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DECOMPOSITION OF SPINOR GROUPS BY THE INVOLUTION σ' IN EXCEPTIONAL LIE GROUPS

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Introduction

The compact exceptional Lie groups F_4, E_6, E_7 and E_8 have spinor groups as a subgroup as follows:

$$\begin{array}{l}
F_4 \supset Spin(9) \supset Spin(8) \supset Spin(7) \supset \cdots \supset Spin(1) \ni 1 \\
\cap \\
E_6 \supset Spin(10) \\
\cap \\
E_7 \supset Spin(12) \supset Spin(11) \\
\cap \\
E_8 \supset Ss(16) \supset Spin(15) \supset Spin(14) \supset Spin(13).
\end{array}$$

On the other hand, we know the involution σ' induced an element $\sigma' \in Spin(8) \subset F_4 \subset E_6 \subset E_7 \subset E_8$. Now, in this paper, we determine the group structures of $(Spin(n))^{\sigma'}$ which are the fixed subgroups by the involution σ'. Our results are as follows:

$$\begin{array}{lll}
F_4 & (Spin(9))^{\sigma'} & \cong Spin(8), \\
E_6 & (Spin(10))^{\sigma'} & \cong (Spin(2) \times Spin(8))/\boldsymbol{Z}_2, \\
E_7 & (Spin(11))^{\sigma'} & \cong (Spin(3) \times Spin(8))/\boldsymbol{Z}_2, \\
 & (Spin(12))^{\sigma'} & \cong (Spin(4) \times Spin(8))/\boldsymbol{Z}_2, \\
E_8 & (Spin(13))^{\sigma'} & \cong (Spin(5) \times Spin(8))/\boldsymbol{Z}_2, \\
 & (Spin(14))^{\sigma'} & \cong (Spin(6) \times Spin(8))/\boldsymbol{Z}_2.
\end{array}$$

Needless to say, the spinor groups appeared in the first term have relation

$$Spin(2) \subset Spin(3) \subset Spin(4) \subset Spin(5) \subset Spin(6).$$

One of our aims is to find these groups explicitly in the exceptional groups. In the group E_8, we conjecture that

$$\begin{array}{ll}
(Spin(15))^{\sigma'} & \cong (Spin(7) \times Spin(8))/\boldsymbol{Z}_2, \\
(Ss(16))^{\sigma'} & \cong (Spin(8) \times Spin(8))/(\boldsymbol{Z}_2 \times \boldsymbol{Z}_2),
\end{array}$$

however, we can not realize explicitly.

This paper is closely in connection with the preceding papers [2], [3], [4] and may be a continuation of [2], [3], [4] in some sense.

1. Group F_4

We use the same notation as in [5] (however, some will be rewritten). For example,

- the Cayley algebra $\mathfrak{C} = \boldsymbol{H} \oplus \boldsymbol{H}e_4$,
- the exceptional Jordan algebra $\mathfrak{J} = \{X \in M(3, \mathfrak{C}) \mid X^* = X\}$, the Jordan multiplication $X \circ Y$, the inner product (X, Y) and the elements $E_1, E_2, E_3 \in \mathfrak{J}$,
- the group $F_4 = \{\alpha \in \mathrm{Iso}_{\boldsymbol{R}}(\mathfrak{J}) \mid \alpha(X \circ Y) = \alpha X \circ \alpha Y\}$, and the element $\sigma \in F_4$: $\sigma X = DXD$, $D = \mathrm{diag}(1, -1, -1)$, $X \in \mathfrak{J}$ and the element $\sigma' \in F_4$: $\sigma' X = D'XD'$, $D' = \mathrm{diag}(-1, -1, 1)$, $X \in \mathfrak{J}$,
- the groups $SO(8) = SO(\mathfrak{C})$ and $Spin(8) = \{(\alpha_1, \alpha_2, \alpha_3) \in SO(8) \times SO(8) \times SO(8) \mid (\alpha_1 x)(\alpha_2 y) = \overline{\alpha_3 (\overline{xy})}\}$.

Proposition 1.1. $(F_4)_{E_1} \cong Spin(9)$.

Proof. We define a 9-dimensional $\boldsymbol{R}$-vector space V^9 by

$$V^9 = \{X \in \mathfrak{J} \mid E_1 \circ X = 0,\ \mathrm{tr}(X) = 0\} = \left\{ \begin{pmatrix} 0 & 0 & 0 \\ 0 & \xi & x \\ 0 & \overline{x} & -\xi \end{pmatrix} \;\middle|\; \xi \in \boldsymbol{R},\ x \in \mathfrak{C} \right\}$$

with the norm $1/2(X, X) = \xi^2 + \overline{x}x$. Let $SO(9) = SO(V^9)$. Then, we have $(F_4)_{E_1}/\boldsymbol{Z}_2 \cong SO(9)$, $\boldsymbol{Z}_2 = \{1, \sigma\}$. Therefore, $(F_4)_{E_1}$ is isomorphic to $Spin(9)$ as a double covering group of $SO(9)$. (In detail, see [5], [8]). □

Now, we shall determine the group structure of $(Spin(9))^{\sigma'}$.

Theorem 1.2. $(Spin(9))^{\sigma'} \cong Spin(8)$.

