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THE BEHAVIOR OF THE SECOND PLURI-GENUS OF NORMAL SURFACE SINGULARITIES OF TYPE ${}_*A_n, {}_*D_n, {}_*E_n, {}_*\widetilde{A_n}, {}_*\widetilde{D_n}$ AND ${}_*\widetilde{E_n}$

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Abstract. In this paper, we study the behavior of the second pluri-genus δ_2 of normal surface singularities of type ${}_*A_n, {}_*D_n, {}_*E_n, {}_*\widetilde{A_n}, {}_*\widetilde{D_n}$ and ${}_*\widetilde{E_n}$. We obtain main result which corresponds to that of [7], i.e., main result is a δ_2-version of Ohyanagi's result.

1. Introduction.

Let (X, x) be a normal surface singularity and $\pi : (\widetilde{X}, A) \to (X, x)$ a resolution of the singularity (X, x). The geometric genus of the singularity (X, x) is defined by

$$p_g(X, x) = \dim_{\mathbb{C}}(R^1\pi_*\mathscr{O}_{\widetilde{X}})_x.$$

In [12], Watanabe introduced pluri-genera$\{\delta_m(X, x)\}_{m\in\mathbb{N}}$ which carry more precise information of (X, x). Suppose V is a Stein neighborhood of x in X. Let K be the canonical line bundle of $X - \{x\}$. He defined the m-th L^2-pluri-genus $\delta_m(X, x)$ by

$$\delta_m(X, x) = \dim_{\mathbb{C}} \Gamma(V - \{x\}, \mathscr{O}(mK))/L^{2/m}(V - \{x\}),$$

where $L^{2/m}(V - \{x\})$ denotes the set of $L^{2/m}$-integrable m-ple holomorphic 2-forms on $V - \{x\}$. Note that $p_g(X, x) = \delta_1(X, x)$. Recently, Okuma showed relations between the second pluri-genus δ_2 and Arnold's modality as follows:

Theorem (Okuma [9]). *Let (X, x) be a Gorenstein surface singularity and $\pi : (\widetilde{X}, A) \to (X, x)$ the minimal good resolution. Then we have the following: $\delta_2(X, x) = 0$ if and only if (X, x) is a rational double point; $\delta_2(X, x) = 1$ if and only if (X, x) is a simple elliptic or cusp singularity or a singularity whose weighted dual graph is of type $D(b_1, b_2, b_3)$; $\delta_2(X, x) = 2$ if and only if (X, x) is a minimally elliptic singularity whose weighted dual graph is of type either ${}_*\widetilde{D_n}$, ${}_*\widetilde{E_6}$, ${}_*\widetilde{E_7}$ or ${}_*\widetilde{E_8}$. In particular, if (X, x) is a hypersurface singularity, then we have the following:(X, x) is a simple (resp. unimodular, bimodular) singularity if and only if $\delta_2(X, x) = 0$ (resp.$1, 2$), where*

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$D(b_1, b_2, b_3)$ is the star-shaped graph which consists of four rational curves such that the self-intersection number of the central curve is -1.

In this paper, we study the precise behavior of the second pluri-genus δ_2 of normal surface singularities of type ${}_*A_n, {}_*D_n, {}_*E_n, {}_*\widetilde{A_n}, {}_*\widetilde{D_n}$ and ${}_*\widetilde{E_n}$. In Sect. 2, we recall normal surface singularities of type ${}_*A_n, {}_*D_n, {}_*E_n, {}_*\widetilde{A_n}$, ${}_*\widetilde{D_n}$ and ${}_*\widetilde{E_n}$ and δ_2-formula. In Sect. 3, we show the following.

Theorem A. *Let (X, x) be a normal surface singularity and $\pi : (\widetilde{X}, A) \to (X, x)$ the minimal resolution of the singularity (X, x). Assume that the weighted dual graph of (X, x) is of type either ${}_*A_n$, ${}_*D_n$, ${}_*E_6$, ${}_*E_7$ or ${}_*E_8$. Then we have $\delta_2(X, x) = 0$.*

Theorem B. *Let (X, x) be a rational singularity and $\pi : (\widetilde{X}, A) \to (X, x)$ the minimal resolution of the singularity (X, x).*
(1) *Assume that the weighted dual graph of (X, x) is of type ${}_*\widetilde{D_n}, n \geq 4$. Then we have $\delta_2(X, x) = 1$.*
(2) *Assume that the weighted dual graph of (X, x) is of type either ${}_*\widetilde{E_6}$, ${}_*\widetilde{E_7}$ or ${}_*\widetilde{E_8}$. Then we have $\delta_2(X, x) = 0$ or 1.*

In [8], we see that the second pluri-genus controls the weighted dual graphs. Theorem A, B give examples of ([8], Lemma 3.4, 3.12). By Theorem (Okuma [9]) above, we have that $\delta_2(X, x) = 2$ if (X, x) is a minimally elliptic singularity whose weighted dual graph is of type either ${}_*\widetilde{D_n}, {}_*\widetilde{E_6}, {}_*\widetilde{E_7}$ or ${}_*\widetilde{E_8}$. We can obtain this by showing a special case of ([9], Corollary 2.6):for a minimally elliptic singularity which satisfies some condition, the second pluri-genus is determined by some self-intersection number V^2. We have this directly from ([9], Corollary 2.6). However, we give a different proof. Hence we obtain the precise behavior of the second pluri-genus of normal surface singularities of type ${}_*\widetilde{D_n}, {}_*\widetilde{E_6}, {}_*\widetilde{E_7}$ and ${}_*\widetilde{E_8}$.

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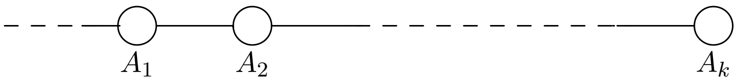
2. Preliminaries.

Let (X, x) be a normal surface singularity, $\pi : (\widetilde{X}, A) \to (X, x)$ the minimal good resolution and $A = \cup A_i, 1 \leq i \leq n$ the decomposition of the exceptional set A into irreducible components. Then we define the following.

Definition 2.1. A *string* S in A is a chain of non-singular rational curves $A_1, \ldots, A_k$ so that $A_i \cdot A_{i+1} = 1$ for $i = 1, \ldots, k-1$, and these account for all intersections in A among the A_i's, except that A_1 intersects exactly one other curve.



