MULTIPLE STRUCTURES ON $\mathbf{P}^1$: RATIONAL ROPES

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Abstract. Here we study the theory of rational ropes (multiple structures on $\mathbf{P}^1$ whose ideal sheaf, $F$, has square zero) introduced by K. Chandler. $F$ is a vector bundle on $\mathbf{P}^1$ and here we show that several properties of the rope depend on the splitting type of $F$. We study the moduli space of all rational ropes with $F$ as ideal sheaf.

1. Introduction

K. Chandler introduced the following definition ([2]). Let $Y$ be a smooth projective curve and $x$ a positive integer. Let $X$ be an algebraic scheme such that $X_{\text{red}} = Y$ and the ideal sheaf $I_{Y,X}$ of $Y$ in $X$ satisfies $I_{Y,X}^2 = 0$. Thus $I_{Y,X}$ is the conormal sheaf of $Y$ in $X$ and it may be seen as an $O_Y$-sheaf. Set $F := I_{Y,X}$ when seen as an $O_Y$-sheaf. Assume that $F$ has no torsion; this is equivalent to require that the one-dimensional scheme $X$ is locally Cohen-Macaulay. Since $Y$ is a smooth curve, $F$ is locally free. It is called the conormal module of $X$. Set $x := \text{rank}(F)$. The scheme $X$ is called a $(x+1)$-rope over $Y$ or with $Y$ as support. A 2-rope is a ribbon in the sense of [1]. Ribbons were studied in details in [1], [3] and [4]. The aim of this paper is to extend several of their results (with appropriate definitions) to the case of ropes. The main difference is that a $t$-rope is not Gorenstein if $t \geq 3$. We will mainly be interested in the case $Y = \mathbf{P}^1$. In this case we will call $X$ a rational rope. It is easy to describe all ropes with a fixed vector bundle $F$ over $\mathbf{P}^1$ as conormal module. Several geometric properties of the rope depend only from the splitting type of $F$. In some cases (e.g. $F$ spanned) the rope $X$ is uniquely determined by $F$, every vector bundle on $X$ is a direct sum of line bundles and the Brill-Noether theory of vector bundles on $X$ is trivial (see 3.5, 3.6, 4.6 and 4.7). An arbitrary rational rope has a maximal subrope (perhaps reduced to $X_{\text{red}}$) with spanned conormal module (see section 5). In section 5 we define the blowing ups of a rope, the case of a 2-rope being introduced in [1]. As in [1] and [3] such notion seems to be quite important. In section 6 we compute the number of blowing ups needed to split a rational rope in terms of the restricted cotangent sequence of the rope (see Theorem 6.3).

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2. Foundations

We work over an algebraically closed base field $K$. In this section we collect the easy foundational results on ropes. Let $X$ be a $(x+1)$-rope on the smooth projective curve $Y$ with the rank $x$ vector $F$ on $Y$ as conormal module. By the very definition of conormal module, $I_{Y,X} \cong F$ as coherent $O_Y$-sheaves. Thus we have an exact sequence of $O_X$-modules

$$(1) \quad 0 \rightarrow F \rightarrow O_X \rightarrow O_Y \rightarrow 0$$

The exact sequence (1) is an exact sequence of $O_Y$-modules if and only if there is a retraction $X \rightarrow Y$ and in this case (1) is a split exact sequence of locally free $O_Y$-sheaves. If this is a case, we will say that $X$ is a split rope.

Set $q := p_a(Y)$. We have $\chi(O_X) = \chi(F) + \chi(O_Y) = \deg(F) + (x+1)(1-q)$. Set $g := 1 - \chi(O_X) = (x+1)q - x - \deg(F)$. We have an exact sequence on $Y$

$$(2) \quad 0 \rightarrow F \rightarrow \Omega_X|Y \rightarrow \Omega_Y \rightarrow 0$$

(the restricted cotangent sequence). Hence one may associate to any $(x+1)$-rope on $Y$ an extension class $e_X \in \text{Ext}^1(Y; \Omega_Y, F) \cong H^1(Y, F \otimes \omega_Y^*)$. Since $Y$ is smooth, the exact sequence (2) locally splits. Thus the structure of $(x+1)$-rope is locally split and one can copy [1], p. 724-725, and the general set-up of [6] and obtain the following result.

**Proposition 2.1.** For any rank $x$ vector bundle $F$ on the smooth projective curve $Y$ and every $e \in H^1(Y, F \otimes \omega_Y^*)$ there is a unique $(x+1)$-rope $X$ on $Y$ with $F$ as conormal module and $e$ as associated extension class. Two $(x+1)$-ropes on $Y$ are isomorphic if and only if they have isomorphic conormal modules and proportional extension classes.