Proof. Let $Spin(9) = (F_4)_{E_1}$. Then, the map $\varphi_1 \colon Spin(8) \to (Spin(9))^{\sigma'}$,

$$\varphi_1(\alpha_1, \alpha_2, \alpha_3)X = \begin{pmatrix} \xi_1 & \alpha_3 x_3 & \overline{\alpha_2 x_2} \\ \overline{\alpha_3 x_3} & \xi_2 & \alpha_1 x_1 \\ \alpha_2 x_2 & \overline{\alpha_1 x_1} & \xi_3 \end{pmatrix},\ X \in \mathfrak{J}$$

gives an isomorphism as groups. (In detail, see [3]). □

2. Group E_6

We use the same notation as in [5] (however, some will be rewritten). For example,

- the complex exceptional Jordan algebra $\mathfrak{J}^C = \{X \in M(3, \mathfrak{C}^C) \mid X^* = X\}$, the Freudenthal multiplication $X \times Y$ and the Hermitian inner product $\langle X, Y\rangle$,
- the group $E_6 = \{\alpha \in \mathrm{Iso}_C(\mathfrak{J}^C) \mid \alpha X \times \alpha Y = \tau\alpha\tau(X \times Y),\ \langle \alpha X, \alpha Y\rangle = \langle X, Y\rangle\}$, and the natural inclusion $F_4 \subset E_6$,

- any element ϕ of the Lie algebra $\mathfrak{e}_6$ of the group E_6 is uniquely expressed as $\phi = \delta + i\widetilde{T}$, $\delta \in \mathfrak{f}_4$, $T \in \mathfrak{J}_0$, where $\mathfrak{J}_0 = \{T \in \mathfrak{J} \mid \mathrm{tr}(T) = 0\}$.

Proposition 2.1. $(E_6)_{E_1} \cong Spin(10)$.

Proof. We define a 10-dimensional $\boldsymbol{R}$-vector space V^{10} by

$$V^{10} = \{X \in \mathfrak{J}^C \mid 2E_1 \times X = -\tau X\} = \left\{ \begin{pmatrix} 0 & 0 & 0 \\ 0 & \xi & x \\ 0 & \overline{x} & -\tau\xi \end{pmatrix} \,\middle|\, \xi \in C,\ x \in \mathfrak{C} \right\}$$

with the norm $1/2\langle X, X\rangle = (\tau\xi)\xi + \overline{x}x$. Let $SO(10) = SO(V^{10})$. Then, we have $(E_6)_{E_1}/\boldsymbol{Z}_2 \cong SO(10)$, $\boldsymbol{Z}_2 = \{1, \sigma\}$. Therefore, $(E_6)_{E_1}$ is isomorphic to $Spin(10)$ as a double covering group of $SO(10)$. (In detail, see [5], [8]). □

Lemma 2.2. *For $\nu \in Spin(2) = U(1) = \{\nu \in C \mid (\tau\nu)\nu = 1\}$, we define a C-linear transformation $\phi_1(\nu)$ of $\mathfrak{J}^C$ by*

$$\phi_1(\nu)X = \begin{pmatrix} \xi_1 & \nu x_3 & \nu^{-1}\overline{x}_2 \\ \nu\overline{x}_3 & \nu^2\xi_2 & x_1 \\ \nu^{-1}x_2 & \overline{x}_1 & \nu^{-2}\xi_3 \end{pmatrix}, \ X \in \mathfrak{J}^C.$$

Then, $\phi_1(\nu) \in ((E_6)_{E_1})^{\sigma'}$.

Lemma 2.3. *Any element ϕ of the Lie algebra $((\mathfrak{e}_6)_{E_1})^{\sigma'}$ of the group $((E_6)_{E_1})^{\sigma'}$ is expressed by*

$$\phi = \delta + it(E_2 - E_3)^{\sim},\ \delta \in ((\mathfrak{f}_4)_{E_1})^{\sigma'} = \mathfrak{so}(8),\ t \in \boldsymbol{R}.$$

In particular, we have

$$\dim(((\mathfrak{e}_6)_{E_1})^{\sigma'}) = 28 + 1 = 29.$$

Now, we shall determine the group structure of $(Spin(10))^{\sigma'}$.

Theorem 2.4.

$$(Spin(10))^{\sigma'} \cong (Spin(2) \times Spin(8))/\boldsymbol{Z}_2,\ \boldsymbol{Z}_2 = \{(1,1), (-1,\sigma)\}.$$

Proof. Let $Spin(10) = (E_6)_{E_1}$, $Spin(2) = U(1) \subset ((E_6)_{E_1})^{\sigma'}$ (Lemma 2.2) and $Spin(8) = ((F_4)_{E_1})^{\sigma'} \subset ((E_6)_{E_1})^{\sigma'}$ (Theorem 1.2, Proposition 2.1). Now, we define a map $\varphi\colon Spin(2) \times Spin(8) \to (Spin(10))^{\sigma'}$ by

$$\varphi(\nu, \beta) = \phi_1(\nu)\beta.$$

Then, φ is well-defined: $\varphi(\nu,\beta) \in (Spin(10))^{\sigma'}$. Since $\phi_1(\nu)$ and β are commutative, φ is a homomorphism. $\mathrm{Ker}\,\varphi = \{(1,1), (-1,\sigma)\}$. Since $(Spin(10))^{\sigma'}$ is connected and $\dim(\mathfrak{spin}(2) \oplus \mathfrak{spin}(8)) = 1 + 28 = 29 = \dim((\mathfrak{spin}(10)^{\sigma'}))$ (Lemma 2.3), φ is onto. Thus, we have the isomorphism

$$(Spin(2) \times Spin(8))/\boldsymbol{Z}_2 \cong (Spin(10))^{\sigma'}.$$ □

3. Group E_7

We use the same notation as in [6] (however, some will be rewritten). For example,