The weighted dual graph of (X, x) is called a *star-shaped graph*, if A is not a chain of rational curves, and if $A = A_0 + \sum S_i$, where A_0 is a curve and S_i $(i = 1, \ldots, \beta)$ are the maximal strings. Then A_0 is called the *central curve*, and S_j are called *branches*. Let $S_i = \cup A_{ij}, 1 \leq j \leq r_i$ be the decomposition into irreducible components, where $A_0 \cdot A_{i1} = A_{ij} \cdot A_{ij+1} = 1$. Let g =genus(A_0), $b = -{A_0}^2$ and $b_{ij} = -{A_{ij}}^2$. For each branch S_i, the positive integers e_i and d_i are defined by

$$\frac{d_i}{e_i} = [b_{i1}, \cdots, b_{ir_i}] = b_{i1} - \cfrac{1}{b_{i2} - \cfrac{1}{\cdots - \cfrac{1}{b_{ir_i}}}},$$

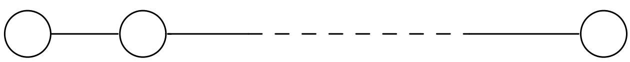
where $e_i < d_i$ and e_i and d_i are relatively prime. We call the set $\{g; b, (d_1, e_1), \cdots, (d_\beta, e_\beta)\}$ the *data of the star-shaped graph.*

We define the divisor $D_m^{(k)}$ on A_0 as follows.

Definition 2.2. For any integers $k \geq 0$ and $m > 0$, we define the divisor $D_m^{(k)}$ on A_0 by $D_m^{(k)} = kD - \sum_{i=1}^{\beta} [(ke_i + m(d_i - 1))/d_i] p_i$, where D is any divisor such that $\mathcal{O}_{A_0}(D)$ is the conormal sheaf of A_0, $p_i = A_0 \cap A_{i1}$, and for any $a \in \mathbb{R}$, $[a]$ is the greatest integer less than or equal to a.

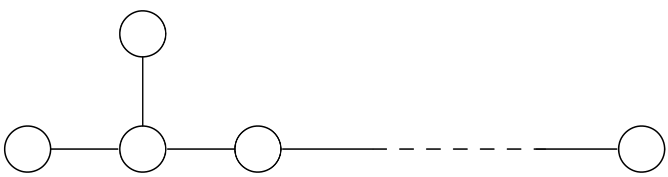
Next we recall normal surface singularities of type ${}_*A_n, {}_*D_n, {}_*E_n, {}_*\widetilde{A_n}, {}_*\widetilde{D_n}$ and ${}_*\widetilde{E_n}$ which Ohyanagi introduced ([7]).

Let A_n, $n \geq 1$, be



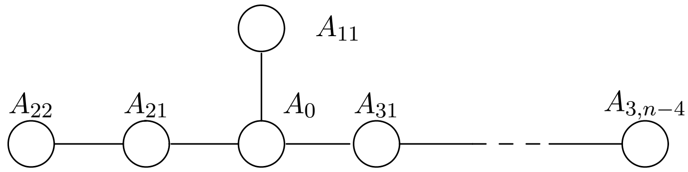
(n-vertices),

D_n, $n \geq 4$,



(n-vertices),

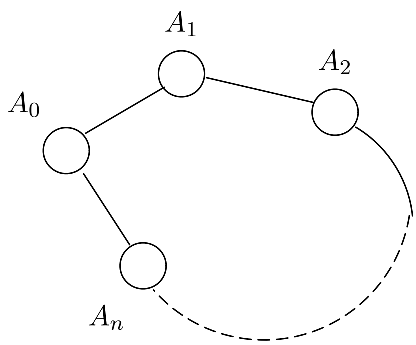
and E_n, $n = 6, 7$ and 8,



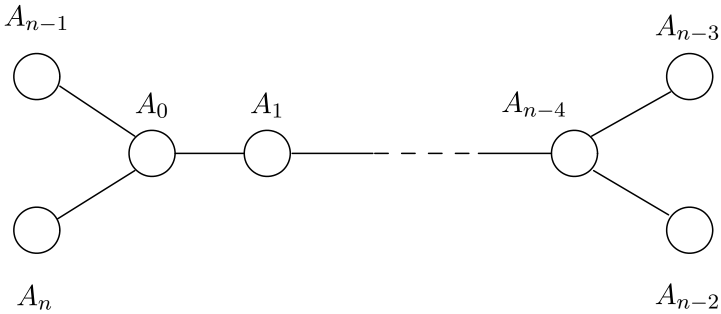
(n-vertices).

where each vertex represents a non-singular ratinal curve with self-intersection number -2. Then, the above graphs are contractible. In [1], Artin has studied the singularities. We say that the singularities are the rational double points. Next, we consider those graphs, contractible, each of which has the same type as the graphs A_n, D_n, E_6, E_7 and E_8 up to the weights. We may assume that each weight of the graphs is less than or equal to -2, since we may assume that $\pi : \widetilde{X} \to X$ is the minimal resolution of (X, x). We can easily check that the graphs are always contractible. Hence we need not feel concern for contractibility of the graphs. We denote them by ${}_*A_n, {}_*D_n, {}_*E_6, {}_*E_7$ and ${}_*E_8$.

