boundary formula (3.1) we have

Lemma 3.1. (1) $\Delta'(\iota_{32}) \equiv [\iota_{16}, \iota_{16}] \bmod 16\rho_{16}$,
(2) $\Delta'(\eta_{32}) \equiv \Sigma\eta^{*\prime} \bmod \omega_{16}, \sigma_{16}\mu_{23}$,
(3) $\Delta'(\eta_{32}^2) \equiv (\Sigma\eta^{*\prime})\eta_{32} \bmod \omega_{16}\eta_{32}, \sigma_{16}\mu_{23}\eta_{32}$,
(4) $\Delta'(\nu_{32}) = \pm(\Sigma^3\lambda - 2\nu_{16}^*)$,

(5) $\Delta'(\nu_{32}^2) = (\Sigma^3\lambda)\nu_{34}$,
(6) $\Delta'(\sigma_{32}) = a\Sigma\sigma^{*\prime} \pm 2\sigma_{16}^*$, *for some odd integer* a,
(7) $\Delta'(\bar{\nu}_{32}) = \Sigma\bar{\nu}^{*\prime}$ *and* $\Delta'(\varepsilon_{32}) = \Sigma\bar{\varepsilon}^{*\prime}$.

Proof. We remark that $P(\iota_{33}) = \pm[\iota_{16}, \iota_{16}]$, where P denotes the boundary homomorphism of the EHP-exact sequence.
(1) is included in (3.1).
(2) By the argument in p. 160 of [14], we have

$$\Delta'(\eta_{32}) = P(\eta_{33}) \equiv \Sigma\eta^{*\prime} \bmod \omega_{16}, \sigma_{16}\mu_{23}.$$

(3) The relation $\Delta'(\eta_{32}^2) \equiv (\Sigma\eta^{*\prime})\eta_{32} \bmod \omega_{16}\eta_{32}, \sigma_{16}\mu_{23}\eta_{32}$ follows immediately from (2).
(4) By Lemma 12.18 of [14], we have

$$\Delta'(\nu_{32}) = \pm P(\nu_{33}) = \pm(\Sigma^3\lambda - 2\nu_{16}^*).$$

(5) The relation $\Delta'(\nu_{32}^2) = (\Sigma^3\lambda)\nu_{34}$ follows immediately from (4).
(6) By the argument in p. 323 of [8], we have

$$\Delta'(\sigma_{32}) = \pm P(\sigma_{33}) \equiv \pm(\Sigma\sigma^{*\prime} - 2\sigma_{16}^*) \bmod \rho_{16}\sigma_{31}.$$

Then by Part III, Proposition 2.3 (4) of [12], we have $\Delta'(\sigma_{32}) = a\Sigma\sigma^{*\prime} \pm 2\sigma_{16}^*$ for some odd integer a.
(7) By (3.4) of [9] we have

$$\Delta'(\bar{\nu}_{32}) = P(\bar{\nu}_{33}) = \Sigma\bar{\nu}^{*\prime} \quad \text{and} \quad \Delta'(\varepsilon_{32}) = P(\varepsilon_{33}) = \Sigma\bar{\varepsilon}^{*\prime}.$$

□

It follows directly from Lemma 3.1 and Table 1 that the homomorphisms $\Delta' : \pi_{i+1}(S^{32}) \to \pi_i(S^{16})$ are monomorphisms for $i \le 39$. Therefore we obtain the following lemma.

Lemma 3.2. *We have the following table of* $\pi_i(X)$ *for* $i \le 39$.

i	$i \le 15$	16	17	18	19	20	21	22
$\pi_i(X)$	0	$\mathbf{Z}$	$\mathbf{Z}_2$	$\mathbf{Z}_2$	$\mathbf{Z}_8$	0	0	$\mathbf{Z}_2$
generator		$j_*(\iota_{16})$	$j_*(\eta_{16})$	$j_*(\eta_{16}^2)$	$j_*(\nu_{16})$			$j_*(\nu_{16}^2)$