- the Freudenthal C-vector space $\mathfrak{P}^C = \mathfrak{J}^C \oplus \mathfrak{J}^C \oplus C \oplus C$, the Hermitian inner product $\langle P, Q \rangle$,
- for $P, Q \in \mathfrak{P}^C$, the C-linear map $P \times Q \colon \mathfrak{P}^C \to \mathfrak{P}^C$,
- the group $E_7 = \{\alpha \in \mathrm{Iso}_C(\mathfrak{P}^C) \mid \alpha(P \times Q)\alpha^{-1} = \alpha P \times \alpha Q,\ \langle \alpha P, \alpha Q \rangle = \langle P, Q \rangle\}$, the natural inclusion $E_6 \subset E_7$ and elements $\sigma, \sigma' \in F_4 \subset E_6 \subset E_7$, $\lambda \in E_7$,
- any element $\varPhi$ of the Lie algebra $\mathfrak{e}_7$ of the group E_7 is uniquely expressed as $\varPhi = \varPhi(\phi, A, -\tau A, \nu)$, $\phi \in \mathfrak{e}_6$, $A \in \mathfrak{J}^C$, $\nu \in i\boldsymbol{R}$.

In the following, the group $((Spin(10))^{\sigma'})_{F_1(x)}$ is defined by

$$((Spin(10))^{\sigma'})_{F_1(x)} = \{\alpha \in (Spin(10))^{\sigma'} \mid \alpha F_1(x) = F_1(x) \text{ for all } x \in \mathfrak{C}\},$$

where $F_1(x) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & x \\ 0 & \overline{x} & 0 \end{pmatrix} \in \mathfrak{J}$.

Proposition 3.1. $((Spin(10))^{\sigma'})_{F_1(x)} \cong Spin(2)$.

Proof. Let $Spin(10) = (E_6)_{E_1}$ and $Spin(2) = U(1) = \{\nu \in C \mid (\tau\nu)\nu = 1\}$. We consider the map $\phi_1 \colon Spin(2) \to ((Spin(10))^{\sigma'})_{F_1(x)}$ defined in Section 2. Then, ϕ_1 is well-defined: $\phi_1(\nu) \in ((Spin(10))^{\sigma'})_{F_1(x)}$. We shall show that ϕ_1 is onto. From $((Spin(10))^{\sigma'})_{F_1(x)} \subset (Spin(10))^{\sigma'}$, we see that for $\alpha \in ((Spin(10))^{\sigma'})_{F_1(x)}$, there exist $\nu \in Spin(2)$ and $\beta \in Spin(8)$ such that $\alpha = \varphi(\nu, \beta)$ (Theorem 2.4). Further, from $\alpha F_1(x) = F_1(x)$ and $\phi_1(\nu)F_1(x) = F_1(x)$, we have $\beta F_1(x) = F_1(x)$. Hence, $\beta = (1, 1, 1)$ or $(1, -1, -1) = \sigma$ by the principle of triality. Hence, $\alpha = \phi_1(\nu)$ or $\phi_1(\nu)\sigma$. However, in the latter case, from $\sigma = \phi_1(-1)$, we have $\alpha = \phi_1(\nu)\phi_1(-1) = \phi_1(-\nu)$. Therefore, ϕ_1 is onto. $\mathrm{Ker}\,\phi_1 = \{1\}$. Thus, we have the isomorphism

$$Spin(2) \cong ((Spin(10))^{\sigma'})_{F_1(x)}. \qquad \square$$

We define C-linear maps $\kappa, \mu \colon \mathfrak{P}^C \to \mathfrak{P}^C$ respectively by

$$\kappa(X, Y, \xi, \eta) = (-\kappa_1 X, \kappa_1 Y, -\xi, \eta),\ \ \kappa_1 X = (E_1, X)E_1 - 4E_1 \times (E_1 \times X),$$
$$\mu(X, Y, \xi, \eta) = (2E_1 \times Y + \eta E_1, 2E_1 \times X + \xi E_1, (E_1, Y), (E_1, X)).$$

Their explicit forms are

$$\kappa(X,Y,\xi,\eta) = \left(\begin{pmatrix} -\xi_1 & 0 & 0 \\ 0 & \xi_2 & x_1 \\ 0 & \overline{x}_1 & \xi_3 \end{pmatrix}, \begin{pmatrix} \eta_1 & 0 & 0 \\ 0 & -\eta_2 & -y_1 \\ 0 & -\overline{y}_1 & -\eta_3 \end{pmatrix}, -\xi, \eta \right),$$

$$\mu(X,Y,\xi,\eta) = \left(\begin{pmatrix} \eta & 0 & 0 \\ 0 & \eta_3 & -y_1 \\ 0 & -\overline{y}_1 & \eta_2 \end{pmatrix}, \begin{pmatrix} \xi & 0 & 0 \\ 0 & \xi_3 & -x_1 \\ 0 & -\overline{x}_1 & \xi_2 \end{pmatrix}, \eta_1, \xi_1 \right).$$

We define subgroup $(E_7)^{\kappa,\mu}$ of E_7 by

$$(E_7)^{\kappa,\mu} = \{\alpha \in E_7 \mid \kappa\alpha = \alpha\kappa,\ \mu\alpha = \alpha\mu\},$$

and also define subgroups $((E_7)^{\kappa,\mu})_{(0,E_1,0,1)}$, $((E_7)^{\kappa,\mu})_{(0,E_1,0,1),(0,-E_1,0,1)}$, $((E_7)^{\kappa,\mu})_{(E_1,0,1,0)}$ and $((E_7)^{\kappa,\mu})_{(E_1,0,1,0),(E_1,0,-1,0)}$ of E_7 by