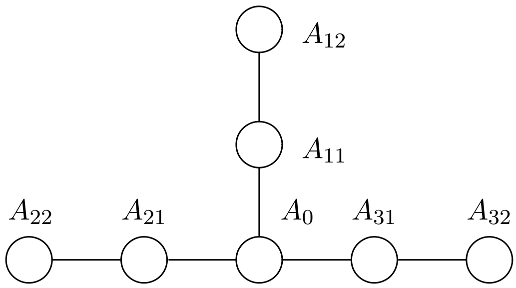
Let $\widetilde{A_n}, n \geq 1$, be



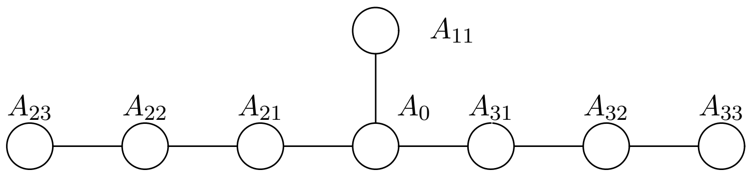
Let $\widetilde{D_n}, n \geq 4$, be



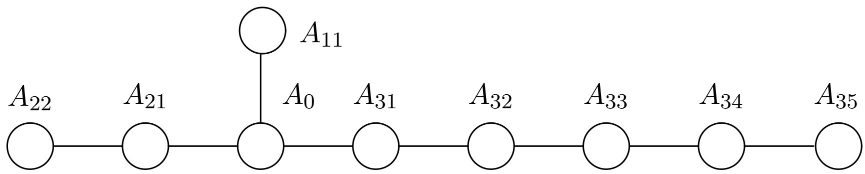
Let $\widetilde{E_6}$ be



Let $\widetilde{E_7}$ be



Let $\widetilde{E_8}$ be



Here we remark that each vertex corresponds to a non-singular rational curve with self-intersection number -2. We consider those graphs, contractible, each of which has the same type as the graphs $\widetilde{A_n}, \widetilde{D_n}, \widetilde{E_6}, \widetilde{E_7}$ and $\widetilde{E_8}$ up to the weights. Assume that each weight is less than or equal to -2. Such a graph is not always contractible. But we can easily check that such graph is contractible if and only if there is at least one vertex whose weight is less than or equal to -3. We denote these contractible graphs by ${}_*\widetilde{A_n}, n \geq 1; {}_*\widetilde{D_n}, n \geq 4; {}_*\widetilde{E_n}, n = 6, 7$ and 8.

Notation 2.3. Let $\mathscr{F}$ be a sheaf of $\mathscr{O}_{\widetilde{X}}$-modules and D a divisor on $\widetilde{X}$. We use the following notation:$\mathscr{F}(D) = \mathscr{F} \otimes \mathscr{O}_{\widetilde{X}}(D)$, $H^i(\mathscr{F}) = H^i(\widetilde{X}, \mathscr{F})$, ${H^i}_A(\mathscr{F}) = {H^i}_A(\widetilde{X}, \mathscr{F})$, $h^i(\mathscr{F}) = \dim_{\mathbb{C}} H^i(\mathscr{F})$ and ${h^i}_A(\mathscr{F}) = \dim_{\mathbb{C}} {H^i}_A(\mathscr{F})$.

We denote by K the canonical divisor on $\widetilde{X}$. The Riemann-Roch theorem implies, for any positive cycle V and any invertible sheaf $\mathscr{F}$ on $\widetilde{X}$, that $\chi(\mathscr{O}_V) = h^0(\mathscr{O}_V) - h^1(\mathscr{O}_V) = -V \cdot (V + K)/2$ and $\chi(\mathscr{O}_V \otimes \mathscr{F}) = h^0(\mathscr{O}_V \otimes \mathscr{F}) - h^1(\mathscr{O}_V \otimes \mathscr{F}) = \mathscr{F} \cdot V + \chi(\mathscr{O}_V)$.

We use the following theorems to study the behavior of the second pluri-genus δ_2 of normal surface singularities of type ${}_*A_n, {}_*D_n, {}_*E_n, {}_*\widetilde{A_n}, {}_*\widetilde{D_n}$ and ${}_*\widetilde{E_n}$.

Theorem 2.1 (Okuma [8], Theorem 2.8)**.** *Let (X, x) be a normal surface singularity and $\pi : (\widetilde{X}, A) \to (X, x)$ the minimal good resolution of the singularity (X, x). Then we have*

$$\delta_2(X, x) = h^1{}_A(\mathscr{O}_{\widetilde{X}}(2K + A)) = h^1(\mathscr{O}_{\widetilde{X}}(-K - A)),$$

where K is the canonical divisor on $\widetilde{X}$.

Theorem 2.2 (Okuma [8], Lemma 3.2)**.** *Let (X, x) be a rational singularity and $\pi : (\widetilde{X}\ , A) \to (X, x)$ the minimal resolution of the singularity (X, x). Assume that the weighted dual graph of (X, x) is a star-shaped graph with the data $\{0; b, (d_1, e_1), \cdots, (d_\beta, e_\beta)\}$. Then we have $\delta_m(X, x) = \sum_{k\geq 0} h^0(\mathscr{O}_{A_0}(mK_{A_0} - D_m^{(k)}))$.*

3. The second pluri-genus of type ${}_*A_n, {}_*D_n, {}_*E_n, {}_*\widetilde{A_n}, {}_*\widetilde{D_n}$ and ${}_*\widetilde{E_n}$.

We follow the notation of the preceding section. Let $A = \cup A_i, 1 \leq i \leq n$, where A_i are its irreducible components, lie on a non-singular complex surface $\widetilde{X}$. Let $G(a_1, \cdots, a_n)$ denote the weighted dual graph associated with a contractible curve $A = \cup A_i$, where $a_i = -{A_i}^2$. We may assume that if A_i is a non-singular rational curve, then $a_i \geq 2$. There are many other combinations of the weights $({a_1}', \cdots, {a_n}')$ which make the graph $G({a_1}', \cdots, {a_n}')$ to be contractible. We denote by $p_g(a_1, \cdots, a_n)$ (resp. $\delta_2(a_1, \cdots, a_n)$) the geometric genus (resp. second pluri-genus) of the singularity (X, x) obtained from a contractible graph $G(a_1, \cdots, a_n)$. Then Ohyanagi decided all those graphs $G(a_1, \cdots, a_n)$ such that $p_g(a_1, \cdots, a_n) = 0$ for any $(a_1, \cdots, a_n)$.

Theorem (Ohyanagi [7], Theorem A, C)**.** *Let (X, x) be a normal surface singularity and $\pi : (\widetilde{X}, A) \to (X, x)$ the minimal resolution of the singularity (X, x). Assume that the weighted dual graph of (X, x) is of type either ${}_*A_n$, ${}_*D_n$, ${}_*E_6$, ${}_*E_7$ or ${}_*E_8$. Then we have $p_g(X, x) = 0$. Moreover, all graphs $G(a_1, \cdots, a_n)$ such that $p_g(a_1, \cdots, a_n) = 0$ for any $(a_1, \cdots, a_n)$ are of type ${}_*A_n$, ${}_*D_n$, ${}_*E_6$, ${}_*E_7$, ${}_*E_8$.*

We obtain the following theorem which corresponds to (Ohyanagi [7], Theorem A, C).

