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MULTIPLE STRUCTURES ON $\mathbf{P}^1$: RATIONAL ROPES

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Abstract. Here we study the theory of rational ropes (multiple structures on $\mathbf{P}^1$ whose ideal sheaf, F, has square zero) introduced by K. Chandler. F is a vector bundle on $\mathbf{P}^1$ and here we show that several properties of the rope depend on the splitting type of F. We study the moduli space of all rational ropes with F as ideal sheaf.

1. Introduction

K. Chandler introduced the following definition ([2]). Let Y be smooth projective curve and x a positive integer. Let X be an algebraic scheme such that $X_{\mathrm{red}} = Y$ and the ideal sheaf $\boldsymbol{I}_{Y,X}$ of Y in X satisfies $\boldsymbol{I}_{Y,X}{}^2 = 0$. Thus $\boldsymbol{I}_{Y,X}$ is the conormal sheaf of Y in X and it may be seen as an $\boldsymbol{O}_Y$-sheaf. Set $F := \boldsymbol{I}_{Y,X}$ when seen as an $\boldsymbol{O}_Y$-sheaf. Assume that F has no torsion; this is equivalent to require that the one-dimensional scheme X is locally Cohen-Macaulay. Since Y is a smooth curve, F is locally free. It is called the conormal module of X. Set $x := \mathrm{rank}(F)$. The scheme X is called a $(x+1)$-rope over Y or with Y as support. A 2-rope is a ribbon in the sense of [1]. Ribbons were studied in details in [1], [3] and [4]. The aim of this paper is to extend several of their results (with appropriate definitions) to the case of ropes. The main difference is that a t-rope is not Gorenstein if $t \geq 3$. We will mainly be interested in the case $Y = \mathbf{P}^1$. In this case we will call X a rational rope. It is easy to describe all ropes with a fixed vector bundle F over $\mathbf{P}^1$ as conormal module. Several geometric properties of the rope depend only from the splitting type of F. In some cases (e.g. F spanned) the rope X is uniquely determined by F, every vector bundle on X is a direct sum of line bundles and the Brill-Noether theory of vector bundles on X is trivial (see 3.5, 3.6, 4.6 and 4.7). An arbitrary rational rope has a maximal subrope (perhaps reduced to X_{red}) with spanned conormal module (see section 5). In section 5 we define the blowing ups of a rope, the case of a 2-rope being introduced in [1]. As in [1] and [3] such notion seems to be quite important. In section 6 we compute the number of blowing ups needed to split a rational rope in terms of the restricted cotangent sequence of the rope (see Theorem 6.3).

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2. Foundations

We work over an algebraically closed base field $\mathbf{K}$. In this section we collect the easy foundational results on ropes. Let X be a $(x+1)$-rope on the smooth projective curve Y with the rank x vector F on Y as conormal module. By the very definition of conormal module, $\boldsymbol{I}_{Y,X} \cong F$ as coherent $\boldsymbol{O}_Y$-sheaves. Thus we have an exact sequence of $\boldsymbol{O}_X$-modules

$$0 \to F \to \boldsymbol{O}_X \to \boldsymbol{O}_Y \to 0 \tag{1}$$

The exact sequence (1) is an exact sequence of $\boldsymbol{O}_Y$-modules if and only if there is a retraction $X \to Y$ and in this case (1) is a split exact sequence of locally free $\boldsymbol{O}_Y$-sheaves. If this is a case, we will say that X is a split rope. Set $q := \mathrm{p_a}(Y)$. We have $\chi(\boldsymbol{O}_X) = \chi(F) + \chi(\boldsymbol{O}_Y) = \deg(F) + (x+1)(1-q)$. Set $g := 1 - \chi(\boldsymbol{O}_X) = (x+1)q - x - \deg(F)$. We have an exact sequence on Y

$$0 \to F \to \Omega_X|Y \to \Omega_Y \to 0 \tag{2}$$

(the restricted cotangent sequence). Hence one may associate to any $(x+1)$-rope on Y an extension class $e_X \in \mathrm{Ext}^1(Y; \Omega_Y, F) \cong \mathrm{H}^1(Y, F \otimes {\omega_Y}^*)$. Since Y is smooth, the exact sequence (2) locally splits. Thus the structure of $(x+1)$-rope is locally split and one can copy [1], p. 724-725, and the general set-up of [6] and obtain the following result.