$$\begin{aligned} &((E_7)^{\kappa,\mu})_{(0,E_1,0,1)} \\ &\qquad = \{\alpha \in (E_7)^{\kappa,\mu} \mid \alpha(0,E_1,0,1) = (0,E_1,0,1)\}, \\ &((E_7)^{\kappa,\mu})_{(0,E_1,0,1),(0,-E_1,0,1)} \\ &\qquad = \left\{ \alpha \in (E_7)^{\kappa,\mu} \;\middle|\; \begin{array}{l} \alpha(0,E_1,0,1) = (0,E_1,0,1) \\ \alpha(0,-E_1,0,1) = (0,-E_1,0,1) \end{array} \right\}, \\ &((E_7)^{\kappa,\mu})_{(E_1,0,1,0)} \\ &\qquad = \{\alpha \in (E_7)^{\kappa,\mu} \mid \alpha(E_1,0,1,0) = (E_1,0,1,0)\}, \\ &((E_7)^{\kappa,\mu})_{(E_1,0,1,0),(E_1,0,-1,0)} \\ &\qquad = \left\{ \alpha \in (E_7)^{\kappa,\mu} \;\middle|\; \begin{array}{l} \alpha(E_1,0,1,0) = (E_1,0,1,0) \\ \alpha(E_1,0,-1,0) = (E_1,0,-1,0) \end{array} \right\}. \end{aligned}$$

Proposition 3.2. (1) $((E_7)^{\kappa,\mu})_{(E_1,0,1,0)} = ((E_7)^{\kappa,\mu})_{(0,E_1,0,1)}$.

(2) $((E_7)^{\kappa,\mu})_{(E_1,0,1,0),(E_1,0,-1,0)} = ((E_7)^{\kappa,\mu})_{(0,E_1,0,1),(0,-E_1,0,1)}$.

Proof. (1) For $\alpha \in ((E_7)^{\kappa,\mu})_{(E_1,0,1,0)}$, we have

$$\alpha(0,E_1,0,1) = \alpha\mu(E_1,0,1,0) = \mu\alpha(E_1,0,1,0) = \mu(E_1,0,1,0) = (0,E_1,0,1).$$

Hence, $\alpha \in ((E_7)^{\kappa,\mu})_{(0,E_1,0,1)}$. The converse is also proved.

(2) It is proved in a way similar to (1). □

Proposition 3.3. $((E_7)^{\kappa,\mu})_{(0,E_1,0,1),(0,-E_1,0,1)} \cong Spin(10)$.

Proof. If $\alpha \in E_7$ satisfies $\alpha(0,E_1,0,1) = (0,E_1,0,1)$ and $\alpha(0,-E_1,0,1) = (0,-E_1,0,1)$, then we have $\alpha(0,0,0,1) = (0,0,0,1)$ and $\alpha(0,E_1,0,0) = (0,E_1,0,0)$. From the first condition, we see that $\alpha \in E_6$. Moreover, from the second condition, we have $\alpha \in (E_6)_{E_1} = Spin(10)$. The proof of the converse is trivial because κ,μ are defined by using E_1. □

Proposition 3.4. $((E_7)^{\kappa,\mu})_{(0,E_1,0,1)} \cong Spin(11)$.

Proof. We define an 11-dimensional $\boldsymbol{R}$-vector space V^{11} by

$$
\begin{aligned}
V^{11} &= \{P \in \mathfrak{P}^C \mid \kappa P = P,\ \mu\tau\lambda P = P,\ P \times (0, E_1, 0, 1) = 0\} \\
&= \left\{ \left(\begin{pmatrix} 0 & 0 & 0 \\ 0 & \xi & x \\ 0 & \overline{x} & -\tau\xi \end{pmatrix}, \begin{pmatrix} \eta & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, 0, \tau\eta \right) \;\middle|\; x \in \mathfrak{C},\ \xi \in C,\ \eta \in i\boldsymbol{R} \right\}
\end{aligned}
$$

with the norm

$$
(P, P)_\mu = \frac{1}{2}(\mu P, \lambda P) = (\tau\eta)\eta + \overline{x}x + (\tau\xi)\xi.
$$

Let $SO(11) = SO(V^{11})$. Then, we have $((E_7)^{\kappa,\mu})_{(0,E_1,0,1)}/\boldsymbol{Z}_2 \cong SO(11)$, $\boldsymbol{Z}_2 = \{1, \sigma\}$. Therefore, $((E_7)^{\kappa,\mu})_{(0,E_1,0,1)}$ is isomorphic to $Spin(11)$ as a double covering group of $SO(11)$. (In detail, see [6], [8]). □

Now, we shall consider the following group

$$
\begin{aligned}
&((Spin(11))^{\sigma'})_{(0,F_1(y),0,0)} \\
&= \{\alpha \in (Spin(11))^{\sigma'} \mid \alpha(0, F_1(y), 0, 0) = (0, F_1(y), 0, 0) \text{ for all } y \in \mathfrak{C}\}.
\end{aligned}
$$

Lemma 3.5. *The Lie algebra $((\mathfrak{spin}(11))^{\sigma'})_{(0,F_1(y),0,0)}$ of the group $((Spin(11))^{\sigma'})_{(0,F_1(y),0,0)}$ is given by*

$$
\begin{aligned}
&((\mathfrak{spin}(11))^{\sigma'})_{(0,F_1(y),0,0)} \\
&= \left\{ \varPhi\left(i\begin{pmatrix} 0 & 0 & 0 \\ 0 & \epsilon & 0 \\ 0 & 0 & -\epsilon \end{pmatrix}^{\sim}, \begin{pmatrix} 0 & 0 & 0 \\ 0 & \rho & 0 \\ 0 & 0 & \tau\rho \end{pmatrix}, -\tau\begin{pmatrix} 0 & 0 & 0 \\ 0 & \rho & 0 \\ 0 & 0 & \tau\rho \end{pmatrix}, 0 \right) \right. \\
&\qquad \left. \;\middle|\; \epsilon \in \boldsymbol{R},\ \rho \in C \right\}.
\end{aligned}
$$