23	24	25	26	27
$\mathbf{Z}_{16}$	$\mathbf{Z}_2 \oplus \mathbf{Z}_2$	$\mathbf{Z}_2 \oplus \mathbf{Z}_2 \oplus \mathbf{Z}_2$	$\mathbf{Z}_2$	$\mathbf{Z}_8$
$j_*(\sigma_{16})$	$j_*(\bar{\nu}_{16}), j_*(\varepsilon_{16})$	$j_*(\nu_{16}^3), j_*(\mu_{16}), j_*(\eta_{16}\varepsilon_{17})$	$j_*(\eta_{16}\mu_{17})$	$j_*(\zeta_{16})$

28	29	30	31	32
0	0	$\mathbf{Z}_2 \oplus \mathbf{Z}_2$	$\mathbf{Z}_{32} \oplus \mathbf{Z}_2$	$\mathbf{Z}_2 \oplus \mathbf{Z}_2 \oplus \mathbf{Z}_2$
		$j_*(\sigma_{16}^2), j_*(\kappa_{16})$	$j_*(\rho_{16}), j_*(\bar{\varepsilon}_{16})$	$j_*(\eta_{16}^*), j_*(\omega_{16}), j_*(\sigma_{16}\mu_{23})$

33
$\mathbf{Z}_2 \oplus \mathbf{Z}_2 \oplus \mathbf{Z}_2 \oplus \mathbf{Z}_2 \oplus \mathbf{Z}_2$
$j_*(\eta^*_{16}\eta_{32}), j_*(\varepsilon^*_{16}), j_*(\sigma_{16}\eta_{23}\mu_{24}), j_*(\nu_{16}\kappa_{19}), j_*(\bar{\mu}_{16})$

34	35	36
$\mathbf{Z}_8 \oplus \mathbf{Z}_8 \oplus \mathbf{Z}_2$	$\mathbf{Z}_8 \oplus \mathbf{Z}_2 \oplus \mathbf{Z}_2$	$\mathbf{Z}_8$
$j_*(\nu^*_{16}), j_*(\xi_{16}), j_*(\eta_{16}\bar{\mu}_{17})$	$j_*(\bar{\zeta}_{16}), j_*(\omega_{16}\nu_{32}), j_*(\bar{\sigma}_{16})$	$j_*(\bar{\kappa}_{16})$

37	38
$\mathbf{Z}_2 \oplus \mathbf{Z}_2 \oplus \mathbf{Z}_2$	$\mathbf{Z}_{16} \oplus \mathbf{Z}_2 \oplus \mathbf{Z}_2 \oplus \mathbf{Z}_2$
$j_*(\eta_{16}\bar{\kappa}_{17}), j_*(\sigma^3_{16}), j_*(\nu^*_{16}\nu_{34})$	$j_*(\sigma^*_{16}), j_*(\omega_{16}\nu^2_{32}), j_*(\varepsilon_{16}\kappa_{24}), j_*(\nu_{16}\bar{\sigma}_{19})$

39
$\mathbf{Z}_{16} \oplus \mathbf{Z}_8 \oplus \mathbf{Z}_2 \oplus \mathbf{Z}_2 \oplus \mathbf{Z}_2 \oplus \mathbf{Z}_2$
$j_*(\bar{\rho}_{16}), j_*(\nu_{16}\bar{\kappa}_{19}), j_*(\phi_{16}), j_*(\psi_{16}), j_*(\bar{\varepsilon}^*_{16}), j_*(\bar{\nu}^*_{16})$

4. Some calculations

We consider the homotopy exact sequence

$$\cdots \to \pi_i(X) \xrightarrow{i_*} \pi_i(S^9) \xrightarrow{p_*} \pi_i(E_6/F_4) \xrightarrow{\Delta} \pi_{i-1}(X) \xrightarrow{i_*} \cdots$$

associated with the 2-local fibration

$$X \xrightarrow{i} S^9 \xrightarrow{p} E_6/F_4.$$

Then for the inclusion $j : S^{16} \to X$, we have

$$i_*j_*(\iota_{16}) = \sigma_9. \tag{4.1}$$

By use of this formula, we calculate $i_* : \pi_n(X) \to \pi_n(S^9)$ for $n \leq 39$.