Proposition 2.1. *For any rank x vector bundle F on the smooth projective curve Y and every $e \in \mathrm{H}^1(Y, F \otimes {\omega_Y}^*)$ there is a unique $(x+1)$-rope X on Y with F as conormal module and e as associated extension class. Two $(x+1)$-ropes on Y are isomorphic if and only if they have isomorphic conormal modules and proportional extension classes.*

Since ${\boldsymbol{I}_{Y,X}}^2 = 0$, from (1) we obtain the exact sequence

$$0 \to \mathrm{H}^1(Y, F) \to \mathrm{Pic}(F) \to \mathrm{Pic}(Y) \to 0 \tag{3}$$

For every rank r vector bundle L on X the sheaf $\boldsymbol{I}_{Y,X} \otimes L$ is a rank xr vector bundle on Y isomorphic to $F \otimes (L|Y)$. Set $c := \deg(L|Y)$. We have $\deg(F \otimes (L|Y)) = cx + r(\deg(F))$. Thus $\chi(L) = \chi(\boldsymbol{I}_{Y,X} \otimes L) + \chi(L|Y) = (x+1)c + (r+1)\deg(F) + (xr+1)(1-q) = (x+1)(\deg(L|Y)) + 1 - g$.

For every coherent sheaf L on X set $\deg(L) = \chi(L) - \chi(\boldsymbol{O}_X)$.

Let X be any $(x+1)$-rope with a smooth curve Y as support, F as conormal module and $e_X \in \mathrm{Ext}^1(Y; \Omega_Y, F) \cong \mathrm{H}^1(Y, F \otimes {\omega_Y}^*)$ as extension class. Let T be any scheme. The description of all morphisms $f : X \to T$ given in [1], Th. 1.6 and part (1) of Th. 1.8, in the case $x = 1$ works verbatim in the general case and we have the following result.

Proposition 2.2. *Let T be any algebraic scheme. Let X be any $(x+1)$-rope with a smooth curve Y as support, F as conormal module and $e_X \in \mathrm{Ext}^1(Y;\Omega_Y,F)$ as an extension class. Let $\gamma : \Omega_X|Y \to \Omega_Y$ be the surjective map appearing in (2). Fix a morphism $f : Y \to T$. The set of all morphisms $h : X \to T$ extending f is in one-to-one correspondence with the set of all splittings of the exact sequence $df^*(e_X)$, i.e. with the set of all maps of sheaves $u : \Omega_T|Y \to \Omega_Y$ such that $\gamma \circ u = df$.*

Notice that any morphism $h : X \to T$ extending f induces a map $\alpha_f : f^*(\boldsymbol{I}_{f(Y)}/(\boldsymbol{I}_{f(Y)})^2) \to F$. As in part (1) of [1], Th. 1.8, we have the following result.

Proposition 2.3. *The morphism h is a closed immersion if and only if f is a closed immersion and α_f is surjective.*

Remark 2.4. Proposition 2.3 gives a very strong criterion to say when a $(x+1)$-rope X over a smooth curve Y may be embedded in a prescribed $(y+1)$-rope T over Y. For all pairs of integers (x,y) with $x < y$ and all vector bundles (F,G) on Y with $\mathrm{rank}(F) = x$ and $\mathrm{rank}(G) = y$ there is a triple (X,T,j) such that:

(i) X is a $(x+1)$-rope over Y with conormal module F;
(ii) T is a $(y+1)$-rope over Y with conormal module G;
(iii) $j : X \to T$ is a closed immersion such that $j|Y$ is the identity

if and only if F is a quotient of G.

If $Y = \mathbf{P}^1$ we will say that the rope is rational. Set $D{:=}\ \mathbf{P}^1$. Now we will apply Proposition 2.2 to study the elliptic ropes over $\mathbf{P}^1$ and the finite maps with elliptic ropes as target.

Definition 2.5. Let C be a $(z+1)$-rope over D. We will say that C is an elliptic rope if it has negative type (i.e. the splitting type $a_1 \geq \cdots \geq a_z$ of the conormal module of C has $a_1 < 0$) and $\mathrm{p_a}(C) = 1$. By the genus formula for rational $(z+1)$-ropes, these conditions are equivalent to $a_z = -2$ and $a_i = -1$ for $1 \leq i < z$.

Remark 2.6. By Remark 3.5 below every elliptic $(z+1)$-rope C over D is a split rope. Hence for any integer $z \geq 1$ there is a unique elliptic $(z+1)$-rope over D. By its very definition the conormal module of an elliptic $(z+1)$-rope is semistable if and only if $z = 1$. Set $G{:=}\ \boldsymbol{O}_D(-2) \oplus \boldsymbol{O}_D(-1)^{\oplus(z-1)}$. The restricted cotangent sequence of C splits. Take any integer $t \geq 2$ and any degree t morphism $f : D \to D$. We have $f^*(G) \cong \boldsymbol{O}_D(-2t) \oplus \boldsymbol{O}_D(-t)^{\oplus(z-1)}$, $f^*(\Omega_D) \cong \boldsymbol{O}_D(-2t)$ and $f^*(\Omega_C|D) \cong \boldsymbol{O}_D(-2t)^{\oplus 2} \oplus \boldsymbol{O}_D(-t)^{\oplus(z-1)}$. We have a map $df : f^*(\Omega_C|D) \cong \boldsymbol{O}_D(-2t) \to \Omega_D \cong \boldsymbol{O}_D(-2)$. Hence Proposition 2.2 gives the following result.