In particular, we have

$$
\dim(((\mathfrak{spin}(11))^{\sigma'})_{(0,F_1(y),0,0)}) = 3.
$$

Lemma 3.6. *For $a \in \boldsymbol{R}$, the maps $\alpha_k(a)\colon \mathfrak{P}^C \to \mathfrak{P}^C$, $k = 1, 2, 3$ defined by*

$$
\alpha_k(a)\begin{pmatrix} X \\ Y \\ \xi \\ \eta \end{pmatrix} = \begin{pmatrix} (1 + (\cos a - 1)p_k)X - 2(\sin a)E_k \times Y + \eta(\sin a)E_k \\ 2(\sin a)E_k \times X + (1 + (\cos a - 1)p_k)Y - \xi(\sin a)E_k \\ ((\sin a)E_k, Y) + (\cos a)\xi \\ (-(\sin a)E_k, X) + (\cos a)\eta \end{pmatrix}
$$

belong to the group E_7, where $p_k\colon \mathfrak{J}^C \to \mathfrak{J}^C$ is defined by

$$
p_k(X) = (X, E_k)E_k + 4E_k \times (E_k \times X),\ X \in \mathfrak{J}^C.
$$

$\alpha_1(a), \alpha_2(b), \alpha_3(c)$ $(a, b, c \in \boldsymbol{R})$ commute with each other.

Proof. For $\varPhi_k(a) = \varPhi(0, aE_k, -aE_k, 0) \in \mathfrak{e}_7$, we have $\alpha_k(a) = \exp \varPhi_k(a) \in E_7$. Since $[\varPhi_k(a), \varPhi_l(b)] = 0$, $k \neq l$, $\alpha_k(a)$ and $\alpha_l(b)$ are commutative. $\square$

Lemma 3.7. *$((Spin(11))^{\sigma'})_{(0,F_1(y),0,0)}/Spin(2) \simeq S^2$.*
In particular, $((Spin(11))^{\sigma'})_{(0,F_1(y),0,0)}$ is connected.

Proof. We define a 3-dimensional $\boldsymbol{R}$-vector space W^3 by

$$\begin{aligned} W^3 &= \{P \in \mathfrak{P}^C \mid \kappa P = -P,\ \mu\tau\lambda P = -P,\ \sigma' P = P,\ P \times (E_1, 0, 1, 0) = 0\} \\ &= \left\{ P = \left(\begin{pmatrix} i\xi & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 0 & \eta & 0 \\ 0 & 0 & -\tau\eta \end{pmatrix}, -i\xi, 0 \right) \,\middle|\, \xi \in \boldsymbol{R},\ \eta \in C \right\} \end{aligned}$$

with the norm

$$(P, P)_\mu = -\frac{1}{2}(\mu P, \lambda P) = \xi^2 + (\tau\eta)\eta.$$

Then, $S^2 = \{P \in W^3 \mid (P, P)_\mu = 1\}$ is a 2-dimensional sphere. The group $((Spin(11))^{\sigma'})_{(0,F_1(y),0,0)}$ acts on S^2. We shall show that this action is transitive. To show this, it is sufficient to show that any element $P \in S^2$ can be transformed to $(-iE_1, 0, i, 0) \in S^2$ under the action of $((Spin(11))^{\sigma'})_{(0,F_1(y),0,0)}$. Now, for a given

$$P = \left(\begin{pmatrix} i\xi & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 0 & \eta & 0 \\ 0 & 0 & -\tau\eta \end{pmatrix}, -i\xi, 0 \right) \in S^2,$$

choose $a \in \boldsymbol{R}$, $0 \leq a < \pi/2$ such that $\tan 2a = -\dfrac{2i\xi}{\tau\eta - \eta}$ (if $\tau\eta - \eta = 0$, then let $a = \pi/4$). Operate $\alpha_{23}(a) := \alpha_2(a)\alpha_3(a) = \exp(\varPhi(0, a(E_2+E_3), -a(E_2+E_3), 0)) \in ((Spin(11))^{\sigma'})_{(0,F_1(y),0,0)}$ (Lemmas 3.5, 3.6) on P. Then, we have the ξ-term of $\alpha_{23}(a)P$ is $-((\cos 2a)(i\xi) + 1/2(\sin 2a)(\tau\eta - \eta)) = 0$. Hence,

$$\alpha_{23}(a)P = \left(0, \begin{pmatrix} 0 & 0 & 0 \\ 0 & \zeta & 0 \\ 0 & 0 & -\tau\zeta \end{pmatrix}, 0, 0 \right) = P_1,\ \zeta \in C,\ (\tau\zeta)\zeta = 1.$$

From $(\tau\zeta)\zeta = 1$, $\zeta \in C$, we can put $\zeta = e^{i\theta}$, $0 \leq \theta < 2\pi$. Let $\nu = e^{-i\theta/2}$, and operate $\phi_1(\nu) \in ((Spin(10))^{\sigma'})_{F_1(x)}$ (Lemma 2.2) $(\subset ((Spin(11)^{\sigma'})_{(0,F_1(x),0,0)})$ on P_1. Then,

$$\phi_1(\nu)P_1 = (0, E_2 - E_3, 0, 0) = P_2.$$

Moreover, operate $\phi_1(e^{i\pi/4})$ on P_2,

$$\phi_1(e^{i\pi/4})P_2 = (0, i(E_2 + E_3), 0, 0) = P_3.$$

Operate again $\alpha_{23}(\pi/4)$ on P_3. Then, we have

$$\alpha_{23}(\pi/4)P_3 = (-iE_1, 0, i, 0).$$

This shows the transitivity. The isotropy subgroup of $((Spin(11))^{\sigma'})_{(0,F_1(y),0,0)}$ at $(-iE_1, 0, i, 0)$ is $((Spin(10))^{\sigma'})_{F_1(y)}$ (Propositions 3.2 (2), 3.3, 3.4) $=$ $Spin(2)$. Thus, we have the homeomorphism

$$((Spin(11))^{\sigma'})_{(0,F_1(y),0,0)}/Spin(2) \simeq S^2. \qquad \square$$

Proposition 3.8. $((Spin(11))^{\sigma'})_{(0,F_1(y),0,0)} \cong Spin(3)$.

