EVEN-DIMENSIONAL MANIFOLDS STRUCTURED BY A CONSTANT $T$-PARALLEL CONNECTION

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Abstract. Geometrical and structural properties are proved for even-dimensional manifolds which are equipped with a constant $T$-parallel connection.

1. Introduction

Manifolds structured by a $T$-parallel connection have been defined in [17] and have also been studied in [13]. The present paper continues the study of the structural properties of manifolds endowed with a $T$-parallel connection in the presence of additional geometric structures; as such the present investigation can be situated in the prolongation of the recent publications [3] [4] [5]. A general discussion of the geometrical structures which appear here and in the sequel can be found in the standard references [16] and [26] which also contain more background information and additional references (see also [1] [7] [20] for further reading).

Let now $M$ be a $2m$-dimensional $C^\infty$-manifold and $e_a (a \in \{1, \ldots , 2m\})$ an orthonormal vector basis. We recall that if $M$ carries a globally defined vector field $T$ and the connection forms satisfy

$$\theta^a_b = \langle T, e_b \wedge e_a \rangle,$$

where $\wedge$ denotes the wedge product of vector fields, then one says that $M$ is structured by a $T$-parallel connection. In the present paper we assume in addition that $T$ is constant. Introducing the notation $\beta = \|T\|^2$, $\beta$ will be called the structural pfaffian. Defining $2t = \|T\|^2$, we consequently see that this quantity is also constant.

For the above mentioned structure, we prove the following properties:

(i): $M$ is a hyperbolic space-form, i.e. for the curvature forms $\Theta^a_b$ one has that

$$\Theta^a_b = -2t \, \omega^a \wedge \omega^b,$$

where $\{\omega^a\}$ denotes the cobasis of the vector basis $\{e_a\}$;

(ii): $M$ carries a locally conformal symplectic form $\Omega$ having $\beta (= T^b)$ as covector of Lee [9], i.e.

$$d\Omega = 2\beta \wedge \Omega,$$

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and $T$ defines a relative conformal transformation [19] [12] of $\Omega$, i.e.

\[ d(\mathcal{L}_T \Omega) = 8t\beta \wedge \Omega; \]

(iii): $T$ is torse forming [23] (see also [12] [19] [21]); moreover, with $T$ there is associated a second vector field $X$ which defines an infinitesimal automorphism [10] (see also [11]) of $\Omega$, i.e.

\[ \mathcal{L}_X \Omega = 0; \]

(iv): both vector fields $T$ and $X$ turn out to be biconcircular (in the sense of Okumura [14], see also [24]) and exterior concurrent [18]. In addition, $T$ has also the property to be an affine vector field [16], i.e.

\[ \mathcal{L}_T \nabla T = 0. \]

Finally, if we define the function $s$ by $s = \langle T, X \rangle$, one also finds that

\[ ds = -s\beta, \]

and one further derives that

\[
\begin{align*}
\text{grad } s &= 2ts^2, \\
\text{div } \text{grad } s &= 2t(2 - tm)s,
\end{align*}
\]

which shows that $s$ is an isoparametric function [22].

In Section 4 we consider some properties of the tangent bundle manifold $TM$ having the manifold $M$, studied in Section 3, as basis. On $TM$ the canonical vector field $V(V^a) (a = 1, \ldots, 2m)$ is called the Liouville vector field [6]. We will denote the adapted cobasis in $TM$ by $\mathcal{B}^* = \{\omega^a, dV^a\}$. Then, the complete lift $\Omega^C$ [25] of the 2-form $\Omega$ is given by

\[ \Omega^C = \sum_{a=1}^{m} (dV^a \wedge \omega^a^* + \omega^a \wedge dV^a^*), \quad a^* = a + m. \]

One can deduce that

\[ d\Omega^C = \beta \wedge \Omega^C, \]

which shows that the 2-form $\Omega^C$ is, just as $\Omega$, also a conformal symplectic form. Next, since the Liouville vector field $V$ is given by

\[ V = \sum_{a=1}^{2m} V^a \frac{\partial}{\partial V^a}, \]

the basic 1-form $\mu$ (also called the Liouville form) associated with the canonical vector field $V$ (i.e. $\mu = V^b$) can be written as [8]

\[ \mu = \sum_{a=1}^{2m} V^a \omega^a. \]
Taking the Lie differential of $\Omega^C$, one finds that

$$\mathcal{L}_V \Omega^C = \Omega^C,$$

which expresses that the 2-form $\Omega^C$ is a homogeneous 2-form of class 1 \cite{8} on $TM$. Some further properties of the tangent bundle manifold $TM$ are also discussed.