Theorem 3.1. *Let (X, x) be a normal surface singularity and $\pi : (\widetilde{X}, A) \to (X, x)$ the minimal resolution of the singularity (X, x). Assume that the weighted dual graph of (X, x) is of type either ${}_*A_n$, ${}_*D_n$, ${}_*E_6$, ${}_*E_7$ or ${}_*E_8$.*

Then we have $\delta_2(X,x) = 0$. Moreover, all graphs $G(a_1, \cdots, a_n)$ such that $\delta_2(a_1, \cdots, a_n) = 0$ for any $(a_1, \cdots, a_n)$ are of type $_*A_n$, $_*D_n$, $_*E_6$, $_*E_7$, $_*E_8$

Proof. By ([7], Theorem 2.2), we have that each of the graphs $_*A_n, n \geq 1; {}_*D_n, n \geq 4; {}_*E_n, n = 6, 7$ and 8, is the weighted dual graph for a rational singularity. Let (X,x) be a normal surface singularity whose weighted dual graph is of type $_*A_n$. Since (X,x) is cyclic quotient singularity, we have $\delta_2(X,x) = 0$ ([12], Theorem 3.9). Let (X,x) be a rational singularity whose weighted dual graph is a star-shaped graph with three branches. For any integers $k \geq 0$ and $m > 0$, we put $c(m,k) = -2m - bk + \sum_{i=1}^{3}[(ke_i + m(d_i - 1))/d_i]$. Using Theorem 2.2, if $b(= -{A_0}^2) \geq 3$, then $\delta_2(X,x) = 0$. We may assume that $b = 2$.

Let (X,x) be a normal surface singularity whose weighted dual graph is of type $_*D_n$. We have the following.

Claim 1. Let $b_i \geq 2, 1 \leq i \leq r$, be integers. We put $d/e = [b_1, \cdots, b_r]$, where d and e are relatively prime and $r \geq 2$. For any integer $k \geq 0$, we have that

$$\left[\frac{ke-2}{d}\right] \leq \begin{cases} \left[\frac{kr-2}{r+1}\right] & \text{if } k \not\equiv r \pmod{(r+1)}, \\ 1 + \left[\frac{kr-2}{r+1}\right] & \text{if } k \equiv r \pmod{(r+1)}. \end{cases}$$

Proof of Claim 1. Since $e/d \leq r/(r+1)$, we have $[(ke-2)/d] \leq [kr/(r+1) - 2/d]$. By comparing $[kr/(r+1) - 2/d]$ with $[(kr-2)/(r+1)]$, we obtain the assertion.

Hence we have that

$$c(2,k) \leq \begin{cases} 2 - 2k + 2\left[\frac{k-2}{2}\right] + \left[\frac{kr-2}{r+1}\right] & \text{if } k \not\equiv r \pmod{(r+1)}, \\ 3 - 2k + 2\left[\frac{k-2}{2}\right] + \left[\frac{kr-2}{r+1}\right] & \text{if } k \equiv r \pmod{(r+1)}. \end{cases}$$

If $k \not\equiv r \pmod{(r+1)}$, $c(2,k) \leq -2/(r+1) - k/(r+1)$, so we have $c(2,k) < 0$. If $k \equiv r \pmod{(r+1)}$, $c(2,k) \leq (r-1)/(r+1) - k/(r+1)$, so we have $c(2,k) < 0$. By Theorem 2.2, we have $\delta_2(X,x) = 0$. We can easily check the following.

Claim 2. Let $b_i \geq 2, (i = 1, 2)$ be integers. we put $d/e = [b_1, b_2]$, where d and e are relatively prime. Assume that $d/e \neq [2, 2]$. For any integer $k \geq 0$, we have that

$$\left[\frac{ke-2}{d}\right] \leq \begin{cases} \left[\frac{3k-2}{5}\right] & \text{if } k \not\equiv 2 \pmod{5}, \\ 1 + \left[\frac{3k-2}{5}\right] & \text{if } k \equiv 2 \pmod{5}. \end{cases}$$

In a similar way, by using Claim 1 and Claim 2, we can prove that if (X,x) is a nomal surface singularity whose weighted dual graph is of type either $_*E_6$, $_*E_7$ or $_*E_8$, then we have $\delta_2(X,x) = 0$ (cf. Proof of Theorem

3.3). Let $G(a_1, \cdots, a_n)$ be a graph such that $\delta_2(a_1, \cdots, a_n) = 0$ for any $(a_1, \cdots, a_n)$. Since it is well-known that we have the inequality $\delta_1 \leq \delta_2$ ([10], Lemma 1.66), $G(a_1, \cdots, a_n)$ is a graph such that $\delta_1(a_1, \cdots, a_n) = 0$ for any $(a_1, \cdots, a_n)$. By Theorem (Ohyanagi [7]) in Sect. 3, $G(a_1, \cdots, a_n)$ is of type either ${}_*A_n$, ${}_*D_n$, ${}_*E_6$, ${}_*E_7$ or ${}_*E_8$. Hence, by the argument above, all graphs $G(a_1, \cdots, a_n)$ such that $\delta_2(a_1, \cdots, a_n) = 0$ for any $(a_1, \cdots, a_n)$ are of type ${}_*A_n$, ${}_*D_n$, ${}_*E_6$, ${}_*E_7$, ${}_*E_8$. □

The following example is normal surface singularities of type ${}_*D_n, n \geq 4$, ${}_*E_6$, ${}_*E_7$ and ${}_*E_8$ with the third pluri-genus $\delta_3 \neq 0$.

Example 3.1. There exists a rational singularity (X, x) whose weighted dual graph is a star-shaped graph with the data $\{0; 2, (3,1), (3,1), (3,1)\}$ (resp. $\{0; 2, (3,1), (3,2), (3,2)\}$, $\{0; 2, (3,1), (3,2), (4,3)\}$, $\{0; 2, (3,1), (3,2), (5,4)\}$),i.e., (X, x) is of type ${}_*D_4$ (resp. ${}_*E_6$, ${}_*E_7$, ${}_*E_8$). By Theorem 2.2, we have $\delta_3(X, x) = 1$. Hence all graphs of type ${}_*A_n, {}_*D_n, n \geq 4$, ${}_*E_6$, ${}_*E_7$ and ${}_*E_8$ are not characterized by the third pluri-genus δ_3.