Proof. Since $((Spin(11))^{\sigma'})_{(0,F_1(y),0,0)}$ is connected (Lemma 3.7), we can define a homomorphism $\pi\colon ((Spin(11))^{\sigma'})_{(0,F_1(y),0,0)} \to SO(3) = SO(W^3)$ by

$$\pi(\alpha) = \alpha|W^3.$$

$\operatorname{Ker}\pi = \{1, \sigma\} = \boldsymbol{Z}_2$. Since $\dim(((\mathfrak{spin}(11))^{\sigma'})_{(0,F_1(y),0,0)}) = 3$ (Lemma 3.5) $= \dim(\mathfrak{so}(3))$, π is onto. Hence, $((Spin(11))^{\sigma'})_{(0,F_1(y),0,0)}/\boldsymbol{Z}_2 \cong SO(3)$. Therefore, $((Spin(11))^{\sigma'})_{(0,F_1(y),0,0)}$ is isomorphic to $Spin(3)$ as a double covering group of $SO(3)$. $\square$

Lemma 3.9. *The Lie algebra $(\mathfrak{spin}(11))^{\sigma'}$ of the group $(Spin(11))^{\sigma'}$ is given by*

$$\begin{aligned}
&(\mathfrak{spin}(11))^{\sigma'} \\
&= \left\{ \Phi\left(D + i\begin{pmatrix} 0 & 0 & 0 \\ 0 & \epsilon & 0 \\ 0 & 0 & -\epsilon \end{pmatrix}^{\sim}, \begin{pmatrix} 0 & 0 & 0 \\ 0 & \rho & 0 \\ 0 & 0 & \tau\rho \end{pmatrix}, -\tau\begin{pmatrix} 0 & 0 & 0 \\ 0 & \rho & 0 \\ 0 & 0 & \tau\rho \end{pmatrix}, 0 \right) \right. \\
&\qquad \left. \middle| \; D \in \mathfrak{so}(8),\ \epsilon \in \boldsymbol{R},\ \rho \in C \right\}.
\end{aligned}$$

In particular, we have

$$\dim((\mathfrak{spin}(11))^{\sigma'}) = 28 + 3 = 31.$$

Now, we shall determine the group structure of $(Spin(11))^{\sigma'}$.

Theorem 3.10.

$$(Spin(11))^{\sigma'} \cong (Spin(3) \times Spin(8))/\boldsymbol{Z}_2, \;\; \boldsymbol{Z}_2 = \{(1,1), (-1,\sigma)\}.$$

Proof. Let $Spin(11) = ((E_7)^{\kappa,\mu})_{(0,E_1,0,1)}$, $Spin(3) = ((Spin(11))^{\sigma'})_{(0,F_1(y),0,0)}$ and $Spin(8) = ((F_4)_{E_1})^{\sigma'} \subset ((E_6)_{E_1})^{\sigma'} = (((E_7)^{\kappa,\mu})_{(E_1,0,1,0),(E_1,0,-1,0)})^{\sigma'} \subset (((E_7)^{\kappa,\mu})_{(E_1,0,1,0)})^{\sigma'}$ (Theorem 1.2, Propositions 3.2, 3.3, 3.4). Now, we define a map $\varphi\colon Spin(3) \times Spin(8) \to (Spin(11))^{\sigma'}$ by

$$\varphi(\alpha, \beta) = \alpha\beta.$$

Then, φ is well-defined: $\varphi(\alpha, \beta) \in (Spin(11))^{\sigma'}$. Since $[\varPhi_D, \varPhi_3] = 0$ for $\varPhi_D = \varPhi(D, 0, 0, 0) \in \mathfrak{spin}(8)$, $\varPhi_3 \in \mathfrak{spin}(3) = ((\mathfrak{spin}(11))^{\sigma'})_{(0,F_1(y),0,0)}$ (Proposition 3.8), we have $\alpha\beta = \beta\alpha$. Hence, φ is a homomorphism. $\operatorname{Ker}\varphi = \{(1,1), (-1,\sigma)\} = \boldsymbol{Z}_2$. Since $(Spin(11))^{\sigma'}$ is connected and $\dim(\mathfrak{spin}(3) \oplus \mathfrak{spin}(8)) = 3$ (Lemma 3.5) $+28 = 31 = \dim((\mathfrak{spin}(11))^{\sigma'})$ (Lemma 3.9), φ is onto. Thus, we have the isomorphism

$$(Spin(3) \times Spin(8))/\boldsymbol{Z}_2 \cong (Spin(11))^{\sigma'}. \qquad \square$$

Proposition 3.11. $(E_7)^{\kappa,\mu} \cong Spin(12)$.