2. Preliminaries

Let $(M, g)$ be a Riemannian $C^\infty$-manifold and let $\nabla$ be the Levi-Civita operator with respect to the metric tensor $g$. Let $\Gamma TM = \Xi(M)$ be the set of sections of the tangent bundle, and

$$\flat : TM \to T^* M \quad \text{and} \quad \sharp : TM \leftarrow T^* M$$

the classical isomorphisms defined by $g$ (i.e. $\flat$ is the index lowering operator, and $\sharp$ is the index raising operator).

Following \cite{16}, we denote by

$$A^q(M, TM) = \Gamma \text{Hom}(\Lambda^q TM, TM),$$

the set of vector valued $q$-forms ($q \leq \dim M$), and we write for the covariant derivative operator with respect to $\nabla$

$$d^\nabla : A^q(M, TM) \to A^{q+1}(M, TM).$$

It should be noticed that in general $d^\nabla^2 = d^\nabla \circ d^\nabla \neq 0$, unlike $d^2 = d \circ d = 0$. We denote by $dp \in A^1(M, TM)$ the canonical vector valued 1-form of $M$, which is also called the soldering form of $M$ \cite{2}. Since $\nabla$ is symmetric one has that $d^\nabla(dp) = 0$.

A vector field $Z \in \Xi(M)$ which satisfies

$$(1) \quad d^\nabla(\nabla Z) = \nabla^2 Z = \pi \wedge dp \in A^2(M, TM); \quad \pi \in \Lambda^1 M$$

is defined to be an exterior concurrent vector field \cite{17} (see also \cite{13}). The 1-form $\pi$ in (4) is called the concurrence form and is defined by

$$(2) \quad \pi = \lambda Z^\flat, \quad \lambda \in \Lambda^0 M.$$ 

Let $\mathcal{O} = \text{vect}\{e_a | a = 1, \ldots, 2m\}$ be a local field of adapted vectorial frames over $M$ and let $\mathcal{O}^* = \text{covect}\{\omega^a\}$ be its associated coframe. Then the soldering form $dp$ is expressed by

$$(3) \quad dp = \sum_{a=1}^{2m} \omega^a \otimes e_a,$$
and E. Cartan’s structure equations can be written in indexless manner are

\( \nabla e = \theta \otimes e, \)  
\( d\omega = -\theta \wedge \omega, \)  
\( d\theta = -\theta \wedge \theta + \Theta. \)

In the above equations \( \theta \) (respectively \( \Theta \)) are the local connection forms in the tangent bundle \( TM \) (respectively the curvature 2-forms on \( M \)).

3. Manifolds with constant \( T \)-parallel connection

Let \( (M, g) \) be a \( 2m \)-dimensional \( C^\infty \)-manifold and

\[ T = T^a e_a, \]

be a globally defined vector field. Let \( \theta^a_b \) \((a, b \in \{1, \ldots, 2m\})\) be the local connection forms in the tangent bundle \( TM \). Then, by reference to [17] [13], \( (M, g) \) is said to be structured by a \( T \)-parallel connection if the connection forms \( \theta \) satisfy

\[ \theta^a_b = \langle T, e_b \wedge e_a \rangle, \]

where \( \wedge \) means the wedge product of vector fields. Making use of Cartan’s structure equations (4), we can see that

\[ \theta^a_b = T^b \omega^a - T^a \omega^b. \]

In consequence of (8), the equations (4) take the form

\[ \nabla e_a = T^a dp - \omega^a \otimes T. \]

In the sequel we assume in addition that \( T^a \) are the components of a constant vector field \( T \), called the structure vector field of \( M \).

Let

\[ T^b = \beta = \sum_{a=1}^{2m} T^a \omega^a \]

be the dual form of \( T \), then by E. Cartan’s structure equations (5) one derives that

\[ d\omega^a = \beta \wedge \omega^a. \]

Hence, by (11) it follows that all the elements \( \omega^a \) of the covector basis \( O^* \) are exterior recurrent forms [2]. Consequently, the pfaffian \( \beta \) can be seen to be in fact a closed form, i.e.

\[ d\beta = dT^b = 0. \]