Lemma 4.1. (1) $\sigma_9\eta^*_{16} \equiv \phi_9 \bmod \sigma_9^2\mu_{23}, 4\nu_9\bar{\kappa}_{12}$,
(2) $\sigma_9\eta^*_{16}\eta_{32} \equiv \delta_9 \bmod \bar{\mu}_9\sigma_{26}, \sigma_9^2\eta_{23}\mu_{24}$,
(3) $\sigma_9(\bar{\varepsilon}^*_{16} + \bar{\nu}^*_{16}) = \phi_9\sigma_{32}$.

Proof. (1) is obtained in Part I, Proposition 3.4 (7) of [12].
(2) By (1), we have

$$\sigma_9\eta^*_{16}\eta_{32} \equiv \phi_9\eta_{32} \bmod \sigma_9^2\eta_{23}\mu_{24}.$$

By Part I, Proposition 3.5 (9) of [12], we have

$$\phi_9\eta_{32} \equiv \delta_9 \bmod \bar{\mu}_9\sigma_{26}, \nu_9\eta_{12}\bar{\kappa}_{13}.$$

Since $\nu_9\eta_{12} = 0$ ((5.9) of [14]), we have

$$\sigma_9\eta^*_{16}\eta_{32} \equiv \delta_9 \bmod \bar{\mu}_9\sigma_{26}, \sigma_9^2\eta_{23}\mu_{24}.$$

(3) By the definition of $\bar{\varepsilon}^*_{16}$ (see (3.4) of [9]) and (1) we have

$$\sigma_9(\bar{\varepsilon}^*_{16} + \bar{\nu}^*_{16}) = \sigma_9\eta^*_{16}\sigma_{32} \equiv \phi_9\sigma_{32} \bmod \sigma_9^2\mu_{23}\sigma_{32}, 4\nu_9\bar{\kappa}_{12}\sigma_{32}.$$

By (2.3) of [9], we have

$$\sigma_9^2\mu_{23}\sigma_{32} = 0.$$

By Part III, Proposition 2.2 (5) of [12], we have

$$4\nu_9\bar{\kappa}_{12}\sigma_{32} = 0.$$

Hence we have $\sigma_9(\bar{\varepsilon}^*_{16} + \bar{\nu}^*_{16}) = \phi_9\sigma_{32}$. □

For $i_* : \pi_n(X) \to \pi_n(S^9)$, we have the following.

Lemma 4.2. (1) *The homomorphisms $i_* : \pi_n(X) \to \pi_n(S^9)$ are epimorphisms for $n = 16, 21, 22, 39$. For the other values of n $(17 \leq n \leq 38)$, we have the following table of the cokernels of i_*.*

n	17	18	19	20	23	24
Coker i_*	$\mathbf{Z}_2 \oplus \mathbf{Z}_2$	$\mathbf{Z}_2 \oplus \mathbf{Z}_2 \oplus \mathbf{Z}_2$	$\mathbf{Z}_2$	$\mathbf{Z}_8 \oplus \mathbf{Z}_2$	$\mathbf{Z}_4$	$\mathbf{Z}_{16} \oplus \mathbf{Z}_2$
generator	$\bar{\nu}_9, \varepsilon_9$	$\nu_9^3, \mu_9, \eta_9\varepsilon_{10}$	$\eta_9\mu_{10}$	$\zeta_9, \bar{\nu}_9\nu_{17}$	κ_9	$\rho', \bar{\varepsilon}_9$

25	26	27	28	29	30	31
$\mathbf{Z}_2$	$\mathbf{Z}_2 \oplus \mathbf{Z}_2 \oplus \mathbf{Z}_2$	$\mathbf{Z}_2$	$\mathbf{Z}_8 \oplus \mathbf{Z}_2$	$\mathbf{Z}_8$	$\mathbf{Z}_2$	$\mathbf{Z}_2 \oplus \mathbf{Z}_2$
$\mu_9\sigma_{18}$	$\nu_9\kappa_{12}, \bar{\mu}_9, \eta_9\mu_{10}\sigma_{19}$	$\eta_9\bar{\mu}_{10}$	$\bar{\zeta}_9, \bar{\sigma}_9$	$\bar{\kappa}_9$	$\eta_9\bar{\kappa}_{10}$	$\varepsilon_9\kappa_{17}, \nu_9\bar{\sigma}_{12}$