Next we consider the second pluri-genus of type ${}_*\widetilde{A_n}, {}_*\widetilde{D_n}$, ${}_*\widetilde{E_6}$, ${}_*\widetilde{E_7}$ and ${}_*\widetilde{E_8}$. Let (X, x) be a normal surface singularity and $\pi : (\widetilde{X}, A) \to (X, x)$ the minimal resolution of the singularity (X, x). If the weighted dual graph of (X, x) is of type ${}_*\widetilde{A_n}$, then (X, x) is a cusp singularity. Hence we have $\delta_2(X, x) = 1$ (cf. [12], Theorem 1.16). Assume that the weighted dual graph of (X, x) is of type either ${}_*\widetilde{D_n}$, ${}_*\widetilde{E_6}$, ${}_*\widetilde{E_7}$ or ${}_*\widetilde{E_8}$. Then (X, x) is a rational or minimally elliptic singularity ([7], Theorem 4.1.3, 4.2.1, 4.3.1, 4.4.1). We have the following lemma directly from ([9], Corollary 2.6). However, we give a different proof for it. We can use the method of proof to prove (1) of Theorem B in Introduction.

Lemma 3.1. *Let (X, x) be a minimally elliptic singularity and $\pi : (\widetilde{X}, A) \to (X, x)$ the minimal good resolution of the singularity (X, x). Let Z_K be a cycle such that $\mathscr{O}_{\widetilde{X}}(K) \cong \mathscr{O}_{\widetilde{X}}(-Z_K)$ and $V = Z_K - A$. Assume that (X, x) is not a simple elliptic or cusp singularity. Then we have that*

$$\delta_2(X, x) = 1 + h^1(\mathscr{O}_V(-K - A)) = V \cdot (K + A) = -V^2,$$

where K is the canonical divisor on $\widetilde{X}$.

Proof. If π is the minimal, by the assumption and ([4], Theorem 3.10), Z_k is the fundamental cycle and we have $h^1(\mathscr{O}_{\widetilde{X}}) = h^1(\mathscr{O}_{Z_K}) = 1$ and $V \cdot (K+A) = -V^2$. Hence it is enough to prove that $\delta_2(X, x) = 1 + h^1(\mathscr{O}_V(-K - A)) = V \cdot (K + A)$. In the minimal resolution of a minimally elliptic singularity, for $0 < W < Z_K$, we have the strict inequality $h^1(\mathscr{O}_W) < h^1(\mathscr{O}_{Z_K})$ (cf. Némethi, [6]). By the assumption and the adjunction formula (cf.[4]), we have $A < Z_K$. Since $0 < V, A < Z_K$, we have that $H^1(\mathscr{O}_V) = 0$ and

$\chi(\mathscr{O}_A) = 1$. By ([4], Theorem 3.4) and the Riemann-Roch theorem, we have that $\chi(\mathscr{O}_{Z_K}) = \chi(\mathscr{O}_V) + \chi(\mathscr{O}_A) - V \cdot A = 0$. Since $-V \cdot A = (-Z_K + A) \cdot A = (K + A) \cdot A$, we have $V \cdot A = 2$. Hence we have $\chi(\mathscr{O}_V) = 1$. From a sheaf exact sequence

$$0 \to \mathscr{O}_{\widetilde{X}}(-K - A - V) \to \mathscr{O}_{\widetilde{X}}(-K - A) \to \mathscr{O}_V(-K - A) \to 0,$$

we have an exact sequence

$$0 \to H^0(\mathscr{O}_{\widetilde{X}}(-K - A - V)) \to H^0(\mathscr{O}_{\widetilde{X}}(-K - A)) \to H^0(\mathscr{O}_V(-K - A))$$

$$\to H^1(\mathscr{O}_{\widetilde{X}}(-K - A - V)) \to H^1(\mathscr{O}_{\widetilde{X}}(-K - A)) \to H^1(\mathscr{O}_V(-K - A)) \to 0.$$

Since $\mathscr{O}_{\widetilde{X}}(-K - A - V) \cong \mathscr{O}_{\widetilde{X}}$, we have $h^1(\mathscr{O}_{\widetilde{X}}(-K - A - V)) = 1$. Also, since $(2K + A + V) \cdot A_i = K \cdot A_i \geq 0$ for all $A_i \subset Supp(V)$ and $H^1(\mathscr{O}_V) = 0$, it follows from ([5], (11.1)) that $H^1(\mathscr{O}_V(2K + A + V)) = 0$. By the Serre duality, the Riemann-Roch theorem and Theorem 2.1, we have that

$$\delta_2(X, x) = h^1(\mathscr{O}_{\widetilde{X}}(-K - A)) = 1 + h^1(\mathscr{O}_V(-K - A)) = V \cdot (K + A).$$

If π is not minimal, by the assumption and ([4], Proposition 3.5), (X, x) is a singularity whose weighted dual graph is of type $D(b_1, b_2, b_3)$. Then we have $Z_K = 2A_0 + \sum_{i=1}^{3} A_i$. Since $A_0 \cong \mathbb{P}^1$ and $K \cdot A_0 = -1$, we have $H^1(\mathscr{O}_{A_0}(K)) = 0$ and $\chi(\mathscr{O}_{A_0}) = 1$. In a similar way, we have $\delta_2(X, x) = 1 + h^1(\mathscr{O}_V(-K - A)) = V \cdot (K + A) = -V^2$. □

Theorem (Okuma [9]). *Let (X, x) be a minimally elliptic singularity and $\pi : (\widetilde{X}, A) \to (X, x)$ the minimal resolution of the singularity (X, x). Assume that the weighted dual graph of (X, x) is of type either ${}_*\widetilde{D_n}, {}_*\widetilde{E_6}, {}_*\widetilde{E_7}$ or ${}_*\widetilde{E_8}$, then $\delta_2(X, x) = 2$.*

Proof. Let (X, x) be a minimally elliptic singularity whose weighted dual graph is of type ${}_*\widetilde{D_n}$ (resp. ${}_*\widetilde{E_6}$, ${}_*\widetilde{E_7}$, ${}_*\widetilde{E_8}$). We put $V = \sum_{i=0}^{n-4} A_i$ (resp. $2A_0 + \sum_{i=1}^{3} A_{i1}$, $3A_0 + 2(A_{21} + A_{31}) + A_{11} + A_{22} + A_{32}$, $5A_0 + 4A_{31} + 3(A_{21} + A_{32}) + 2(A_{11} + A_{33}) + A_{22} + A_{34}$). Then we have $V^2 = -2$ and $Z_K = V + A$. By Lemma 3.1, we have that $\delta_2(X, x) = 2$. □

Hence we consider the precise behavior of the second pluri-genus δ_2 of rational singularities of type ${}_*\widetilde{D_n}, {}_*\widetilde{E_6}$, ${}_*\widetilde{E_7}$ and ${}_*\widetilde{E_8}$. We obtain the following theorem with respect to type ${}_*\widetilde{D_n}$,