Under the present conditions, by (8) and (11) one finds that

\[ d\theta^a_b = \beta \wedge \theta^a_b. \]
which expresses that all the connection forms $\theta^a_b$ are exterior recurrent [2] with $\beta$ as recurrence form. Under these conditions, the structure equations (6) involving the curvature forms $\Theta^a_b$ are expressed by

\begin{equation}
\Theta^a_b = -2t \omega^a \wedge \omega^b,
\end{equation}

where we have set

\begin{equation}
2t = \|T\|^2 = \text{const.}
\end{equation}

It is well known that the equation (14) thus shows that the manifold $M$ under consideration is a space form of hyperbolic type. We remark that in view of (11), one derives that

\begin{equation}
d\Theta^a_b = 2\beta \wedge \Theta^a_b,
\end{equation}

which means that all curvature forms are exterior recurrent; we therefore agree to call $\beta$ the basic pfaffian on $M$.

In another perspective, we consider on $M$ the local almost symplectic form $\Omega$ given by

\begin{equation}
\Omega = \sum_{a=1}^{m} \omega^a \wedge \omega^{a^*}, \quad a^* = a + m.
\end{equation}

Taking the exterior derivative of $\Omega$, and in view of (11), one finds that

\begin{equation}
d\Omega = 2\beta \wedge \Omega,
\end{equation}

which shows that $\Omega$ is a locally conformal symplectic form having $\beta$ as covector of Lee [9].

Taking first the Lie derivative of $\Omega$ with respect to the vector field $T$, we get

\begin{equation}
\mathcal{L}_T \Omega = \sum_{a=1}^{m} \mathcal{L}_T \omega^a \wedge \omega^{a^*} + \sum_{a=1}^{m} \omega^a \wedge \mathcal{L}_T \omega^{a^*},
\end{equation}

where $\mathcal{L}_T \omega^a$ can be calculated as follows.

\[\mathcal{L}_T \omega^a = (i(T) \circ d + d \circ i(T)) \omega^a \quad (a = 1, \cdots, 2m)\]

Taking into account equation (11) for $d\omega^a$ and the definition (15) of $2t$, it follows that

\[\mathcal{L}_T \omega^a = 2t\omega^a - 2T^a \beta, \quad (a = 1, \cdots, 2m).\]

Continuing now the calculation of $\mathcal{L}_T \Omega$ leads to

\[\mathcal{L}_T \Omega = 4t\Omega + 2\beta \wedge ^bT,
\]

where

\[^bT = -i_T \Omega = \sum_{a=1}^{m} \left( T^{a^*} \omega^a - T^a \omega^{a^*} \right).\]
Exterior differentiation of $L_{T} \Omega$ gives
$$d(L_{T} \Omega) = 4td\Omega + 2d\beta \wedge bT - 2\beta \wedge d(bT).$$
One can verify directly that $d(bT) = 0$, and recalling that the 1-form $\beta = T^b$ is closed, the above expression reduces to
$$d(L_{T} \Omega) = 4td\Omega.$$
Replacing $d\Omega$ through equation (18), finally yields
$$d(L_{T} \Omega) = 8t\beta \wedge \Omega.$$ 
Hence, following a known definition [19] (see also [12]), the above equation means that $T$ defines a relative conformal transformation of $\Omega$.

Further, consider the vector field
$$X = \sum_{a=1}^{2m} X^a e_a.$$ 
Taking the Lie differential of $\Omega$ w.r.t. $X$, yields
$$L_X \Omega = -\sum_{a=1}^{m} (dX^a + \beta X^a) \land \omega^a + \sum_{a=1}^{m} (dX^a + \beta X^a) \land \omega^a.$$ 
Therefore, the necessary and sufficient condition for $X$ to define an infinitesimal automorphism [10] (see also [11]) of $\Omega$, namely
$$L_X \Omega = 0,$$
can be seen to be
$$dX^a + \beta X^a = 0.$$ 
We now introduce the notation
$$\alpha = X^b = \sum_{a=1}^{2m} X^a \omega^a$$
for the dual form of $X$.
Taking the exterior derivative of (24) gives
$$d\alpha = \sum_{a=1}^{2m} dX^a \land \omega^a + \sum_{a=1}^{2m} X^a d\omega^a.$$ 
Replacing in the above formula $dX^a$ using (23), and $d\omega^a$ using (11), yields
$$d\alpha = -\sum_{a=1}^{2m} \beta X^a \land \omega^a + \sum_{a=1}^{2m} X^a \beta \land \omega^a.$$
From this it follows that

\[(25) \quad d\alpha = 0,\]

which shows that \(X\) is also a closed vector field.