32	33	34
$\mathbf{Z}_{16} \oplus \mathbf{Z}_8 \oplus \mathbf{Z}_2$	$\mathbf{Z}_2 \oplus \mathbf{Z}_2$	$\mathbf{Z}_2 \oplus \mathbf{Z}_2$
$\bar{\rho}_9, \nu_9\bar{\kappa}_{12}, \bar{\kappa}_9\nu_{29} - \nu_9\bar{\kappa}_{12}$	$\bar{\mu}_9\sigma_{26}, \bar{\sigma}'_9$	$\mu_{3,9}, \eta_9\bar{\mu}_{10}\sigma_{27}$

35	36	37	38
$\mathbf{Z}_2 \oplus \mathbf{Z}_2 \oplus \mathbf{Z}_2 \oplus \mathbf{Z}_2$	$\mathbf{Z}_8 \oplus \mathbf{Z}_2$	$\mathbf{Z}_2 \oplus \mathbf{Z}_2$	$\mathbf{Z}_2$
$\bar{\kappa}_9\nu_{29}^2, \bar{\sigma}_9\sigma_{28}, \nu_9^2\bar{\kappa}_{15}, \eta_9\mu_{3,10}$	$\zeta_{3,9}, \bar{\nu}_9\bar{\sigma}_{17}$	$\bar{\nu}_9\bar{\kappa}_{17}, \varepsilon_9\bar{\kappa}_{17}$	$\eta_9\bar{\varepsilon}_{10}\bar{\kappa}_{18}$

(2) *For $n = 16, 31, 34$, we have the following table of the kernels of $i_* : \pi_n(X) \to \pi_n(S^9)$.*

n	16	31	34
Ker i_*	$\mathbf{Z}$	$\mathbf{Z}_2$	$\mathbf{Z}_2$
generator	$16j_*(\iota_{16})$	$16j_*(\rho_{16})$	$4j_*(\nu^*_{16})$

For other values of n $(n \leq 38)$, the homomorphisms i_* are monomorphisms.

Proof. (1) By (4.1), Lemmas 3.2 and 4.1, we obtain the results easily.
(2) By (4.1), Lemmas 3.2 and 4.1, we can determine Ker$\{i_* : \pi_n(X) \to \pi_n(S^9)\}$ easily except for the case $n = 31$.

We consider $i_* : \pi_{31}(X) \to \pi_{31}(S^9)$ where

$$\pi_{31}(S^9) = \{\sigma_9\rho_{16}, \varepsilon_9\kappa_{17}, \nu_9\bar{\sigma}_{12}, \sigma_9\bar{\varepsilon}_{16}\} \cong \mathbf{Z}_{16} \oplus \mathbf{Z}_2 \oplus \mathbf{Z}_2 \oplus \mathbf{Z}_2.$$

For the case $g = [\iota_{16}, \iota_{16}] + 16\rho_{16}$, we have $\pi_{31}(X) = \{j_*(\rho_{16}), j_*(\bar{\varepsilon}_{16})\} \cong \mathbf{Z}_{32} \oplus \mathbf{Z}_2$ and $j_*([\iota_{16}, \iota_{16}]) = 16j_*(\rho_{16})$. By (4.1) we have $i_*j_*(\rho_{16}) = \sigma_9\rho_{16}$ and $i_*j_*(\bar{\varepsilon}_{16}) = \sigma_9\bar{\varepsilon}_{16}$. By (4.1) and the argument in Section 2, we have $i_*j_*([\iota_{16}, \iota_{16}]) = \sigma_9[\iota_{16}, \iota_{16}] = 0$. Hence we have $\mathrm{Ker}\{i_* : \pi_{31}(X) \to \pi_{31}(S^9)\} = \{16j_*(\rho_{16})\} \cong \mathbf{Z}_2$ and $j_*([\iota_{16}, \iota_{16}]) = 16j_*(\rho_{16})$.

For the case $g = [\iota_{16}, \iota_{16}]$, we have $\pi_{31}(X) = \{j_*(\rho_{16}), j_*(\bar{\varepsilon}_{16})\} \cong \mathbf{Z}_{32} \oplus \mathbf{Z}_2$. Then by the above argument, we have $\mathrm{Ker}\{i_* : \pi_{31}(X) \to \pi_{31}(S^9)\} = \{16j_*(\rho_{16})\} \cong \mathbf{Z}_2$. □

The following lemma will be used later.