Since $I_{Y,X}^2 = 0$, from (1) we obtain the exact sequence

$$(3) \quad 0 \rightarrow H^1(Y, F) \rightarrow \text{Pic}(F) \rightarrow \text{Pic}(Y) \rightarrow 0$$

For every rank $r$ vector bundle $L$ on $X$ the sheaf $I_{Y,X} \otimes L$ is a rank $xr$ vector bundle on $Y$ isomorphic to $F \otimes (L|Y)$. Set $c := \deg(L|Y)$. We have $\deg(F \otimes (L|Y)) = cx + r(\deg(F))$. Thus $\chi(L) = \chi(I_{Y,X} \otimes L) + \chi(L|Y) = (x+1)c + (r+1)\deg(F) + (xr+1)(1-q) = (x+1)(\deg(L|Y)) + 1 - g$.

For every coherent sheaf $L$ on $X$ set $\deg(L) = \chi(L) - \chi(O_X)$.

Let $X$ be any $(x+1)$-rope with a smooth curve $Y$ as support, $F$ as conormal module and $e_X \in \text{Ext}^1(Y; \Omega_Y, F) \cong H^1(Y, F \otimes \omega_Y^*)$ as extension class. Let $T$ be any scheme. The description of all morphisms $f : X \rightarrow T$ given in [1], Th. 1.6 and part (1) of Th. 1.8, in the case $x = 1$ works verbatim in the general case and we have the following result.
Proposition 2.2. Let $T$ be any algebraic scheme. Let $X$ be any $(x+1)$-rope with a smooth curve $Y$ as support, $F$ as conormal module and $e_X \in \text{Ext}^1(Y; \Omega_Y, F)$ as an extension class. Let $\gamma : \Omega_X|Y \to \Omega_Y$ be the surjective map appearing in (2). Fix a morphism $f : Y \to T$. The set of all morphisms $h : X \to T$ extending $f$ is in one-to-one correspondence with the set of all splittings of the exact sequence $df^*(e_X)$, i.e. with the set of all maps of sheaves $u : \Omega_T|Y \to \Omega_Y$ such that $\gamma \circ u = df$.

Notice that any morphism $h : X \to T$ extending $f$ induces a map $\alpha_f : f^*(I_{f(Y)}/(I_{f(Y)})^2) \to F$. As in part (1) of [1], Th. 1.8, we have the following result.

Proposition 2.3. The morphism $h$ is a closed immersion if and only if $f$ is a closed immersion and $\alpha_f$ is surjective.

Remark 2.4. Proposition 2.3 gives a very strong criterion to say when a $(x+1)$-rope $X$ over a smooth curve $Y$ be embedded in a prescribed $(y+1)$-rope $T$ over $Y$. For all pairs of integers $(x, y)$ with $x < y$ and all vector bundles $(F, G)$ on $Y$ with $\text{rank}(F) = x$ and $\text{rank}(G) = y$ there is a triple $(X, T, j)$ such that:

(i) $X$ is a $(x+1)$-rope over $Y$ with conormal module $F$;
(ii) $T$ is a $(y+1)$-rope over $Y$ with conormal module $G$;
(iii) $j : X \to T$ is a closed immersion such that $j|Y$ is the identity if and only if $F$ is a quotient of $G$.

If $Y = \mathbb{P}^1$ we will say that the rope is rational. Set $D := \mathbb{P}^1$. Now we will apply Proposition 2.2 to study the elliptic ropes over $\mathbb{P}^1$ and the finite maps with elliptic ropes as target.

Definition 2.5. Let $C$ be a $(z+1)$-rope over $D$. We will say that $C$ is an elliptic rope if it has negative type (i.e. the splitting type $a_1 \geq \cdots \geq a_z$ of the conormal module of $C$ has $a_1 < 0$) and $p_n(C) = 1$. By the genus formula for rational $(z+1)$-ropes, these conditions are equivalent to $a_z = -2$ and $a_i = -1$ for $1 \leq i < z$.

Remark 2.6. By Remark 3.5 below every elliptic $(z+1)$-rope $C$ over $D$ is a split rope. Hence for any integer $z \geq 1$ there is a unique elliptic $(z+1)$-rope over $D$. By its very definition the conormal module of an elliptic $(z+1)$-rope is semistable if and only if $z = 1$. Set $G := O_D(-2) \oplus O_D(-1)^{\oplus(z-1)}$. The restricted cotangent sequence of $C$ splits. Take any integer $t \geq 2$ and any degree $t$ morphism $f : D \to D$. We have $f^*(G) \cong O_D(-2t) \oplus O_D(-t)^{\oplus(z-1)}$, $f^*(\Omega_D) \cong O_D(-2t)$ and $f^*(\Omega_C|D) \cong O_D(-2t)^{\oplus2} \oplus O_D(-t)^{\oplus(z-1)}$. We have a map $df : f^*(\Omega_C|D) \cong O_D(-2t) \to \Omega_D \cong O_D(-2)$. Hence Proposition 2.2 gives the following result.
Proposition 2.7. Fix integers $x, z$ with $x > z \geq 1$ and let $X$ be a $(x + 1)$-rope over $D$ with conormal module $F$. Let $f : D \to D$ be a degree $t$ morphism and $C$ an elliptic $(z + 1)$-rope. Assume $h^0(D, F(2t)) \neq 0$. Then there is a morphism $u : X \to C$ lifting $f$ and with $u(X)$ not contained in $D$.