Proposition 2.7. *Fix integers x, z with $x > z \geq 1$ and let X be a $(x+1)$-rope over D with conormal module F. Let $f : D \to D$ be a degree t morphism and C an elliptic $(z+1)$-rope. Assume $\mathrm{h}^0(D, F(2t)) \neq 0$. Then there is a morphism $u : X \to C$ lifting f and with $u(X)$ not contained in D.*

3. Rational ropes and the splitting type of F

Let X be a rational $(x+1)$-rope with conormal module $F := \boldsymbol{O}_D(a_1) \oplus \cdots \oplus \boldsymbol{O}_D(a_x)$ with $a_1 \geq \cdots \geq a_x$. We have $\mathrm{p_a}(X) = -\sum_{1 \leq i \leq x} a_i - x$. We will say that X has negative type if $a_1 < 0$. If X has negative type we will call the integer $-a_1$ the negative level of X. The deformation theory of a split rope is equivalent to the deformation theory of the vector bundle F on D. We will say that a rational rope is rigid if its conormal module F is rigid as a vector bundle on D, i.e. if $a_x \geq a_1 - 1$. We will say that a rational rope is semistable if its conormal module is a semistable vector bundle on D, i.e. if $a_x = a_1$. Since the multiplicative structure of F is trivial, every $\boldsymbol{O}_D$-subsheaf J of F is an $\boldsymbol{O}_X$-ideal subsheaf of $\boldsymbol{O}_X$ and hence it defines a closed subscheme $\mathrm{Spec}(\boldsymbol{O}_X/J)$ of X with D as support. In particular for every integer i with $1 \leq i \leq x$ the vector bundle $F_i := \boldsymbol{O}_D(a_1) \oplus \cdots \oplus \boldsymbol{O}_D(a_i)$ is a subbundle of F and any inclusion of F_i into F defines a closed subscheme $\mathrm{Spec}(\boldsymbol{O}_X/F_i)$ of X. However, unless $a_i > a_j$ for all pairs (i,j) with $i < j$, these subschemes are not uniquely determined by F. Call y the number of different integers in the set $\{a_1, \ldots, a_x\}$, say $\{a_1, \ldots, a_x\} = \{b_1, \ldots, b_y\}$ with $b_i > b_j$ if $i < j$ and with b_i appearing r_i times in the weakly decreasing sequence $a_1 \geq a_2 \geq \cdots \geq a_x$. The vector bundles $F(i) := \bigoplus_{1 \leq j \leq i} \boldsymbol{O}_D(b_j)^{\oplus r_j}$ are uniquely determined by F; they give the Harder-Narasimhan filtration of F. Set $X(i) := \mathrm{Spec}(\boldsymbol{O}_X/F(i))$.

Remark 3.1. Let X be a rational $(x+1)$-rope of negative type. Call $c := -a_1$ the negative level of X. For every $L \in \mathrm{Pic}(X)$ with $\deg(L) < (x+1)c$ the restriction map $\mathrm{H}^0(X, L) \to \mathrm{H}^0(D, L|D)$ is injective.

Remark 3.2. Let X be a rational $(x+1)$-rope of negative type. Call c the negative level of X. Fix $L, R \in \mathrm{Pic}(X)$ with $\deg(L) < (x+1)c$ and $\deg(R) < (x+1)c$. Since D is reduced and connected, the pairing $\mathrm{H}^0(D, L|D) \otimes \mathrm{H}^0(D, M|D) \to \mathrm{H}^0(D, L \otimes M|D)$ is non-degenerate in both variables. Since the restriction maps $\mathrm{H}^0(X, L) \to \mathrm{H}^0(D, L|D)$ and $\mathrm{H}^0(X, M) \to \mathrm{H}^0(D, M|D)$ are injective (Remark 3.1) the pairing $\alpha : \mathrm{H}^0(X, L) \otimes \mathrm{H}^0(X, M) \to \mathrm{H}^0(X, L \otimes M)$ is non-degenerate in both variables. Hence by a classical lemma of Hopf we have $\dim(\mathrm{Im}(\alpha)) \geq \mathrm{h}^0(X, L) + \mathrm{h}^0(X, M) - 1$ and in particular $\mathrm{h}^0(X, L \otimes M) \geq \mathrm{h}^0(X, L) + \mathrm{h}^0(X, M) - 1$.

From Remark 3.1 we immediately obtain the following result.