Proof. We define a 12-dimensional $\boldsymbol{R}$-vector space V^{12} by

$$\begin{aligned} V^{12} &= \{P \in \mathfrak{P}^C \mid \kappa P = P,\ \mu\tau\lambda P = P\} \\ &= \left\{ \left(\begin{pmatrix} 0 & 0 & 0 \\ 0 & \xi & x \\ 0 & \overline{x} & -\tau\xi \end{pmatrix}, \begin{pmatrix} \eta & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, 0, \tau\eta \right) \middle|\ x \in \mathfrak{C},\ \xi,\ \eta \in C \right\} \end{aligned}$$

with the norm

$$(P, P)_\mu = \frac{1}{2}(\mu P, \lambda P) = (\tau\eta)\eta + \overline{x}x + (\tau\xi)\xi.$$

Let $SO(12) = SO(V^{12})$. Then, we have $(E_7)^{\kappa,\mu}/\boldsymbol{Z}_2 \cong SO(12)$, $\boldsymbol{Z}_2 = \{1, \sigma\}$. Therefore, $(E_7)^{\kappa,\mu}$ is isomorphic to $Spin(12)$ as a double covering group of $SO(12)$. (In detail, see [6], [8]). $\square$

Now, we shall consider the following group

$$\begin{aligned} &((Spin(12))^{\sigma'})_{(0,F_1(y),0,0)} \\ &= \{\alpha \in (Spin(12))^{\sigma'} \mid \alpha(0, F_1(y), 0, 0) = (0, F_1(y), 0, 0) \text{ for all } y \in \mathfrak{C}\}. \end{aligned}$$

Lemma 3.12. *The Lie algebra* $((\mathfrak{spin}(12))^{\sigma'})_{(0,F_1(y),0,0)}$ *of the group* $((Spin(12))^{\sigma'})_{(0,F_1(y),0,0)}$ *is given by*

$$\begin{aligned} &((\mathfrak{spin}(12))^{\sigma'})_{(0,F_1(y),0,0)} \\ &= \left\{ \varPhi\left(i \begin{pmatrix} \epsilon_1 & 0 & 0 \\ 0 & \epsilon_2 & 0 \\ 0 & 0 & \epsilon_3 \end{pmatrix}^{\sim}, \begin{pmatrix} 0 & 0 & 0 \\ 0 & \rho_2 & 0 \\ 0 & 0 & \rho_3 \end{pmatrix}, -\tau \begin{pmatrix} 0 & 0 & 0 \\ 0 & \rho_2 & 0 \\ 0 & 0 & \rho_3 \end{pmatrix}, -\frac{3}{2}i\epsilon_1 \right) \right. \\ &\qquad \left. \middle|\ \epsilon_i \in \boldsymbol{R},\ \epsilon_1 + \epsilon_2 + \epsilon_3 = 0,\ \rho_i \in C \right\}. \end{aligned}$$

In particular, we have

$$\dim(((\mathfrak{spin}(12))^{\sigma'})_{(0,F_1(y),0,0)}) = 6.$$

Lemma 3.13. *For $t \in \boldsymbol{R}$, the map $\alpha(t)\colon \mathfrak{P}^C \to \mathfrak{P}^C$ defined by*

$$\alpha(t)(X, Y, \xi, \eta) = \left(\begin{pmatrix} e^{2it}\xi_1 & e^{it}x_3 & e^{it}\overline{x}_2 \\ e^{it}\overline{x}_3 & \xi_2 & x_1 \\ e^{it}x_2 & \overline{x}_1 & \xi_3 \end{pmatrix}, \begin{pmatrix} e^{-2it}\eta_1 & e^{-it}y_3 & e^{-it}\overline{y}_2 \\ e^{-it}\overline{y}_3 & \eta_2 & y_1 \\ e^{-it}y_2 & \overline{y}_1 & \eta_3 \end{pmatrix}, e^{-2it}\xi, e^{2it}\eta \right)$$

belongs to the group $((Spin(12))^{\sigma'})_{(0,F_1(y),0,0)}$.

Proof. For $\Phi = \Phi(2itE_1 \vee E_1, 0, 0, -2it) \in ((\mathfrak{spin}(12))^{\sigma'})_{(0,F_1(y),0,0)}$ (Lemma 3.12), we have $\alpha(t) = \exp \Phi \in ((Spin(12)^{\sigma'})_{(0,F_1(y),0,0)}$. □

Lemma 3.14. $((Spin(12))^{\sigma'})_{(0,F_1(y),0,0)}/Spin(3) \simeq S^3$.
In particular, $((Spin(12))^{\sigma'})_{(0,F_1(y),0,0)}$ is connected.

Proof. We define a 4-dimensional $\boldsymbol{R}$-vector space W^4 by

$$\begin{aligned} W^4 &= \{P \in \mathfrak{P}^C \mid \kappa P = -P,\ \mu\tau\lambda P = -P,\ \sigma' P = P\} \\ &= \left\{ P = \left(\begin{pmatrix} \xi & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 0 & \eta & 0 \\ 0 & 0 & -\tau\eta \end{pmatrix}, \tau\xi, 0 \right) \,\middle|\, \xi, \eta \in C \right\} \end{aligned}$$

with the norm

$$(P, P)_\mu = -\frac{1}{2}(\mu P, \lambda P) = (\tau\xi)\xi + (\tau\eta)\eta.$$

Then, $S^3 = \{P \in W^4 \mid (P, P)_\mu = 1\}$ is a 3-dimensional sphere. The group $((Spin(12))^{\sigma'})_{(0,F_1(y),0,0)}$ acts on S^3. We shall show that this action is transitive. To show this, it is sufficient to show that any element $P \in S^3$ can be transformed to $(E_1, 0, 1, 0) \in S^3$ under the action of $((Spin(12))^{\sigma'})_{(0,F_1(y),0,0)}$. Now, for a given