Lemma 4.3. *Let (E, p, B) be a fibration, F a fiber $p^{-1}(*)$ and Δ the boundary homomorphism in the homotopy exact sequence of the fibration. Then for any element α of $\pi_{i+1}(B)$, we have*

$$\alpha \in \{p, i, \Delta(\alpha)\}.$$

Proof. Let E_+^{i+1} (resp. E_-^{i+1}) be the upper-(resp. lower-)hemisphere of S^{i+1}. Since $p_* : \pi_{i+1}(E, F) \to \pi_{i+1}(B, *)$ is an isomorphism, there exists $a : (E_+^{i+1}, S^i) \to (E, F)$ such that $p \circ a$ and $a|_{S^i}$ are representatives of α and $\Delta(\alpha)$ respectively. Then we define a map $\tilde{a} : S^{i+1} \to E \bigcup CF$ by $\tilde{a}|_{E_+^{i+1}} = a$ and $\tilde{a}|_{E_-^{i+1}}(x, t) = (a|_{S^i}(x), 1 - 2t) \in CF$. We define a map $\bar{p} : E \bigcup CF \to B$ by $\bar{p}|_E = p$ and $\bar{p}|_{CF} = *$. Then by the definition of Toda bracket, $\bar{p} \circ \tilde{a}$ represents an element of $\{p, i, \Delta(\alpha)\}$. Since $\bar{p} \circ \tilde{a} \simeq p \circ a$, we have $\alpha \in \{p, i, \Delta(\alpha)\}$. □

Let us state our main result.

Theorem 4.4. *We have the following table of $\pi_i(E_6/F_4)$ for $i \leq 39$.*

i	$i \leq 8$	9	10	11	12	13	14	15
$\pi_i(E_6/F_4)$	0	$\mathbf{Z}$	$\mathbf{Z}_2$	$\mathbf{Z}_2$	$\mathbf{Z}_8$	0	0	$\mathbf{Z}_2$
generator		$p_*(\iota_9)$	$p_*(\eta_9)$	$p_*(\eta_9^2)$	$p_*(\nu_9)$			$p_*(\nu_9^2)$

16	17	18	19
0	$\mathbf{Z} \oplus \mathbf{Z}_2 \oplus \mathbf{Z}_2$	$\mathbf{Z}_2 \oplus \mathbf{Z}_2 \oplus \mathbf{Z}_2$	$\mathbf{Z}_2$
	$[16j_*(\iota_{16})], p_*(\bar{\nu}_9), p_*(\varepsilon_9)$	$p_*(\nu_9^3), p_*(\mu_9), p_*(\eta_9\varepsilon_{10})$	$p_*(\eta_9\mu_{10})$

20	21	22	23	24	25
$\mathbf{Z}_8 \oplus \mathbf{Z}_2$	0	0	$\mathbf{Z}_4$	$\mathbf{Z}_{16} \oplus \mathbf{Z}_2$	$\mathbf{Z}_2$
$p_*(\zeta_9), p_*(\bar{\nu}_9\nu_{17})$			$p_*(\kappa_9)$	$p_*(\rho'), p_*(\bar{\varepsilon}_9)$	$p_*(\mu_9\sigma_{18})$

26	27	28	29
$\mathbf{Z}_2 \oplus \mathbf{Z}_2 \oplus \mathbf{Z}_2$	$\mathbf{Z}_2$	$\mathbf{Z}_8 \oplus \mathbf{Z}_2$	$\mathbf{Z}_8$
$p_*(\nu_9\kappa_{12}), p_*(\bar{\mu}_9), p_*(\eta_9\mu_{10}\sigma_{19})$	$p_*(\eta_9\bar{\mu}_{10})$	$p_*(\bar{\zeta}_9), p_*(\bar{\sigma}_9)$	$p_*(\bar{\kappa}_9)$

30	31
$\mathbf{Z}_2$	$\mathbf{Z}_2 \oplus \mathbf{Z}_2$
$p_*(\eta_9\bar{\kappa}_{10})$	$p_*(\varepsilon_9\kappa_{17}), p_*(\nu_9\bar{\sigma}_{12})$