Proposition 3.3 (Clifford's inequality)**.** *Let X be a rational $(x+1)$-rope of negative type. Let $0 > a_1 \geq \cdots \geq a_x$ be the splitting type of the conormal module of X. For every $L \in \mathrm{Pic}(X)$ with $0 \leq \deg(L) \leq (x+1)(-a_1-1)$ we have $\mathrm{h}^0(X, L) - 1 \leq \deg(L)/(x+1)$.*

Remark 3.4. Let X be a rational $(x+1)$-rope of negative type. Using Remark 3.1 we see that X splits if and only if there is $L \in \mathrm{Pic}(X)$ such that $\deg(L) \leq x+1$ and $\mathrm{h}^0(X, L) \geq 2$.

Remark 3.5. Let X be a rational rope with conormal module F. Assume that F has splitting type $a_1 \geq \cdots \geq a_x$ with $a_x \geq -1$. By (3) we have $\mathrm{Pic}(X) \cong \mathbf{Z}$ and every line bundle L on X is uniquely determined by its restriction to X_{red}, i.e. by the unique integer d such that $\deg(L) = (x+1)d$.

Remark 3.6. Let X be a rational $(x+1)$-rope whose conormal module has splitting type $a_1 \geq \cdots \geq a_x$ with $a_x \geq 0$. By Remark 3.5 we have $\mathrm{Pic}(X) \cong \mathbf{Z}$. Call $L(t)$ the unique line bundle on X with $\deg(L(t)) = (x+1)t$. The sequence of integers $\mathrm{h}^0(X, L(-t))$, $t \geq 0$, uniquely determines all the integers $a_1, \dots, a_x$; if $a_x = 0$ to obtain this observation we use either that every regular function on D is constant and hence that the restriction map $\mathrm{H}^0(X, \boldsymbol{O}_X) \to \mathrm{H}^0(D, \boldsymbol{O}_D)$ is surjective or that X is a split rope.

Let X be a rational $(x+1)$-ropes and F its conormal module. If X is not of negative type, then $\mathrm{h}^0(X, \boldsymbol{O}_X) \geq 2$ by the exact sequence (1). The finite dimensional $\mathbf{K}$-vector space $\mathrm{H}^0(X, \boldsymbol{O}_X)$ has a $\mathbf{K}$-algebra structure for which it is a local ring whose maximal ideal $\mathbf{m}$ has $\mathbf{m}^2 = 0$. As a $\mathbf{K}$-vector space we have $\mathbf{m} \cong \mathrm{H}^0(D, F)$. For any coherent sheaf E on X the $\mathbf{K}$-vector spaces $\mathrm{H}^0(X, E)$ and $\mathrm{H}^1(X, E)$ are $\mathrm{H}^0(X, \boldsymbol{O}_X)$-modules.

4. Rational ropes, their moduli spaces and decomposition of F

In this section we will study the moduli space of all rational $(x+1)$-ropes with fixed arithmetic genus (i.e. with conormal module of fixed degree) or with conormal module of fixed splitting type. Let $F := \boldsymbol{O}_D(a_1) \oplus \cdots \oplus \boldsymbol{O}_D(a_x)$ be a rank x vector bundle on D with $a_1 \geq \cdots \geq a_x$. Let $F(\geq t)$ (resp. $F(\leq t)$, resp. $F(>t)$, resp. $F(<t)$) be the direct sum of all factors $\boldsymbol{O}_D(a_i)$ of F with $a_i \geq t$ (resp. $\leq t$, resp. $> t$, resp. $< t$). Since $(\boldsymbol{I}_{D,X})^2 = 0$, for any $\boldsymbol{O}_D$-subbundle G of F there is a uniquely determined rational rope with F/G as conormal module. If we take $F(\geq t)$ (resp. $F(>t)$) as G we will call $X(<t)$ (resp. $X(\leq t)$) the corresponding rope.

Remark 4.1. Let F be a rank x vector bundle on D. By Remark 2.2 the set $\mathbf{S}(F)$ of all non-split rational ropes with F as conormal module are parametrized one-to-one by $\mathbf{P}(\mathrm{H}^1(D, F(2)))$. Hence if $F = F(\geq -1)$, then every $(x+1)$-rope with F as conormal module is split.

Fix a $(x+1)$-rope X over D with conormal module F and let $e_X \in \mathrm{H}^1(D, F(2))$ the corresponding extension class, uniquely determined up to a multiplicative non-zero constant (see Proposition 2.1). There is an exact sequence of $\boldsymbol{O}_X$-modules

$$0 \to F(\geq 0) \to \boldsymbol{O}_X \to \boldsymbol{O}_{X(<0)} \to 0 \tag{4}$$

Notice that $\mathrm{H}^1(D, F(2)) \cong \mathrm{H}^1(D, F(<0)(2))$. This isomorphism maps the extension class e_X onto an extension class $e_{X(<0)}$ of $X(<0)$.