Theorem 3.2. *Let (X, x) be a rational singularity and $\pi : (\widetilde{X}, A) \to (X, x)$ the minimal resolution of the singularity (X, x). Assume that the weighted dual graph of (X, x) is of type ${}_*\widetilde{D_n}$. Then we have $\delta_2(X, x) = 1$.*

Proof. We follow the proof of Lemma 3.1. Let $V = \sum_{i=0}^{n-4} A_i$. Since (X, x) is rational and the natural map $H^1(\mathcal{O}_{\widetilde{X}}) \to H^1(\mathcal{O}_V)$ is surjective, we have $H^1(\mathcal{O}_V) = 0$. Consider a sheaf exact sequence

$$0 \to \mathcal{O}_{\widetilde{X}}(-K - A - V) \to \mathcal{O}_{\widetilde{X}}(-K - A) \to \mathcal{O}_V(-K - A) \to 0.$$

Since (X, x) is rational and $(-K - A - V) \cdot A_i \geq 0$ for all A_i, it follows from ([5], (12.1)) that $H^1(\mathcal{O}_{\widetilde{X}}(-K - A - V)) = 0$. Also, since $(2K + A + V) \cdot A_i = 0$ for all $A_i \subset Supp(V)$ and $H^1(\mathcal{O}_V) = 0$, it follows from ([5], (11.1)) that $H^1(\mathcal{O}_V(2K + A + V)) = 0$. As in the proof of Lemma 3.1, we have that

$$\delta_2(X, x) = h^1(\mathcal{O}_{\widetilde{X}}(-K - A)) = -\chi(\mathcal{O}_V(-K - A)) = V \cdot (K + A) - 1 = 1$$

□

Also, we obtain the following theorem.

Theorem 3.3. *Let (X, x) be a rational singularity and $\pi : (\widetilde{X}, A) \to (X, x)$ the minimal resolution of the singularity (X, x).*

(A) *If the weighted dual graph of (X, x) is of type ${}_*\widetilde{E_6}$, then we have the following*:
(1) *if $b = 2$, $\delta_2(X, x) = 1$;*
(2) *otherwise, $\delta_2(X, x) = 0$.*

(B) *If the weighted dual graph of (X, x) is of type ${}_*\widetilde{E_7}$, then we have the following*:
(1) *if $b = b_{21} = b_{31} = 2$, $\delta_2(X, x) = 1$;*
(2) *otherwise, $\delta_2(X, x) = 0$.*

(C) *If the weighted dual graph of (X, x) is of type ${}_*\widetilde{E_8}$, then we have the following*:
(1) *if $b = b_{11} = b_{21} = b_{31} = b_{32} = b_{33} = 2$, $\delta_2(X, x) = 1$;*
(2) *otherwise, $\delta_2(X, x) = 0$.*

Proof. We follow the proof of Theorem 3.1. We may assume that $b = 2$. Let (X, x) be a rational singularity whose weighted dual graph is of type ${}_*\widetilde{E_6}$. We put $d_i/e_i = [-A_{i1}{}^2, -A_{i2}{}^2]$, $(i = 1, 2, 3)$. We may assume that $e_1/d_1 \leq e_2/d_2 \leq e_3/d_3$ without loss of generality. We can easily check the following.

Claim 3. Let $b_i \geq 2, (i = 1, 2)$ be integers. We put $d/e = [b_1, b_2]$, where d and e are relatively prime. Assume that $d/e \neq [2, c], (c \geq 2, c \in \mathbb{Z})$. For any integer $k \geq 0$, we have that

$$\left[\frac{ke - 2}{d}\right] \leq \begin{cases} \left[\frac{2k-2}{5}\right] & \text{if } k \not\equiv 3 \pmod 5, \\ 1 + \left[\frac{2k-2}{5}\right] & \text{if } k \equiv 3 \pmod 5. \end{cases}$$

Since $p_g(X,x)=0$, by ([7], Theorem 4.2.1), Claim 1 and Claim 3, we have the following:

$$c(2,k) \leq \begin{cases} 2-2k+\left[\frac{2k-2}{5}\right]+2\left[\frac{2k-2}{3}\right] & \text{if } k \not\equiv 2(\text{mod } 3) \text{ and } k \not\equiv 3(\text{mod } 5), \\ 3-2k+\left[\frac{2k-2}{5}\right]+2\left[\frac{2k-2}{3}\right] & \text{if } k \not\equiv 2(\text{mod } 3) \text{ and } k \equiv 3(\text{mod } 5), \\ 4-2k+\left[\frac{2k-2}{5}\right]+2\left[\frac{2k-2}{3}\right] & \text{if } k \equiv 2(\text{mod } 3) \text{ and } k \not\equiv 3(\text{mod } 5), \\ 5-2k+\left[\frac{2k-2}{5}\right]+2\left[\frac{2k-2}{3}\right] & \text{if } k \equiv 2(\text{mod } 3) \text{ and } k \equiv 3(\text{mod } 5). \end{cases}$$

Hence we have that $c(2,1)\leq 0$, $c(2,2)\leq 0$ and $c(2,k)<0$ for $k\neq 1,2$. When $k=2$, we have that

$$\left[\frac{2(e_3-1)}{d_3}\right] \leq_{(a)} \left[\frac{4}{3}-\frac{2}{d_3}\right] \leq_{(b)} 1+\left[\frac{4}{3}-\frac{2}{3}\right]=1.$$

Then there is not an associated continued fraction which the equalities (a) and (b) are simultaneously satisfied. In fact, we have that

$$\frac{2(e_3-1)}{d_3} \leq \frac{2(-{A_{32}}^2)-2}{2(-{A_{32}}^2)-1} < 1, \ \textit{i.e.}, \ \left[\frac{2(e_3-1)}{d_3}\right] < 1.$$

Hence we have $c(2,2)<0$. By Theorem 2.2, $\delta_2(X,x)=h^0(\mathscr{O}_{A_0}(2K_{A_0}-D_2^{(1)}))\leq 1$. By considering continued fractions which satisfy $c(2,1)=0$, we can easily check (A).