32	33
$\mathbf{Z}_{32} \oplus \mathbf{Z}_8 \oplus \mathbf{Z}_2$	$\mathbf{Z}_2 \oplus \mathbf{Z}_2$
$[16j_*(\iota_{16})]\rho_{17}, p_*(\nu_9\bar{\kappa}_{12}), p_*(\bar{\kappa}_9\nu_{29} - \nu_9\bar{\kappa}_{12})$	$p_*(\bar{\mu}_9\sigma_{26}), p_*(\bar{\sigma}'_9)$

34
$\mathbf{Z}_2 \oplus \mathbf{Z}_2$
$p_*(\mu_{3,9}), p_*(\eta_9\bar{\mu}_{10}\sigma_{27})$

35
$\mathbf{Z}_4 \oplus \mathbf{Z}_2 \oplus \mathbf{Z}_2 \oplus \mathbf{Z}_2$
$[4j_*(\nu_{16}^*)], p_*(\bar{\sigma}_9\sigma_{28}), p_*((a+1)\bar{\kappa}_9\nu_{29}^2 + a\nu_9^2\bar{\kappa}_{15}), p_*(\eta_9\mu_{3,10})$

36	37	38	39
$\mathbf{Z}_8 \oplus \mathbf{Z}_2$	$\mathbf{Z}_2 \oplus \mathbf{Z}_2$	$\mathbf{Z}_2$	0
$p_*(\zeta_{3,9}), p_*(\bar{\nu}_9\bar{\sigma}_{17})$	$p_*(\bar{\nu}_9\bar{\kappa}_{17}), p_*(\varepsilon_9\bar{\kappa}_{17})$	$p_*(\eta_9\bar{\varepsilon}_{10}\bar{\kappa}_{18})$	

Here we denote by $[\alpha]$ an element of $\pi_i(E_6/F_4)$ such that $\Delta([\alpha]) = \alpha \in \pi_{i-1}(X)$. The following relations hold:

$$2[16j_*(\iota_{16})]\rho_{17} = -p_*(\bar{\rho}_9),$$
$$2[4j_*(\nu_{16}^*)] \equiv p_*(a\bar{\kappa}_9\nu_{29}^2 + (a+1)\nu_9^2\bar{\kappa}_{15}) \bmod p_*(\bar{\sigma}_9\sigma_{28}),$$

where $a = 0$ or 1.

Proof. By Lemma 4.2, we can determine $\pi_i(E_6/F_4)$ for $i \leq 39$ easily except for the case $i = 32, 35$.

Consider the case $i = 32$; by Lemma 4.2, we have an exact sequence

$$0 \to \mathbf{Z}_{16} \oplus \mathbf{Z}_8 \oplus \mathbf{Z}_2 \xrightarrow{p_*} \pi_{32}(E_6/F_4) \xrightarrow{\Delta} \mathbf{Z}_2 \to 0,$$

where $\mathbf{Z}_{16} \oplus \mathbf{Z}_8 \oplus \mathbf{Z}_2$ is generated by $\bar{\rho}_9, \nu_9\bar{\kappa}_{12}, \bar{\kappa}_9\nu_{29} - \nu_9\bar{\kappa}_{12}$ and $\mathbf{Z}_2$ is generated by $16j_*(\rho_{16})$. For $p_*(\rho') \in \pi_{24}(E_6/F_4)$, we have

$$\begin{array}{rcll} p_*(\rho') & \in & p \circ \{\sigma_9, 16\iota_{16}, \sigma_{16}\} & \text{by the definition of } \rho' \text{ ([14])} \\ & = & -\{p, \sigma_9, 16\iota_{16}\} \circ \sigma_{17} & \\ & = & -\{p, i_*j_*(\iota_{16}), 16\iota_{16}\} \circ \sigma_{17} & \text{by (4.1)} \\ & \supset & -\{p, i, 16j_*(\iota_{16})\} \circ \sigma_{17} & \\ & = & -\{p, i, \Delta[16j_*(\iota_{16})]\} \circ \sigma_{17} & \\ & \ni & -[16j_*(\iota_{16})]\sigma_{17} & \text{by Lemma 4.3.} \end{array}$$