Lemma 4.2. *The inclusion of $X(<0)$ into X has a retraction. The exact sequence* (4) *is an exact sequence of $\boldsymbol{O}_{X(<0)}$-modules and it splits as an exact sequence of $\boldsymbol{O}_{X(<0)}$-modules.*

Proof. The lemma follows from the construction of a rope from its extension class considered in Proposition 2.1. □

Lemma 4.3. *The restriction map $\rho : \mathrm{Pic}(X) \to \mathrm{Pic}(X(<0))$ is an isomorphism.*

Proof. The injectivity of ρ follows from Remark 3.5. The surjectivity of ρ follows from the existence of a retraction of X onto $X(<0)$ (Lemma 4.2). □

Remark 4.4. Fix $L \in \mathrm{Pic}(X)$ and set $t := \deg(L|D)$. Thus $\deg(L) = (x+1)t$. First assume $t < 0$. Then by the exact sequence (4) we obtain $\mathrm{h}^0(X(<0), L|X(<0)) = 0$ and $\mathrm{h}^0(X, L) = \mathrm{h}^0(D, F(t)) = \mathrm{h}^0(X, F(\geq 0)(t))$. If $t = 0$ we have $\mathrm{h}^0(D, F(\geq 0)) \leq \mathrm{h}^0(X, L) \leq \mathrm{h}^0(X, F(\geq 0)) + 1$. Now assume $t > 0$. We have $\mathrm{h}^1(D, F(\geq 0)(t)) = 0$. Hence from (4) we obtain $\mathrm{h}^0(X, L) = \mathrm{h}^0(X(<0), L|X(<0)) + \mathrm{h}^0(D, F(t))$. Thus the Brill-Noether theory of X is essentially determined by the Brill-Noether theory of $X(<0)$.

The proof of Lemma 4.3 gives verbatim the following result.

Lemma 4.5. *The restriction map from the set of all isomorphism classes of vector bundles on X to the set of all isomorphism classes of vector bundles on $X(<0)$ is an isomorphism.*

Corollary 4.6. *Assume $X(<0) = D$, i.e. assume $a_x \geq 0$. Then every vector bundle, E, on X is a direct sum of line bundles and E is uniquely determined by $E|D$: if $E|D \cong \bigoplus_{1 \leq i \leq r} \boldsymbol{O}_D(b_i)$, then $E \cong \bigoplus_{1 \leq i \leq r} \boldsymbol{O}_X(b_i)$, where $\boldsymbol{O}_X(c)$, $c \in \mathbf{Z}$, is the unique line bundle on X with $\boldsymbol{O}_X(c)|D \cong \boldsymbol{O}_D(c)$, i.e. the unique line bundle on X with degree $(x+1)c$. We have $\mathrm{h}^0(X, \boldsymbol{O}_X(c)) = \mathrm{h}^0(D, F(c)) + c + 1$ and $\mathrm{h}^1(X, \boldsymbol{O}_X(c)) = 0$ for every $c \geq 0$. We have $\mathrm{h}^0(X, \boldsymbol{O}_X(c)) = 0$ if and only if $c < -a_1$ and $\mathrm{h}^1(X, \boldsymbol{O}_X(c)) = 0$ if and only if $c \geq -1$.*

Remark 4.7. Corollary 4.6 gives a complete description of the Brill-Noether theory of vector bundles on any rational rope of non-negative type. It is remarkable that every vector bundle on a rational rope of non-negative type is a direct sum of line bundles. We do not know any other locally Cohen-Macaulay positive-dimensional projective scheme Z with this property and Z_{red} irreducible. Some (but not all) the reduced and connected projective curves, T, with $\mathrm{p_a}(T) = 0$ have this property.

Remark 4.8. Let X (resp. Z) be a split rational $(x+1)$-rope with conormal module F (resp. G). Z is the flat limit of a flat family of ropes isomorphic to X if and only if the vector bundle G on D is a specialization of G.

Proposition 4.9. *Fix rank x vector bundles F and G on D with G specialization of F. Assume $\mathrm{h}^1(D, F(2)) = \mathrm{h}^1(D, G(2))$. Then there is an irreducible family of $(x+1)$-ropes, say parametrized by an irreducucible variety T, whose general member has conormal module isomorphic to F and such that every non-split rational $(x+1)$-ribbon with G as conormal module occurs for at least one value of T.*

Proof. The result follows from the definition of specializations of vector bundles on D, Remark 4.2 and the theory of the relative Ext-functor ([5]). □