3. Rational ropes and the splitting type of $F$

Let $X$ be a rational $(x + 1)$-rope with conormal module $F := \mathcal{O}_D(a_1) \oplus \cdots \oplus \mathcal{O}_D(a_x)$ with $a_1 \geq \cdots \geq a_x$. We have $p_a(X) = -\sum_{1 \leq i \leq x} a_i - x$. We will say that $X$ has negative type if $a_1 < 0$. If $X$ has negative type we will call the integer $-a_1$ the negative level of $X$. The deformation theory of a split rope is equivalent to the deformation theory of the vector bundle $F$ on $D$. We will say that a rational rope is rigid if its conormal module $F$ is rigid as a vector bundle on $D$, i.e. if $a_x \geq a_1 - 1$. We will say that a rational rope is semistable if its conormal module is a semistable vector bundle on $D$, i.e. if $a_x = a_1$. Since the multiplicative structure of $F$ is trivial, every $\mathcal{O}_D$-subsheaf $J$ of $F$ is an $\mathcal{O}_X$-ideal subsheaf of $\mathcal{O}_X$ and hence it defines a closed subscheme $\text{Spec}(\mathcal{O}_X/J)$ of $X$ with $D$ as support. In particular for every integer $i$ with $1 \leq i \leq x$ the vector bundle $F_i := \mathcal{O}_D(a_1) \oplus \cdots \oplus \mathcal{O}_D(a_i)$ is a subbundle of $F$ and any inclusion of $F_i$ into $F$ defines a closed subscheme $\text{Spec}(\mathcal{O}_X/F_i)$ of $X$. However, unless $a_i > a_j$ for all pairs $(i, j)$ with $i < j$, these subschemes are not uniquely determined by $F$. Call $y$ the number of different integers in the set $\{a_1, \ldots, a_x\}$, say $\{a_1, \ldots, a_x\} = \{b_1, \ldots, b_y\}$ with $b_i > b_j$ if $i < j$ and with $b_i$ appearing $r_i$ times in the weakly decreasing sequence $a_1 \geq a_2 \geq \cdots \geq a_x$. The vector bundles $F(i) := \bigoplus_{1 \leq j \leq y} \mathcal{O}_D(b_j)^{\oplus r_j}$ are uniquely determined by $F$; they give the Harder-Narasimhan filtration of $F$. Set $X(i) := \text{Spec}(\mathcal{O}_X/F(i))$.

Remark 3.1. Let $X$ be a rational $(x + 1)$-rope of negative type. Call $c := -a_1$ the negative level of $X$. For every $L \in \text{Pic}(X)$ with $\deg(L) < (x + 1)c$ the restriction map $H^0(X, L) \to H^0(D, L|D)$ is injective.

Remark 3.2. Let $X$ be a rational $(x + 1)$-rope of negative type. Call $c$ the negative level of $X$. Fix $L, R \in \text{Pic}(X)$ with $\deg(L) < (x + 1)c$ and $\deg(R) < (x + 1)c$. Since $D$ is reduced and connected, the pairing $H^0(D, L|D) \otimes H^0(D, M|D) \to H^0(D, L \otimes M|D)$ is non-degenerate in both variables. Since the restriction maps $H^0(X, L) \to H^0(D, L|D)$ and $H^0(X, M) \to H^0(D, M|D)$ are injective (Remark 3.1) the pairing $\alpha : H^0(X, L) \otimes H^0(X, M) \to H^0(X, L \otimes M)$ is non-degenerate in both variables. Hence by a classical lemma of Hopf we have $\dim(\text{Im}(\alpha)) \geq h^0(X, L) + h^0(X, M) - 1$ and in particular $h^0(X, L \otimes M) \geq h^0(X, L) + h^0(X, M) - 1$.

From Remark 3.1 we immediately obtain the following result.
Proposition 3.3 (Clifford’s inequality). Let $X$ be a rational $(x+1)$-rope of negative type. Let $0 > a_1 \geq \cdots \geq a_x$ be the splitting type of the conormal module of $X$. For every $L \in \text{Pic}(X)$ with $0 \leq \text{deg}(L) \leq (x+1)(-a_1-1)$ we have $h^0(X,L)-1 \leq \text{deg}(L)/(x+1)$.