By Lemma 10.7 of [14] and (4.1), we have

$$p_*\pi_{17}(S^9)\sigma_{17} + \pi_{17}(E_6/F_4)16\sigma_{17} = \{p_*(\bar{\nu}_9\sigma_{17}), p_*(\varepsilon_9\bar{\sigma}_{17}), p_*(\sigma_9\eta_{16}\sigma_{17})\} = 0.$$

Hence we have $p_*(\rho') = -[16j_*(\iota_{16})]\sigma_{17}$. Then we have

$$\begin{array}{rcll} p_*(\bar{\rho}_9) & \in & \{p_*(\rho'), 16\iota_{24}, \sigma_{24}\} & \text{by (3.2) of [9]} \\ & = & \{-[16j_*(\iota_{16})]\sigma_{17}, 16\iota_{24}, \sigma_{24}\} & \\ & \supset & -[16j_*(\iota_{16})] \circ \{\sigma_{17}, 16\iota_{24}, \sigma_{24}\} & \\ & \ni & -[16j_*(\iota_{16})] \circ \Sigma^8\rho' & \text{by the definition of } \rho' \text{ ([14])} \\ & = & -[16j_*(\iota_{16})] \circ 2\rho_{17} & \text{by Lemma 10.9 of [14].} \end{array}$$

The indeterminacy of $\{p_*(\rho'), 16\iota_{24}, \sigma_{24}\}$ is

$$p_*(\rho') \circ \pi_{32}(S^{24}) + \pi_{25}(E_6/F_4) \circ \sigma_{25} = \{p_*(\rho'\varepsilon_{24}), p_*(\rho'\bar{\nu}_{24}), p_*(\mu_9\sigma_{18}^2)\}.$$

By Part III, Proposition 2.2 (1) of [12] and (4.1), we have

$$p_*(\rho'\bar{\nu}_{24}) = 0$$

and

$$p_*(\rho'\varepsilon_{24}) = p_*(\sigma_9^2\mu_{23}) = 0.$$

By Lemma 2.1 (4) of [13], we have

$$\mu_9\sigma_{18}^2 \equiv \rho'\eta_{24}\sigma_{25} + \sigma_9^2\mu_{23} \bmod \sigma_9\nu_{16}^3\sigma_{25}, \sigma_9\eta_{16}\varepsilon_{17}\sigma_{25}.$$

By (7.20) of [14], we have $\sigma_9\nu_{16}^3\sigma_{25} = 0$. By Lemma 10.7 of [14], we have $\sigma_9\eta_{16}\varepsilon_{17}\sigma_{25} = 0$. By Lemma 6.4 of [14] and Part III, Proposition 2.2 (1) of [12], we have $\rho'\eta_{14}\sigma_{25} = \rho'\bar{\nu}_{24} + \rho'\varepsilon_{24} = \sigma_9^2\mu_{23}$. Hence we have

$$\mu_9\sigma_{18}^2 = 2\sigma_9^2\mu_{23} = 0.$$

So we have

$$p_*(\rho') \circ \pi_{32}(S^{24}) + \pi_{25}(E_6/F_4) \circ \sigma_{25} = 0$$

and

$$p_*(\bar{\rho}_9) = -2[16j_*(\iota_{16})]\rho_{17}.$$

Therefore we have

$$\pi_{32}(E_6/F_4) = \{[16j_*(\iota_{16})]\rho_{17}, p_*(\nu_9\bar{\kappa}_{12}), p_*(\bar{\kappa}_9\nu_{29} - \nu_9\bar{\kappa}_{12})\} \cong \mathbf{Z}_{32} \oplus \mathbf{Z}_8 \oplus \mathbf{Z}_2.$$

Consider the case $i = 35$; by Lemma 4.2, we have an exact sequence

$$0 \to \mathbf{Z}_2 \oplus \mathbf{Z}_2 \oplus \mathbf{Z}_2 \oplus \mathbf{Z}_2 \xrightarrow{p_*} \pi_{35}(E_6/F_4) \xrightarrow{\Delta} \mathbf{Z}_2 \to 0,$$