5. Blowing ups of a rope

Now we extend to the case of ropes the definition of blowing up introduced in [1] for ribbons. For simplicity we consider only rational ropes but the same definitions are obtained taking as D any smooth projective curve. Fix $P \in D$, a vector bundle E on D and a surjection $u : E \to \boldsymbol{O}_P$. The surjection $u : E \to \boldsymbol{O}_P$ is uniquely determined by its restriction $u|\{P\} : E|\{P\} \to \mathbf{K}$, where $E|\{P\}$ is the fiber of E over P. Conversely, any surjective linear map $E|\{P\} \to \mathbf{K}$ induces a surjection $E \to \boldsymbol{O}_P$. Set $G := \mathrm{Ker}(u)$. Since P is a Cartier divisor of D, G is a vector bundle on D. We will say that G is obtained from E making a negative elementary transformation supported by P. We have $\mathrm{rank}(G) = \mathrm{rank}(E)$ and $\deg(G) = \deg(E) - 1$. For every $\lambda \in \mathbf{K}\backslash\{0\}$ we have $\mathrm{Ker}(\lambda u) \cong \mathrm{Ker}(u)$. For every subsheaf A of E with $\mathrm{rank}(A) = \mathrm{rank}(E)$ and $\deg(A) = \deg(E) - 1$ there is a unique $Q \in D$ such that $E/A \cong \boldsymbol{O}_Q$; A is obtained from E making a negative elementary transformation supported by Q. We will say that G^* is obtained from E^* making a positive elementary transformation supported by P; more precisely, G^* is obtained from E^* making the positive elementary transformation dual to the negative elementary transformation associated to u. E^* is a subsheaf of G^*, $\mathrm{rank}(G^*) = \mathrm{rank}(E^*)$, $\deg(G^*) = \deg(E^*) + 1$

and $G^*/E^* \cong \boldsymbol{O}_P$. Conversely, for every vector bundle H and any inclusion $j : E^* \to H$ with $H/E^* \cong \boldsymbol{O}_P$ there is a unique (up to a non-zero multiplicative constant) surjection $u : E \to \boldsymbol{O}_P$ such that H is isomorphic to the positive elementary transformation associated to u. Let X be a rational $(x+1)$-rope with conormal module F. Take any surjection $u : F^* \to \boldsymbol{O}_P$. We will prove the existence of a unique rational $(x+1)$-rope $X(u)$ with $\mathrm{Ker}(u)^*$ as conormal module and equipped with a proper morphism $\phi_u : X(u) \to X$. For every $\lambda \in \mathbf{K}\backslash\{0\}$ we will obtain $X(\lambda u) \cong X(u)$ and, modulo this isomorphism, $\phi_u = \phi_{\lambda u}$. However, in general $X(u)$ will depend on the inclusion of $\mathrm{Ker}(u)$ in F, not just on the isomorphism class of the vector bundle $\mathrm{Ker}(u)$. The surjection u induces an inclusion $j : F \to H$ with $H/j(F) \cong \boldsymbol{O}_P$. The inclusion j induce a map $\alpha : \mathrm{H}^1(D, F(2)) \to \mathrm{H}^1(D, H(2))$. Let $e_X \in \mathrm{H}^1(D, F(2))$ be the extension class (unique up to a non-zero multiplicative constant) associated to X. Let $X(u)$ be the rational $(x+1)$-rope with H as conormal module and $\alpha(e_X)$ as extension class. The rope $X(u)$ is called a blowing up of X or the blowing up of X at one point or the blowing up of X associated to u. Notice that if X is a split rope, then $X(u)$ is a split rope. We have $\mathrm{p_a}(X(u)) = \mathrm{p_a}(X) - 1$. For every $L \in \mathrm{Pic}(X(u))$ the coherent sheaf $\phi_{u*}(L)$ is a rank 1 torsion free sheaf on X whose restriction to $X\backslash\{P\}$ is locally free. Since ϕ_u is finite, we have $\mathrm{h}^0(X, \phi_{u*}(L)) = \mathrm{h}^0(X(u), L)$ and $\mathrm{h}^1(X, \phi_{u*}(L)) = \mathrm{h}^1(X, L)$. Since $\mathrm{p_a}(X(u)) = \mathrm{p_a}(X) - 1$, we obtain $\deg(\phi_{u*}(L)) = \deg(L) + 1$. We may iterate this construction and say when a vector bundle on D is obtained from the vector bundle E making a sequence of t negative elementary transformations, t any positive integer, and when a rational $(x+1)$-rope is obtained from X making a sequence of t blowing ups. Let $\phi : X' \to X$ be the composition of t blowing ups. For any $L \in \mathrm{Pic}(X')$ the coherent sheaf $\phi_*(L)$ is a rank 1 torsion free sheaf on X which is is locally free outside P. Since ϕ is finite, we have $\mathrm{h}^0(X, \phi_*(L)) = \mathrm{h}^0(X(u), L)$ and $\mathrm{h}^1(X, \phi_*(L)) = \mathrm{h}^1(X, L)$. Since $\mathrm{p_a}(X') = \mathrm{p_a}(X) - t$, we obtain $\deg(\phi_*(L)) = \deg(L) + t$. For any fixed vector bundle F on D and any $P \in D$ the set of all isomorphism classes of vector bundles on D obtained from F making a negative elementary transformation supported by P is parametrized by a non-empty open subset of a vector space and in particular it is parametrized by an irreducible variety. Since D is irreducible, the set of all isomorphism classes of vector bundles obtained from F making a negative elementary transformation supported by a point of D is parametrized by an irreducible variety. The same is true for positive elementary transformations. Hence for any rope X we are allowed to say that a rope Y is obtained from X making a sequence of t generic blowing ups. For any rope X let $\gamma(X)$ be the minimal integer t such that there is a split rope obtained from X making a sequence of t blowing ups.