Remark 3.4. Let $X$ be a rational $(x+1)$-rope of negative type. Using Remark 3.1 we see that $X$ splits if and only if there is $L \in \text{Pic}(X)$ such that $\text{deg}(L) \leq x+1$ and $h^0(X,L) \geq 2$.

Remark 3.5. Let $X$ be a rational rope with conormal module $F$. Assume that $F$ has splitting type $a_1 \geq \cdots \geq a_x$ with $a_x \geq -1$. By (3) we have $\text{Pic}(X) \cong \mathbb{Z}$ and every line bundle $L$ on $X$ is uniquely determined by its restriction to $X_{\text{red}}$, i.e. by the unique integer $d$ such that $\text{deg}(L) = (x+1)d$.

Remark 3.6. Let $X$ be a rational $(x+1)$-rope whose conormal module has splitting type $a_1 \geq \cdots \geq a_x$ with $a_x \geq 0$. By Remark 3.5 we have $\text{Pic}(X) \cong \mathbb{Z}$. Call $L(t)$ the unique line bundle on $X$ with $\text{deg}(L(t)) = (x+1)t$. The sequence of integers $h^0(X,L(-t)), t \geq 0$, uniquely determines all the integers $a_1, \ldots, a_x$; if $a_x = 0$ to obtain this observation we use either that every regular function on $D$ is constant and hence that the restriction map $H^0(X,\mathcal{O}_X) \to H^0(D,\mathcal{O}_D)$ is surjective or that $X$ is a split rope.

Let $X$ be a rational $(x+1)$-ropes and $F$ its conormal module. If $X$ is not of negative type, then $h^0(X,\mathcal{O}_X) \geq 2$ by the exact sequence (1). The finite dimensional $K$-vector space $H^0(X,\mathcal{O}_X)$ has a $K$-algebra structure for which it is a local ring whose maximal ideal $\mathfrak{m}$ has $\mathfrak{m}^2 = 0$. As a $K$-vector space we have $\mathfrak{m} \cong H^0(D,F)$. For any coherent sheaf $E$ on $X$ the $K$-vector spaces $H^0(X,E)$ and $H^1(X,E)$ are $H^0(X,\mathcal{O}_X)$-modules.

4. Rational ropes, their moduli spaces and decomposition of $F$

In this section we will study the moduli space of all rational $(x+1)$-ropes with fixed arithmetic genus (i.e. with conormal module of fixed degree) or with conormal module of fixed splitting type. Let $F := \mathcal{O}_D(a_1) \oplus \cdots \oplus \mathcal{O}_D(a_x)$ be a rank $x$ vector bundle on $D$ with $a_1 \geq \cdots \geq a_x$. Let $F(\geq t)$ (resp. $F(\leq t)$, resp. $F(> t)$, resp. $F(< t)$) be the direct sum of all factors $\mathcal{O}_D(a_i)$ of $F$ with $a_i \geq t$ (resp. $\leq t$, resp. $> t$, resp. $< t$). Since $(I_{D,X})^2 = 0$, for any $\mathcal{O}_D$-subbundle $G$ of $F$ there is a uniquely determined rational rope with $F/G$ as conormal module. If we take $F(\geq t)$ (resp. $F(> t)$) as $G$ we will call $X(< t)$ (resp. $X(\leq t)$) the corresponding rope.

Remark 4.1. Let $F$ be a rank $x$ vector bundle on $D$. By Remark 2.2 the set $S(F)$ of all non-split rational ropes with $F$ as conormal module are parametrized one-to-one by $\mathbb{P}(H^1(D,F(2)))$. Hence if $F = F(\geq -1)$, then every $(x+1)$-rope with $F$ as conormal module is split.
Fix a \((x + 1)\)-rope \(X\) over \(D\) with conormal module \(F\) and let \(e_X \in H^1(D, F(2))\) the corresponding extension class, uniquely determined up to a multiplicative non-zero constant (see Proposition 2.1). There is an exact sequence of \(O_X\)-modules
\[
0 \to F(\geq 0) \to O_X \to O_X(\langle 0 \rangle) \to 0
\]
Notice that \(H^1(D, F(2)) \cong H^1(D, F(\langle 0 \rangle)(2))\). This isomorphism maps the extension class \(e_X\) onto an extension class \(e_{X(\langle 0 \rangle)}\) of \(X(\langle 0 \rangle)\).

**Lemma 4.2.** The inclusion of \(X(\langle 0 \rangle)\) into \(X\) has a retraction. The exact sequence (4) is an exact sequence of \(O_X(\langle 0 \rangle)\)-modules and it splits as an exact sequence of \(O_X(\langle 0 \rangle)\)-modules.