Let (X,x) is a rational singularity whose weighted dual graph is of type ${}_*\widetilde{E_7}$. First, assume that $b_{11}\geq 3$. Since $p_g(X,x)=0$, by ([7], Theorem 4.3.1) and Claim 1, we have the following:

$$c(2,k) \leq \begin{cases} 2-2k+\left[\frac{k-2}{3}\right]+2\left[\frac{3k-2}{4}\right] & \text{if } k \not\equiv 3 \pmod 4, \\ 4-2k+\left[\frac{k-2}{3}\right]+2\left[\frac{3k-2}{4}\right] & \text{if } k \equiv 3 \pmod 4. \end{cases}$$

Hence we have that $c(2,2)\leq 0$, $c(2,3)\leq 0$ and $c(2,k)<0$ for $k\neq 2,3$. In a similar way, we have $c(2,3)<0$. By Theorem 2.2, $\delta_2(X,x)=h^0(\mathscr{O}_{A_0}(2K_{A_0}-D_2^{(2)}))\leq 1$. Next, assume that $b_{11}=2$.
(1) Assume that $(b_{21},b_{22},b_{23})=(2,2,2)$. We can easily check the following.

Claim 4. Let $b_i\geq 2, (i=1,2,3)$ be integers. We put $d/e=[b_1,b_2,b_3]$, where d and e are relatively prime. Assume that $d/e\neq[2,2,c],(c\geq 2, c\in\mathbb{Z})$. For any integer $k\geq 0$, we have that

$$\left[\frac{ke-2}{d}\right] \leq \begin{cases} \left[\frac{5k-2}{8}\right] & \text{if } k \not\equiv 5 \pmod 8, \\ 1+\left[\frac{5k-2}{8}\right] & \text{if } k \equiv 5 \pmod 8. \end{cases}$$

Since $p_g(X,x)=0$, by ([7], Theorem 4.3.1) and Claim 4, we have the following:

$$c(2,k) \leq \begin{cases} 2-2k+\left[\frac{k-2}{2}\right]+\left[\frac{3k-2}{4}\right]+\left[\frac{5k-2}{8}\right] & \text{if } k \not\equiv 5 \pmod 8, \\ 3-2k+\left[\frac{k-2}{2}\right]+\left[\frac{3k-2}{4}\right]+\left[\frac{5k-2}{8}\right] & \text{if } k \equiv 5 \pmod 8. \end{cases}$$

Hence we have that $c(2,2) \le 0$ and $c(2,k) < 0$ for $k \ne 2$. By Theorem 2.2, $\delta_2(X,x) = h^0(\mathscr{O}_{A_0}(2K_{A_0} - D_2^{(2)})) \le 1$.
(2) Assume that $(b_{21}, b_{22}, b_{23}) \ne (2,2,2)$. If $(b_{31}, b_{32}, b_{33}) = (2,2,2)$, by (1), we have $\delta_2(X,x) = h^0(\mathscr{O}_{A_0}(2K_{A_0} - D_2^{(2)})) \le 1$. Assume that $(b_{31}, b_{32}, b_{33}) \ne (2,2,2)$. We can easily check the following.

Claim 5. Let $b_i \ge 2, (i = 1,2,3)$ be integers. We put $d/e = [b_1, b_2, b_3]$, where d and e are relatively prime. Assume that $d/e \ne [2,2,2]$. For any integer $k \ge 0$, we have that

$$\left[\frac{ke-2}{d}\right] \le \begin{cases} \left[\frac{5k-2}{7}\right] & \text{if } k \not\equiv 3 \pmod 7, \\ 1 + \left[\frac{5k-2}{7}\right] & \text{if } k \equiv 3 \pmod 7. \end{cases}$$

Since $p_g(X,x) = 0$, by ([7], Theorem 4.3.1), Claim 4 and Claim 5, we have the following:

$$c(2,k) \le \begin{cases} 2 - 2k + \left[\frac{k-2}{2}\right] + \left[\frac{5k-2}{7}\right] + \left[\frac{5k-2}{8}\right] & \text{if } k \not\equiv 5 \pmod 8 \\ & \text{and } k \not\equiv 3 \pmod 7, \\ 3 - 2k + \left[\frac{k-2}{2}\right] + \left[\frac{5k-2}{7}\right] + \left[\frac{5k-2}{8}\right] & \text{if } k \equiv 5 \pmod 8 \\ & \text{and } k \not\equiv 3 \pmod 7, \\ 3 - 2k + \left[\frac{k-2}{2}\right] + \left[\frac{5k-2}{7}\right] + \left[\frac{5k-2}{8}\right] & \text{if } k \not\equiv 5 \pmod 8 \\ & \text{and } k \equiv 3 \pmod 7, \\ 4 - 2k + \left[\frac{k-2}{2}\right] + \left[\frac{5k-2}{7}\right] + \left[\frac{5k-2}{8}\right] & \text{if } k \equiv 5 \pmod 8 \\ & \text{and } k \equiv 3 \pmod 7. \end{cases}$$

Hence we have that $c(2,2) \le 0$ and $c(2,k) < 0$ for $k \ne 2$. By Theorem 2.2, $\delta_2(X,x) = h^0(\mathscr{O}_{A_0}(2K_{A_0} - D_2^{(2)})) \le 1$. Hence, by considering continued fractions which satisfy $c(2,2) = 0$, we can easily check (B).

Let (X,x) is a rational singularity whose weighted dual graph is of type ${}_*\widetilde{E_8}$. First, assume that $b_{11} \ge 3$. Since $p_g(X,x) = 0$, by ([7], Theorem 4.4.1) and Claim 1, we have the following:

$$c(2,k) \le \begin{cases} 2 - 2k + \left[\frac{k-2}{3}\right] + \left[\frac{2k-2}{3}\right] + \left[\frac{5k-2}{6}\right] & \text{if } k \not\equiv 2 \pmod 3 \\ & \text{and } k \not\equiv 5 \pmod 6, \\ 3 - 2k + \left[\frac{k-2}{3}\right] + \left[\frac{2k-2}{3}\right] + \left[\frac{5k-2}{6}\right] & \text{if } k \equiv 2 \pmod 3 \\ & \text{and } k \not\equiv 5 \pmod 6, \\ 3 - 2k + \left[\frac{k-2}{3}\right] + \left[\frac{2k-2}{3}\right] + \left[\frac{5k-2}{6}\right] & \text{if } k \not\equiv 2 \pmod 3 \\ & \text{and } k \equiv 5 \pmod 6, \\ 4 - 2k + \left[\frac{k-2}{3}\right] + \left[\frac{2k-2}{3}\right] + \left[\frac{5k-2}{6}\right] & \text{if } k \equiv 2 \pmod 3 \\ & \text{and } k \equiv 5 \pmod 6. \end{cases}$$

Hence, we have that $c(2,2) \le 0$, $c(2,5) \le 0$ and $c(2,k) < 0$ for $k \ne 2,5$. In a similar way, we have $c(2,2) < 0$ and $c(2,5) < 0$. By Theorem 2.2, $\delta_2(X,x) = 0$. Next, assume that $b_{11} = 2$.
(1) Assume that $(b_{21}, b_{22}) = (2,2)$. We can easily check the following.