where $\mathbf{Z}_2 \oplus \mathbf{Z}_2 \oplus \mathbf{Z}_2 \oplus \mathbf{Z}_2$ is generated by $\bar{\kappa}_9\nu_{29}^2, \bar{\sigma}_9\sigma_{28}, \nu_9^2\bar{\kappa}_{15}, \eta_9\mu_{3,10}$ and $\mathbf{Z}_2$ is generated by $4j_*(\nu_{16}^*)$. By the exactness there exists an element $[4j_*(\nu_{16}^*)] \in \pi_{35}(E_6/F_4)$ such that $\Delta([4j_*(\nu_{16}^*)]) = 4j_*(\nu_{16}^*)$. We consider $\{\sigma_9, 4\nu_{16}^*, 2\iota_{34}\}$. The indeterminacy of the Toda bracket is

$$\sigma_9 \circ \pi_{35}(S^{16}) + 2\pi_{35}(S^9) = \{\sigma_9\bar{\zeta}_{16}, \sigma_9\omega_{16}\nu_{32}, \sigma_9\bar{\sigma}_{16}\}.$$

So the Toda bracket $\{\sigma_9, 4\nu_{16}^*, 2\iota_{34}\}$ is represented by

$$a\bar{\kappa}_9\nu_{29}^2 + b\bar{\sigma}_9\sigma_{28} + c\nu_9^2\bar{\kappa}_{15} + d\eta_9\mu_{3,10} \in \{\sigma_9, 4\nu_{16}^*, 2\iota_{34}\}$$

for some integers a, b, c, d. By Part I, Proposition 3.5 (6) of [12], we have $\bar{\sigma}_{10}\sigma_{29} = 0$. Then the facts that $\langle \sigma, 4\nu^*, 2\iota \rangle = \nu^2\bar{\kappa}$ by Theorem 1 of [4] and that $\pi_{26}^S(S^0) = \{\nu^2\bar{\kappa}, \eta\mu_{3,*}\} \cong \mathbf{Z}_2 \oplus \mathbf{Z}_2$ imply that

$$\{\sigma_9, 4\nu_{16}^*, 2\iota_{34}\} \ni a_1\bar{\kappa}_9\nu_{29}^2 + (a_1+1)\nu_9^2\bar{\kappa}_{15} + a_2\bar{\sigma}_9\sigma_{28},$$

where $a_i = 0$ or 1. We have

$$\begin{array}{rcll}
p_*\{\sigma_9, 4\nu_{16}^*, 2\iota_{34}\} & = & -\{p, \sigma_9, 4\nu_{16}^*\} \circ 2\iota_{35} & \\
& = & -\{p, i_*j_*(\iota_{16}), 4\nu_{16}^*\} \circ 2\iota_{35} & \text{by (4.1)} \\
& \supset & -\{p, i, 4j_*(\nu_{16}^*)\} \circ 2\iota_{35} & \\
& = & -\{p, i, \Delta([4j_*(\nu_{16}^*)])\} \circ 2\iota_{35} & \\
& \ni & [4j_*(\nu_{16}^*)] \circ 2\iota_{35} & \text{by Lemma 4.3.} \\
& = & 2[4j_*(\nu_{16}^*)] &
\end{array}$$

Since $p_*(\sigma_9) = 0$ by (4.1), we have

$$p_*(\sigma_9 \circ \pi_{35}(S^{16}) + 2\pi_{35}(S^9)) = p_*(\{\sigma_9\bar{\zeta}_{16}, \sigma_9\omega_{16}\nu_{32}, \sigma_9\bar{\sigma}_{16}\}) = 0.$$

Therefore we have

$$p_*(a_1\bar{\kappa}_9\nu_{29}^2 + (a_1+1)\nu_9^2\bar{\kappa}_{15} + a_2\bar{\sigma}_9\sigma_{28}) = 2[4j_*(\nu_{16}^*)],$$

where $a_i = 0$ or 1. So we have

$$\begin{aligned}
\pi_{35}(E_6/F_4) &= \{[4j_*(\nu_{16}^*)], p_*((a+1)\bar{\kappa}_9\nu_{29}^2 + a\nu_9^2\bar{\kappa}_{15}), p_*(\bar{\sigma}_9\sigma_{28}), p_*(\eta_9\mu_{3,10})\} \\
&\cong \mathbf{Z}_4 \oplus \mathbf{Z}_2 \oplus \mathbf{Z}_2 \oplus \mathbf{Z}_2.
\end{aligned}$$

□

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