**Proof.** The lemma follows from the construction of a rope from its extension class considered in Proposition 2.1. \(\square\)

**Lemma 4.3.** The restriction map \(\rho : \text{Pic}(X) \to \text{Pic}(X(\langle 0 \rangle))\) is an isomorphism.

**Proof.** The injectivity of \(\rho\) follows from Remark 3.5. The surjectivity of \(\rho\) follows from the existence of a retraction of \(X(\langle 0 \rangle)\) onto \(X(\langle 0 \rangle)\) (Lemma 4.2). \(\square\)

**Remark 4.4.** Fix \(L \in \text{Pic}(X)\) and set \(t := \deg(L|D)\). Thus \(\deg(L) = (x + 1)t\). First assume \(t < 0\). Then by the exact sequence (4) we obtain
\[
h^0(X(\langle 0 \rangle), L|X(\langle 0 \rangle)) = 0 \quad \text{and} \quad h^0(X, L) = h^0(D, F(t)) = h^0(X, F(\geq 0)(t)).
\]
If \(t = 0\) we have \(h^0(D, F(\geq 0)) \leq h^0(X, L) \leq h^0(X, F(\geq 0)) + 1\). Now assume \(t > 0\). We have \(h^1(D, F(\geq 0)(t)) = 0\). Hence from (4) we obtain
\[
h^0(X, L) = h^0(X(\langle 0 \rangle), L|X(\langle 0 \rangle)) + h^0(D, F(t)).
\]
Thus the Brill-Noether theory of \(X\) is essentially determined by the Brill-Noether theory of \(X(\langle 0 \rangle)\).

The proof of Lemma 4.3 gives verbatim the following result.

**Lemma 4.5.** The restriction map from the set of all isomorphism classes of vector bundles on \(X\) to the set of all isomorphism classes of vector bundles on \(X(\langle 0 \rangle)\) is an isomorphism.

**Corollary 4.6.** Assume \(X(\langle 0 \rangle) = D\), i.e. assume \(a_x \geq 0\). Then every vector bundle, \(E\), on \(X\) is a direct sum of line bundles and \(E\) is uniquely determined by \(E|D\): if \(E|D \cong \bigoplus_{1 \leq i \leq r} O_D(b_i)\), then \(E \cong \bigoplus_{1 \leq i \leq r} O_X(b_i)\), where \(O_X(c), c \in \mathbb{Z}\), is the unique line bundle on \(X\) with \(O_X(c)|D \cong O_D(c)\), i.e. the unique line bundle on \(X\) with degree \((x + 1)c\). We have \(h^0(X, O_X(c)) = h^0(D, F(c)) + c + 1\) and \(h^1(X, O_X(c)) = 0\) for every \(c \geq 0\). We have \(h^0(X, O_X(c)) = 0\) if and only if \(c < -a_1\) and \(h^1(X, O_X(c)) = 0\) if and only if \(c \geq -1\).
Remark 4.7. Corollary 4.6 gives a complete description of the Brill-Noether theory of vector bundles on any rational rope of non-negative type. It is remarkable that every vector bundle on a rational rope of non-negative type is a direct sum of line bundles. We do not know any other locally Cohen-Macaulay positive-dimensional projective scheme $Z$ with this property and $Z_{\text{red}}$ irreducible. Some (but not all) the reduced and connected projective curves, $T$, with $p_a(T) = 0$ have this property.

Remark 4.8. Let $X$ (resp. $Z$) be a split rational $(x+1)$-rope with conormal module $F$ (resp. $G$). $Z$ is the flat limit of a flat family of ropes isomorphic to $X$ if and only if the vector bundle $G$ on $D$ is a specialization of $G$.

Proposition 4.9. Fix rank $x$ vector bundles $F$ and $G$ on $D$ with $G$ specialization of $F$. Assume $h^1(D,F(2)) = h^1(D,G(2))$. Then there is an irreducible family of $(x+1)$-ropes, say parametrized by an irreducible variety $T$, whose general member has conormal module isomorphic to $F$ and such that every non-split rational $(x+1)$-ribbon with $G$ as conormal module occurs for at least one value of $T$.

Proof. The result follows from the definition of specializations of vector bundles on $D$, Remark 4.2 and the theory of the relative Ext-functor ([5]).