Further, calculating the covariant differentials of the vector fields \(T\) and \(X\) under consideration and invoking (15), one obtains that

\[(26) \quad \nabla T = 2tdp - 2\beta \otimes T,\]

and

\[(27) \quad \nabla X = sdp - \alpha \otimes T - \beta \otimes X,\]

where we have put

\[(28) \quad s = g(X, T).\]

Equation (26) expresses that the structure vector field \(T\) is torse forming \([23]\) (see also \([12]\) \([19]\) \([21]\)); in this context we will call \(X\) an almost torse forming vector field, and by standard terminology \([21]\) \(2t = ||T||^2\) is the energy of the torse forming vector field \(T\).

Moreover, we notice that any 2 vector fields \(Z, Z' \in \Xi(M)\) satisfy

\[(29) \quad \langle \nabla_Z T, Z' \rangle = \langle \nabla_{Z'} T, Z \rangle,\]

\[\langle \nabla_Z X, Z' \rangle = \langle \nabla_{Z'} T, Z \rangle.\]

According to Okumura \([14]\) (see also \([24]\)), the relations (29) show that \(T\) and \(X\) are gradient vector fields. On the other hand, since \(\nabla\) acts inductively one also derives that

\[(30) \quad d^\nabla (\nabla T) = 2tT^b \wedge dp, \quad (T^b =: \beta)\]

\[(31) \quad d^\nabla (\nabla X) = 2tX^b \wedge dp, \quad (X^b =: \alpha)\]

The above equations mean that both \(T\) and \(X\) are exterior concurrent vector fields \([18]\). Therefore, if \(\mathcal{R}\) denotes the Ricci curvature, it follows from (30), (31) and \([15]\) that

\[(32) \quad \mathcal{R}(T, Z) = -(2m - 1)2tg(T, Z),\]

\[\mathcal{R}(X, Z) = -(2m - 1)2tg(X, Z).\]

We remark that calculating the Lie differential of \(\nabla T\) with respect to \(T\) reveals that

\[(33) \quad \mathcal{L}_T \nabla T = 0,\]

which shows that \(T\) is an affine vector field \([16]\). We recall that with respect to an orthonormal vector basis \(\{e_a\}\) the divergence of a vector field \(Z\) is
calculated according to the formula
\begin{equation}
\text{div } Z = \sum_{a=1}^{2m} \langle \nabla e_a Z, e_a \rangle;
\end{equation}
when applied to the case under consideration, this gives
\begin{equation}
\text{div } T = (2m-1)2t = \text{const.}
\end{equation}
Furthermore, since the components $T^a$ are constant, one finds by differen-
tiation of the equality $s = g(T, X)$ that
\begin{equation}
ds = -s\beta.
\end{equation}
Consequently one may write that
\begin{equation}
\text{grad } s = -sT \implies \|\text{grad } s\|^2 = 2ts^2,
\end{equation}
from which one also derives that
\begin{equation}
\text{div}(\text{grad } s) = 2t(2 - tm)s.
\end{equation}
We remind that a function $f : \mathbb{R}^n \to \mathbb{R}$ is called isoparametric [22] if both \(\|\text{grad } f\|^2\) and \(\text{div}(\text{grad } f)\) are functions of $f$. We may therefore conclude
that $s$ is an isoparametric function.

Summing up, we state the following

**Theorem 3.1.** Let $M(\Omega, T, g)$ be a $2m$-dimensional manifold with almost symplectic form $\Omega$, and structure constant vector field $T$, such that the connection forms satisfy
\[ \theta_b^a = \langle T, e_b \wedge e_a \rangle. \]
Then the following properties hold:

(i): $M$ is a hyperbolic space-form;
(ii): $\Omega$ is a conformally symplectic form and has $\beta(= T^a)$ as covector of Lee;
(iii): the differential of the Lie derivative with respect to $T$ defines a relative conformal transformation of $\Omega$, i.e.
\[ d(L_T \Omega) = 8t\beta \wedge \Omega, \quad 2t = \|T\|^2; \]
(iv): a vector field $X$ which satisfies
\[ dX^a + \beta X^a = 0 \]
defines an infinitesimal automorphism of $\Omega$, i.e.
\[ L_X \Omega = 0; \]
(v): $T$ is a torse forming vector field, as well as an exterior concurrent vector field;
(vi): the vector field $X$ is also an exterior concurrent vector field, and both $\mathcal{T}$ and $X$ are gradient vector fields;
(vii): the scalar $s = \langle \mathcal{T}, X \rangle$ is an isoparametric function.