Thus $\gamma(X) = 0$ if and only if X is a split rope. The next lemma shows that $\gamma(X) < +\infty$ for every rope X.

Lemma 5.1. *Let X be a rational $(x+1)$-rope with conormal module F. Then any rope obtained from X making a sequence of $\mathrm{h}^1(D, F(2))$ generic blowing ups is a split rope. In particular $\gamma(X) \le \mathrm{h}^1(D, F(2))$.*

Proof. Let G be a vector bundle on D and H a vector bundle obtained from G making a general positive elementary transformation. We have $\mathrm{h}^1(X, H) = \max\{0, \mathrm{h}^1(X, G) - 1\}$. Iterating $\mathrm{h}^1(D, F(2))$ times this observation, we conclude. □

In 5.2, 5.3 and 5.4 we will see that the Brill-Noether theory of ropes with low $\gamma(X)$ is quite restricted.

Remark 5.2. Let X be a rational $(x+1)$-rope. Take a split rational rope Z such that there is a sequence $\phi : Z \to X$ of $\gamma(X)$ blowing ups. Let $L \in \mathrm{Pic}(Z)$ the line bundle inducing the splitting of Z. Thus $\deg(L) = x+1$ and $\mathrm{h}^0(Z, L) \ge 2$. If Z is of negative type we have $\mathrm{h}^0(Z, L) = 2$. The generalized line bundle $\phi_*(L)$ on X has degree $x + 1 + \gamma(X)$ and $\mathrm{h}^0(X, \phi_*(L)) = \mathrm{h}^0(Z, L) \ge 2$.

Proposition 5.3. *Let X be a rational $(x+1)$-rope and $\phi : Z \to X$ the composition of z blowing ups with Z split rope. Assume X not split. Let G be the conormal module of Z and call $b_1 \ge \cdots \ge b_x$ the splitting type of G. Assume $b_1 \le -2$ and let t be a positive integer such that $0 < t < -b_1$. There is no spanned line bundle L on X such that $0 < \deg(L) \le t(x+1)$.*

Proof. Assume the existence of $L \in \mathrm{Pic}(X)$ such that $0 < \deg(L) \le t(x+1)$ and set $M := \phi^*(L)$. We have $\deg(L) = \deg(M) = (x+1)y$ with $y = \deg(L|D)$ and $1 \le y \le t$. Let $\beta : X \to \mathbf{P}(\mathrm{H}^0(X, L)^*)$ be the morphism induced by M. Since Z is a split rope and $y \le t < -b_1$, we have $\mathrm{h}^0(Z, M) = y+1$. Since L is spanned, M is spanned by the image, W, of $\phi^*(\mathrm{H}^0(X, M))$ into $\mathrm{H}^0(Z, M)$. Fix a retraction $u : Z \to D$, Since $\mathrm{H}^1(Z, G) \ne 0$, u is not unique (use Remark 3.5), but any two retractions of Z differ by an automorphism of Z whose restriction to D is the identity. The morphism $\alpha : Z \to \mathbf{P}(\mathrm{H}^0(Z, M)^*)$ induced by M has as image a rational normal curve of $\mathbf{P}(\mathrm{H}^0(Z, M)^*)$ and it is the composition of a retraction of Z and a linearly normal embedding of D into $\mathbf{P}(\mathrm{H}^0(Z, M)^*)$. The morphism γ induced by W must be obtained composing a retraction of Y with an embedding of D into $\mathbf{P}^m$, $m := \dim(W) - 1$. Since X is not split, no such morphism γ may factor through ϕ and induce a morphism of X, contradicting the definition of W. □

For any rational rope X there is a split rational rope Z and a morphism $\phi : Z \to X$ with ϕ composition of $\gamma(X)$ blowing ups. By Lemma 5.3 if $\gamma(X)$

is very low the conormal module of any such Z gives strong informations on the Brill-Noether theory of Z. As an immediate corollary of 5.3 we obtain the following result.