5. Blowing ups of a rope

Now we extend to the case of ropes the definition of blowing up introduced in [1] for ribbons. For simplicity we consider only rational ropes but the same definitions are obtained taking as $D$ any smooth projective curve. Fix $P \in D$, a vector bundle $E$ on $D$ and a surjection $u : E \to O_P$. The surjection $u : E \to O_P$ is uniquely determined by its restriction $u|\{P\} : E|\{P\} \to K$, where $E|\{P\}$ is the fiber of $E$ over $P$. Conversely, any surjective linear map $E|\{P\} \to K$ induces a surjection $E \to O_P$. Set $G := \text{Ker}(u)$. Since $P$ is a Cartier divisor of $D$, $G$ is a vector bundle on $D$. We will say that $G$ is obtained from $E$ making a negative elementary transformation supported by $P$. We have $\text{rank}(G) = \text{rank}(E)$ and $\text{deg}(G) = \text{deg}(E) - 1$. For every $\lambda \in K \setminus \{0\}$ we have $\text{Ker}(\lambda u) \cong \text{Ker}(u)$. For every subsheaf $A$ of $E$ with $\text{rank}(A) = \text{rank}(E)$ and $\text{deg}(A) = \text{deg}(E) - 1$ there is a unique $Q \in D$ such that $E/A \cong O_Q$; $A$ is obtained from $E$ making a negative elementary transformation supported by $Q$. We will say that $G^*$ is obtained from $E^*$ making a positive elementary transformation supported by $P$; more precisely, $G^*$ is obtained from $E^*$ making the positive elementary transformation dual to the negative elementary transformation associated to $u$. $E^*$ is a subsheaf of $G^*$, $\text{rank}(G^*) = \text{rank}(E^*)$, $\text{deg}(G^*) = \text{deg}(E^*) + 1$.
and $G^*/E^* \cong O_P$. Conversely, for every vector bundle $H$ and any inclusion $j : E^* \to H$ with $H/E^* \cong O_P$ there is a unique (up to a non-zero multiplicative constant) surjection $u : E \to O_P$ such that $H$ is isomorphic to the positive elementary transformation associated to $u$. Let $X$ be a rational $(x+1)$-rope with conormal module $F$. Take any surjection $u : F^* \to O_P$. We will prove the existence of a unique rational $(x+1)$-rope $X(u)$ with $\text{Ker}(u)^*$ as conormal module and equipped with a proper morphism $\phi_u : X(u) \to X$. For every $\lambda \in \mathbb{K}\setminus\{0\}$ we will obtain $X(\lambda u) \cong X(u)$ and, modulo this isomorphism, $\phi_u = \phi_{\lambda u}$. However, in general $X(u)$ will depend on the inclusion of $\text{Ker}(u)$ in $F$, not just on the isomorphism class of the vector bundle $\text{Ker}(u)$. The surjection $u$ induces an inclusion $j : F \to H$ with $H/j(F) \cong O_P$. The inclusion $j$ induce a map $\alpha : H^1(D, F(2)) \to H^1(D, H(2))$. Let $e_X \in H^1(D, F(2))$ be the extension class (unique up to a non-zero multiplicative constant) associated to $X$. Let $X(u)$ be the rational $(x+1)$-rope with $H$ as conormal module and $\alpha(e_X)$ as extension class. The rope $X(u)$ is called a blowing up of $X$ or the blowing up of $X$ at one point or the blowing up of $X$ associated to $u$. Notice that if $X$ is a split rope, then $X(u)$ is a split rope. We have $p_a(X(u)) = p_a(X) - 1$. For every $L \in \text{Pic}(X(u))$ the coherent sheaf $\phi_{u*}(L)$ is a rank 1 torsion free sheaf on $X$ whose restriction to $X \setminus \{P\}$ is locally free. Since $\phi_u$ is finite, we have $h^0(X, \phi_{u*}(L)) = h^0(X(u), L)$ and $h^1(X, \phi_{u*}(L)) = h^1(X, L)$. Since $p_a(X(u)) = p_a(X) - 1$, we obtain $\deg(\phi_{u*}(L)) = \deg(L) + 1$. We may iterate this construction and say when a vector bundle on $D$ is obtained from the vector bundle $E$ making a sequence of $t$ negative elementary transformations, $t$ any positive integer, and when a rational $(x+1)$-rope is obtained from $X$ making a sequence of $t$ blowing ups. Let $\phi : X' \to X$ be the composition of $t$ blowing ups. For any $L \in \text{Pic}(X')$ the coherent sheaf $\phi_{*}(L)$ is a rank 1 torsion free sheaf on $X$ which is is locally free outside $P$. Since $\phi$ is finite, we have $h^0(X, \phi_{*}(L)) = h^0(X(u), L)$ and $h^1(X, \phi_{*}(L)) = h^1(X, L)$. Since $p_a(X') = p_a(X) - t$, we obtain $\deg(\phi_{*}(L)) = \deg(L) + t$. For any fixed vector bundle $F$ on $D$ and any $P \in D$ the set of all isomorphism classes of vector bundles on $D$ obtained from $F$ making a negative elementary transformation supported by $P$ is parametrized by a non-empty open subset of a vector space and in particular it is parametrized by an irreducible variety. Since $D$ is irreducible, the set of all isomorphism classes of vector bundles obtained from $F$ making a negative elementary transformation supported by a point of $D$ is parametrized by an irreducible variety. The same is true for positive elementary transformations. Hence for any rope $X$ we are allowed to say that a rope $Y$ is obtained from $X$ making a sequence of $t$ generic blowing ups. For any rope $X$ let $\gamma(X)$ be the minimal integer $t$ such that there is a split rope obtained from $X$ making a sequence of $t$ blowing ups.
Thus $\gamma(X) = 0$ if and only if $X$ is a split rope. The next lemma shows that $\gamma(X) < +\infty$ for every rope $X$.