Claim 6. Let $b_i \geq 2, (1 \leq i \leq 5)$ be integers. We put $d/e = [b_1, \cdots, b_5]$, where d and e are relatively prime. Assume that $d/e \neq [2,2,2,2,c], (c \geq 2, c \in \mathbb{Z})$. For any integer $k \geq 0$, we have that

$$\left[\frac{ke-2}{d}\right] \leq \begin{cases} \left[\frac{11k-2}{14}\right] & \text{if } k \not\equiv 9 \pmod{14}, \\ 1 + \left[\frac{11k-2}{14}\right] & \text{if } k \equiv 9 \pmod{14}. \end{cases}$$

Since $p_g(X,x) = 0$, by ([7], Theorem 4.4,1) and Claim 6, we have the following:

$$c(2,k) \leq \begin{cases} 2 - 2k + \left[\frac{k-2}{2}\right] + \left[\frac{2k-2}{3}\right] + \left[\frac{11k-2}{14}\right] & \text{if } k \not\equiv 9 \pmod{14}, \\ 3 - 2k + \left[\frac{k-2}{2}\right] + \left[\frac{2k-2}{3}\right] + \left[\frac{11k-2}{14}\right] & \text{if } k \equiv 9 \pmod{14}, \end{cases}$$

Hence, we have that $c(2,4) \leq 0$ and $c(2,k) < 0$ for $k \neq 4$. By Theorem 2.2, $\delta_2(X,x) = h^0(\mathscr{O}_{A_0}(2K_{A_0} - D_2^{(4)})) \leq 1$.
(2) Assume that $(b_{21}, b_{22}) \neq (2,2)$
Since $p_g(X,x) = 0$, by ([7], Theorem 4,4,1), Claim 1 and Claim 2, we have the following:

$$c(2,k) \leq \begin{cases} 2 - 2k + \left[\frac{k-2}{2}\right] + \left[\frac{3k-2}{5}\right] + \left[\frac{5k-2}{6}\right] & \text{if } k \not\equiv 2 \pmod{5} \\ & \text{and } k \not\equiv 5 \pmod{6}, \\ 3 - 2k + \left[\frac{k-2}{2}\right] + \left[\frac{3k-2}{5}\right] + \left[\frac{5k-2}{6}\right] & \text{if } k \equiv 2 \pmod{5} \\ & \text{and } k \not\equiv 5 \pmod{6}, \\ 3 - 2k + \left[\frac{k-2}{2}\right] + \left[\frac{3k-2}{5}\right] + \left[\frac{5k-2}{6}\right] & \text{if } k \not\equiv 2 \pmod{5} \\ & \text{and } k \equiv 5 \pmod{6}, \\ 4 - 2k + \left[\frac{k-2}{2}\right] + \left[\frac{3k-2}{5}\right] + \left[\frac{5k-2}{6}\right] & \text{if } k \equiv 2 \pmod{5} \\ & \text{and } k \equiv 5 \pmod{6}. \end{cases}$$

Hence, we have that $c(2,2) \leq 0$, $c(2,4) \leq 0$ and $c(2,k) < 0$ for $k \neq 2,4$. In a similar way, we have $c(2,2) < 0$. By Theorem 2.2, $\delta_2(X,x) = h^0(\mathscr{O}_{A_0}(2K_{A_0} - D_2^{(4)})) \leq 1$. Hence, by considering continued fractions which satisfy $c(2,4) = 0$, we can easily check (C). □

Remark. As in the proof of Theorem 3.2, we can give a partial proof of Theorem 3.3. Let (X,x) be a rational singularity whose weighted dual graph is of type $_*\widetilde{E_6}$ (resp. $_*\widetilde{E_7}$, $_*\widetilde{E_8}$). Assume that (X,x) satisfies the condition (1) of (A) (resp. (B), (C)). We put $V = 2A_0 + \sum_{i=1}^{3} A_{i1}$ (resp. $3A_0 + 2(A_{21} + A_{31}) + A_{11} + A_{22} + A_{32}$, $5A_0 + 4A_{31} + 3(A_{21} + A_{32}) + 2(A_{11} + A_{33}) + A_{22} + A_{34}$). Then we have $\delta_2(X,x) = h^1(\mathscr{O}_V(-K - A)) = 1$.

References

[1] M. Artin, *On isolated rational singularities of surfaces*, Amer. J. Math. **88**(1966), 129–136.
[2] H. Grauert, *Über Modifikationen und exzeptionelle analytische Mengen*, Math. Ann. **146**(1962), 331–368.

[3] H. B. Laufer, *On rational singularities*, Amer. J. Math. **94**(1972), 597–608.
[4] ———, *On minimally elliptic singularities*, Amer. J. Math. **99**(1977), 1257–1295.
[5] J. Lipman, *Rational singularities, with applications to algebraic surfaces and unique factorization*, Inst. Hautes Études Sci. Publ. Math. **36**(1969), 195–279.
[6] A. Némethi, *Weakly elliptic Gorenstein singularities of surfaces*, Invent. Math. **137**, No.1(1999), 145–167.
[7] S. Ohyanagi, *On two-dimensional normal singularities of type* $_*A_n, _*D_n$ *and* $_*E_n$, Trans. Amer. Math. Soc. **266**(1981), 57–69.
[8] T. Okuma, *The pluri-genera of surface singularities*, Tôhoku Math. J. **50**(1998), 119–132.
[9] ———, *The second pluri-genus of surface singularities*, Compositio Math. **110**(1998), 263–279.
[10] ———, *Plurigenera of surface singularities*, Nova Science Publishers, Inc., 2000.
[11] T. Tomaru, H. Saito and T. Higuchi, *Pluri-genera* δ_m *of normal surface singularites with* C^* *action*, Sci. Rep. Yokohama Nat. Univ. Sect. *I*, **28**(1981), 35–43.
[12] K. Watanabe, *On plurigenera of normal isolated singularities. I*, Math. Ann. **250**(1980), 65–94.

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