Corollary 5.4. *Let X be a rational non-split $(x+1)$-rope whose conormal module F has spitting type $a_1 \geq \cdots \geq a_x$ with $a_1 \leq -2+\gamma(X)$. Then there is no $L \in \mathrm{Pic}(X)$ with L spanned and $0 < \deg(L) \leq (x+1)(-a_1-2-\gamma(X))$.*

Proof. Let $\phi : Z \to X$ be a composition of $\gamma(X)$ blowing ups with Z split rope. Let G be the conormal module of Z and $b_1 \geq \cdots \geq b_x$ be its splitting type. Since G is obtained from F making $\gamma(X)$ positive elementary transformations, we have $b_1 \leq a_1 + \gamma(X)$. Hence we conclude by 5.3. □

6. Restricted cotangent sequence and splittings

In this section we study the restricted cotangent sequence of a $(x+1)$-rope over D with negative conormal module. Let X be a $(x+1)$-rope over D and F its conormal module. Set $G := \Omega_X|D$. Let $a_1 \geq \cdots \geq a_x$ (resp. $w_1 \geq \cdots \geq w_{x+1}$) be the splitting type of F (resp. G). Consider the restricted cotangent sequence of X:

$$0 \to F \to G \to \boldsymbol{O}_D(-2) \to 0 \tag{5}$$

Remark 6.1. Since every element of $\mathrm{H}^1(D, F(2))$ is an extension class for a rational rope with F as conormal module, for any F and any G fitting in an exact sequence (5) there is a rope X with F as conormal module, G as restricted cotangent bundle and such that (5) is the restricted cotangent sequence. The set of all such pairs (F, G) is completely described in [8] and [7].

Remark 6.2. Assume $a_1 \leq -3$. The exact sequence (5) splits if and only if $w_1 = -2$. Since the extension class of (5) gives the isomorphism class of X, the rope X is a split rope if and only if $w_1 = -2$.

Theorem 6.3. *Assume $a_1 \leq -3$. Then $\gamma(X) = -w_1 - 2$.*

Proof. Since $a_1 \leq -3$, we have $w_1 \leq -2$. If $w_1 = -2$, the result is Remark 6.2. Assume $w_1 < -2$. Take any vector bundle G' obtained from G making a positive elementary transformation. Then either this elementary transformation acts on the subbundle F of G or not, the latter case being the general one. If this elementary transformation acts on F, then it produces a vector bundle F' obtained from F making a positive elementary transformation and fitting in an exact sequence

$$0 \to F' \to G' \to \boldsymbol{O}_D(-2) \to 0 \tag{6}$$

Every sequence of positive elementary transformations gives a sequence of blowing ups of X. It is obvious the existence of a sequence of $-w_1 - 2$

positive elementary stransformations of G such that the vector bundle, H, obtained in this way has splitting type $c_1 \geq \cdots \geq c_{x+1}$ with $c_1 = -2$: at each step we increase by one the higher integer of the splitting type of the corresponding bundle. Furthermore, $-w_1 - 2$ is the minimal length of any such sequence of positive elementary transformations. At the first step we need to prove that as our first positive elementary transformation of G we may choose a positive elementary transformation which induces a positive elementary transformation of F. Since $\mathrm{rank}(G) = \mathrm{rank}(F) + 1$, this is obvious if $w_2 = w_1$. Thus we may assume $w_2 < w_1$. This implies that there is a unique line subbundle of G with degree w_1 and that $\boldsymbol{O}_D(w_1)$ is the first step of the Harder-Narasimhan filtration of G. For any $P \in D$ and any vector bundle H on D, let $H|\{P\}$ be the fiber of H over P; thus $H|\{P\}$ is a $\mathbf{K}$-vector space of dimension $\mathrm{rank}(H)$. Fix $P \in D$. There is a positive elementary transformation of G supported by P which increases the value of w_1 and which induces a positive elementary transformation of F supported by P if and only if $\boldsymbol{O}_D(w_1)|\{P\}$ is contained in the hyperplane $F|\{P\}$ of $G|\{P\}$. The inclusions of F in G and of $\boldsymbol{O}_D(w_1)$ in G induces a map $u : F \oplus \boldsymbol{O}_D(w_1) \to G$. Since to prove the existence of the positive elementary transformation we are looking for we may assume that F does not contain $\boldsymbol{O}_D(w_1)$, the map u is an injective map of sheaves. We have $\mathrm{rank}(F \oplus \boldsymbol{O}_D(w_1)) = \mathrm{rank}(G)$. Since $w_1 < -2$, u cannot be an isomorphism. Thus $\mathrm{Coker}(u)$ is a non-zero skyscraper sheaf. For every $P \in \mathrm{Supp}(\mathrm{Coker}(u))$ we may find a positive elementary transformation of G inducing a positive elementary transformation of F (i.e. inducing an exact sequence (6)) and transforming the subbundle $\boldsymbol{O}_D(w_1)$ of G into the subbundle $\boldsymbol{O}_D(w_1 + 1)$ of G'. Since G' has splitting type $w_1 + 1 > w_2 \geq \cdots \geq w_{x+1}$, we may iterate the proof taking the pair (F', G') instead of the pair (F, G). □

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