**Lemma 5.1.** Let $X$ be a rational $(x + 1)$-rope with conormal module $F$. Then any rope obtained from $X$ making a sequence of $h^1(D, F(2))$ generic blowing ups is a split rope. In particular $\gamma(X) \leq h^1(D, F(2))$.

**Proof.** Let $G$ be a vector bundle on $D$ and $H$ a vector bundle obtained from $G$ making a general positive elementary transformation. We have $h^1(X, H) = \max\{0, h^1(X, G) - 1\}$. Iterating $h^1(D, F(2))$ times this observation, we conclude.  

In 5.2, 5.3 and 5.4 we will see that the Brill-Noether theory of ropes with low $\gamma(X)$ is quite restricted.

**Remark 5.2.** Let $X$ be a rational $(x + 1)$-rope. Take a split rational rope $Z$ such that there is a sequence $\phi : Z \to X$ of $\gamma(X)$ blowing ups. Let $L \in \text{Pic}(Z)$ the line bundle inducing the splitting of $Z$. Thus $\deg(L) = x + 1$ and $h^0(Z, L) \geq 2$. If $Z$ is of negative type we have $h^0(Z, L) = 2$. The generalized line bundle $\phi_*(L)$ on $X$ has degree $x + 1 + \gamma(X)$ and $h^0(X, \phi_*(L)) = h^0(Z, L) \geq 2$.

**Proposition 5.3.** Let $X$ be a rational $(x + 1)$-rope and $\phi : Z \to X$ the composition of $z$ blowing ups with $Z$ split rope. Assume $X$ not split. Let $G$ be the conormal module of $Z$ and call $b_1 \geq \cdots \geq b_x$ the splitting type of $G$. Assume $b_1 \leq -2$ and let $t$ be a positive integer such that $0 < t < -b_1$. There is no spanned line bundle $L$ on $X$ such that $0 < \deg(L) \leq t(x + 1)$.

**Proof.** Assume the existence of $L \in \text{Pic}(X)$ such that $0 < \deg(L) \leq t(x + 1)$ and set $M := \phi_*(L)$. We have $\deg(L) = \deg(M) = (x + 1)y$ with $y = \deg(L|D)$ and $1 \leq y \leq t$. Let $\beta : X \to \mathbf{P}(H^0(X, L)^*)$ be the morphism induced by $M$. Since $Z$ is a split rope and $y \leq t < -b_1$, we have $h^0(Z, M) = y + 1$. Since $L$ is spanned, $M$ is spanned by the image, $W$, of $\phi_*(H^0(X, M))$ into $H^0(Z, M)$. Fix a retraction $u : Z \to D$, Since $H^1(Z, G) \neq 0$, $u$ is not unique (use Remark 3.5), but any two retractions of $Z$ differ by an automorphism of $Z$ whose restriction to $D$ is the identity. The morphism $\alpha : Z \to \mathbf{P}(H^0(Z, M)^*)$ induced by $M$ has as image a rational normal curve of $\mathbf{P}(H^0(Z, M)^*)$ and it is the composition of a retraction of $Z$ and a linearly normal embedding of $D$ into $\mathbf{P}(H^0(Z, M)^*)$. The morphism $\gamma$ induced by $W$ must be obtained composing a retraction of $Y$ with an embedding of $D$ into $\mathbf{P}^m$, $m := \dim(W) - 1$. Since $X$ is not split, no such morphism $\gamma$ may factor through $\phi$ and induce a morphism of $X$, contradicting the definition of $W$.  

For any rational rope $X$ there is a split rational rope $Z$ and a morphism $\phi : Z \to X$ with $\phi$ composition of $\gamma(X)$ blowing ups. By Lemma 5.3 if $\gamma(X)$
is very low the conormal module of any such $Z$ gives strong informations on the Brill-Noether theory of $Z$. As an immediate corollary of 5.3 we obtain the following result.

**Corollary 5.4.** Let $X$ be a rational non-split $(x + 1)$-rope whose conormal module $F$ has splitting type $a_1 \geq \cdots \geq a_x$ with $a_1 \leq -2 + \gamma(X)$. Then there is no $L \in \text{Pic}(X)$ with $L$ spanned and $0 < \deg(L) \leq (x+1)(-a_1-2-\gamma(X))$.