4. Geometry of the tangent bundle

In this section we will discuss some properties of the tangent bundle manifold $TM$ having as basis the manifold $\mathcal{M}$ studied in Section 3. Denote by $V(V^a)$ ($A = 1, \ldots, 2m$) the Liouville vector field (or the canonical vector field on $TM$ [8]). Accordingly, one may consider the set

$$B^* = \{\omega^a, dV^a|a = 1, \ldots, 2m\}$$

as an adapted cobasis in $TM$ (see also [13]). Following [25] the complete lift $\Omega^C$ of the conformal symplectic form $\Omega$ of $\mathcal{M}$ is the 2-form of rank $4m$ on $TM$ given by

$$\Omega^C = \sum_{a=1}^{2m} (dV^a \wedge \omega^a + \omega^a \wedge dV^a), \quad a^* = a + m. \quad (39)$$

On the other hand, the Liouville vector field $V$ is expressed by

$$V = \sum_{a=1}^{2m} V^a \frac{\partial}{\partial V^a}. \quad (40)$$

It is also known that the associated basic 1-form

$$\mu = \sum_{a=1}^{2m} V^a \omega^a \quad (41)$$

is called the Liouville form (see also [8]). (Alternatively, one can also write that $\mu = V^b$.) Then, on behalf of (11), the exterior differential of $\Omega^C$ is given by

$$d\Omega^C = \beta \wedge \Omega^C. \quad (42)$$

Hence, the complete lift $\Omega^C$ of $\Omega$ defines on $TM$ a conformal symplectic structure, as $\Omega$ does on $\mathcal{M}$; this result is meaningful, since it should be stressed that conformal properties are not preserved by complete lifts in general. On behalf of (40) one has that

$$i_V \Omega^C = \sum_{a=1}^{2m} (V^a \omega^{a^*} - V^{a^*} \omega^a), \quad (43)$$

and in view of (42) and (43) one gets

$$\mathcal{L}_V \Omega^C = \Omega^C. \quad (44)$$

Equation (44) shows that $\Omega^C$ is a homogeneous 2-form of class 1 [8] on TM.
Further, taking the exterior differential of the Liouville form $\mu$, one derives by (41) that
\begin{equation}
\label{eq:45}
d\mu = \beta \wedge \mu + \psi,
\end{equation}
where we have introduced the notation
\begin{equation}
\label{eq:46}
\psi = \sum_{a=1}^{2m} dV^a \wedge \omega^a.
\end{equation}
By reference to (46) and (11), it follows that
\begin{equation}
\label{eq:47}
d\psi = \beta \wedge \psi,
\end{equation}
which shows that $\psi$ is an exterior recurrent form with $\beta$ as recurrence form. Since the 2-form $\psi$ is of maximal rank, we will refer to $\psi$ as the canonical conformal symplectic form of $M$. One finally gets that
\begin{equation}
\label{eq:48}
\mathcal{L}_V \psi = \psi,
\end{equation}
which shows that, as $\Omega^C$, the form $\psi$ is also a homogeneous 2-form of class 1.

We remind that the vertical operator $i_V$ in the sense of [6] possesses by definition the following properties (see also [8]):
\begin{equation}
\label{eq:49}
i_V \lambda = 0, \quad i_V \omega^a = 0, \quad i_V dV^a = \omega^a,
\end{equation}
from which one calculates by (46) that
\begin{equation}
\label{eq:50}
i_V \psi = 0.
\end{equation}
Together with (47) we conclude from this that $\psi$ is a Finslerian form [6].

**Theorem 4.1.** Let $TM$ be the tangent bundle manifold having as basis the manifold $M(\Omega, T, \beta)$ considered in Section 3. Let $V$, and $\mu$, be the Liouville vector field, and the Liouville form of $TM$ respectively. One has the following properties:

(i): the complete lift $\Omega^C$ on $TM$ is a conformally symplectic form, and is a homogeneous 2-form of class 1, i.e.
\begin{equation}
\mathcal{L}_V \Omega^C = \Omega^C;
\end{equation}

(ii): $\mu$ satisfies
\begin{equation}
d\mu = \beta \wedge \mu + \sum_{a=1}^{2m} dV^a \wedge \omega^a,
\end{equation}
where
\begin{equation}
\psi = \sum_{a=1}^{2m} dV^a \wedge \omega^a,
\end{equation}
is the canonical conformal symplectic form and turns out to be a Finslerian form.

References


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