**Proof.** Let $\phi : Z \to X$ be a composition of $\gamma(X)$ blowing ups with $Z$ split rope. Let $G$ be the conormal module of $Z$ and $b_1 \geq \cdots \geq b_x$ be its splitting type. Since $G$ is obtained from $F$ making $\gamma(X)$ positive elementary transformations, we have $b_1 \leq a_1 + \gamma(X)$. Hence we conclude by 5.3. \qed

### 6. Restricted cotangent sequence and splittings

In this section we study the restricted cotangent sequence of a $(x + 1)$-rope over $D$ with negative conormal module. Let $X$ be a $(x + 1)$-rope over $D$ and $F$ its conormal module. Set $G := \Omega_X|D$. Let $a_1 \geq \cdots \geq a_x$ (resp. $w_1 \geq \cdots \geq w_{x+1}$) be the splitting type of $F$ (resp. $G$). Consider the restricted cotangent sequence of $X$:

(5) \[ 0 \to F \to G \to O_D(-2) \to 0 \]

**Remark 6.1.** Since every element of $H^1(D, F(2))$ is an extension class for a rational rope with $F$ as conormal module, for any $F$ and any $G$ fitting in an exact sequence (5) there is a rope $X$ with $F$ as conormal module, $G$ as restricted cotangent bundle and such that (5) is the restricted cotangent sequence. The set of all such pairs $(F, G)$ is completely described in [8] and [7].

**Remark 6.2.** Assume $a_1 \leq -3$. The exact sequence (5) splits if and only if $w_1 = -2$. Since the extension class of (5) gives the isomorphism class of $X$, the rope $X$ is a split rope if and only if $w_1 = -2$.

**Theorem 6.3.** Assume $a_1 \leq -3$. Then $\gamma(X) = -w_1 - 2$.

**Proof.** Since $a_1 \leq -3$, we have $w_1 \leq -2$. If $w_1 = -2$, the result is Remark 6.2. Assume $w_1 < -2$. Take any vector bundle $G'$ obtained from $G$ making a positive elementary transformation. Then either this elementary transformation acts on the subbundle $F$ of $G$ or not, the latter case being the general one. If this elementary transformation acts on $F$, then it produces a vector bundle $F'$ obtained from $F$ making a positive elementary transformation and fitting in an exact sequence

(6) \[ 0 \to F' \to G' \to O_D(-2) \to 0 \]

Every sequence of positive elementary transformations gives a sequence of blowing ups of $X$. It is obvious the existence of a sequence of $-w_1 - 2$
positive elementary transformations of $G$ such that the vector bundle, $H$, obtained in this way has splitting type $c_1 \geq \cdots \geq c_{x+1}$ with $c_1 = -2$; at each step we increase by one the higher integer of the splitting type of the corresponding bundle. Furthermore, $-w_1 - 2$ is the minimal length of any such sequence of positive elementary transformations. At the first step we need to prove that as our first positive elementary transformation of $G$ we may choose a positive elementary transformation which induces a positive elementary transformation of $F$. Since rank($G$) = rank($F$) + 1, this is obvious if $w_2 = w_1$. Thus we may assume $w_2 < w_1$. This implies that there is a unique line subbundle of $G$ with degree $w_1$ and that $O_D(w_1)$ is the first step of the Harder-Narasimhan filtration of $G$. For any $P \in D$ and any vector bundle $H$ on $D$, let $H\{|P\}$ be the fiber of $H$ over $P$; thus $H\{|P\}$ is a $K$-vector space of dimension rank($H$). Fix $P \in D$. There is a positive elementary transformation of $G$ supported by $P$ which increases the value of $w_1$ and which induces a positive elementary transformation of $F$ supported by $P$ if and only if $O_D(w_1)\{|P\}$ is contained in the hyperplane $F\{|P\}$ of $G\{|P\}$. The inclusions of $F$ in $G$ and of $O_D(w_1)$ in $G$ induces a map $u : F \oplus O_D(w_1) \to G$. Since to prove the existence of the positive elementary transformation we are looking for we may assume that $F$ does not contain $O_D(w_1)$, the map $u$ is an injective map of sheaves. We have rank($F \oplus O_D(w_1)$) = rank($G$). Since $w_1 < -2$, $u$ cannot be an isomorphism. Thus Coker($u$) is a non-zero skyscraper sheaf. For every $P \in \text{Supp}(\text{Coker}(u))$ we may find a positive elementary transformation of $G$ inducing a positive elementary transformation of $F$ (i.e. inducing an exact sequence (6)) and transforming the subbundle $O_D(w_1)$ of $G$ into the subbundle $O_D(w_1 + 1)$ of $G'$. Since $G'$ has splitting type $w_1 + 1 > w_2 \geq \cdots \geq w_{x+1}$, we may iterate the proof taking the pair $(F', G')$ instead of the pair $(F, G)$. □

References


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