$$P = \left(\begin{pmatrix} \xi & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 0 & \eta & 0 \\ 0 & 0 & -\tau\eta \end{pmatrix}, \tau\xi, 0 \right) \in S^3,$$

choose $t \in \boldsymbol{R}$ such that $e^{2it}\xi \in i\boldsymbol{R}$. Operate $\alpha(t)$ (Lemma 3.13) on P. Then, we have

$$\alpha(t)P = P_1 \in S^2 \subset S^3.$$

Now, since $((Spin(11))^{\sigma'})_{(0,F_1(y),0,0)}$ $(\subset ((Spin(12))^{\sigma'})_{(0,F_1(y),0,0)})$ acts transitively on S^2 (Lemma 3.7), there exists $\beta \in ((Spin(11))^{\sigma'})_{(0,F_1(y),0,0)}$ such that

$$\beta P_1 = (-iE_1, 0, i, 0) = P_2.$$

Operate again $\alpha(\pi/4)$ on P_2. Then, we have

$$\alpha(\pi/4)P_2 = (E_1, 0, 1, 0).$$

This shows the transitivity. The isotropy subgroup of $((Spin(12))^{\sigma'})_{(0,F_1(y),0,0)}$ at $(E_1,0,1,0)$ is $((Spin(11))^{\sigma'})_{(0,F_1(y),0,0)}$ (Propositions 3.2 (1), 3.4, 3.11) $= Spin(3)$. Thus, we have the homeomorphism

$$((Spin(12))^{\sigma'})_{(0,F_1(y),0,0)}/Spin(3) \simeq S^3. \qquad \square$$

Proposition 3.15. $((Spin(12))^{\sigma'})_{(0,F_1(y),0,0)} \cong Spin(4)$.

Proof. Since $((Spin(12))^{\sigma'})_{(0,F_1(y),0,0)}$ is connected (Lemma 3.14), we can define a homomorphism $\pi\colon ((Spin(12))^{\sigma'})_{(0,F_1(y),0,0)} \to SO(4) = SO(W^4)$ by

$$\pi(\alpha) = \alpha|W^4.$$

$\operatorname{Ker}\pi = \{1,\sigma\} = \boldsymbol{Z}_2$. Since $\dim((\mathfrak{spin}(12))^{\sigma'})_{(0,F_1(y),0,0)}) = 6$ (Lemma 3.12) $= \dim(\mathfrak{so}(4))$, π is onto. Hence, $((Spin(12))^{\sigma'})_{(0,F_1(y),0,0)}/\boldsymbol{Z}_2 \cong SO(4)$. Therefore, $((Spin(12))^{\sigma'})_{(0,F_1(y),0,0)}$ is isomorphic to $Spin(4)$ as a double covering group of $SO(4)$. $\square$

Lemma 3.16. *The Lie algebra* $(\mathfrak{spin}(12))^{\sigma'}$ *of the group* $(Spin(12))^{\sigma'}$ *is given by*

$$\begin{aligned}
&(\mathfrak{spin}(12))^{\sigma'}\\
&= \left\{ \Phi\left(D + i\begin{pmatrix} \epsilon_1 & 0 & 0 \\ 0 & \epsilon_2 & 0 \\ 0 & 0 & \epsilon_3 \end{pmatrix}^{\sim}, \begin{pmatrix} 0 & 0 & 0 \\ 0 & \rho_2 & 0 \\ 0 & 0 & \rho_3 \end{pmatrix}, -\tau\begin{pmatrix} 0 & 0 & 0 \\ 0 & \rho_2 & 0 \\ 0 & 0 & \rho_3 \end{pmatrix}, -i\frac{3}{2}\epsilon_1 \right) \right. \\
&\qquad \left. \middle| \; D \in \mathfrak{so}(8),\ \epsilon_i \in \boldsymbol{R},\ \epsilon_1+\epsilon_2+\epsilon_3 = 0,\ \rho_i \in C \right\}.
\end{aligned}$$

In particular, we have

$$\dim((\mathfrak{spin}(12))^{\sigma'}) = 28 + 6 = 34.$$

Now, we shall determine the group structure of $(Spin(12))^{\sigma'}$.

Theorem 3.17.

$$(Spin(12))^{\sigma'} \cong (Spin(4)\times Spin(8))/\boldsymbol{Z}_2,\ \ \boldsymbol{Z}_2 = \{(1,1),(-1,\sigma)\}.$$

Proof. Let $Spin(12) = (E_7)^{\kappa,\mu}$, $Spin(4) = ((Spin(12))^{\sigma'})_{(0,F_1(y),0,0)}$ and $Spin(8) = ((F_4)_{E_1})^{\sigma'} \subset ((E_6)_{E_1})^{\sigma'} = (((E_7)^{\kappa,\mu})_{(E_1,0,1,0),(E_1,0,-1,0)})^{\sigma'} \subset ((E_7)^{\kappa,\mu})^{\sigma'}$ (Theorem 1.2, Propositions 3.2, 3.3, 3.11, 3.15). Now, we define a map $\varphi\colon Spin(4)\times Spin(8) \to (Spin(12))^{\sigma'}$ by

$$\varphi(\alpha,\beta) = \alpha\beta.$$

Then, φ is well-defined: $\varphi(\alpha,\beta) \in (Spin(12))^{\sigma'}$. Since $[\varPhi_D, \varPhi_4] = 0$ for $\varPhi_D = \varPhi(D,0,0,0) \in \mathfrak{spin}(8)$, $\varPhi_4 \in \mathfrak{spin}(4) = ((\mathfrak{spin}(12))^{\sigma'})_{(0,F_1(y),0,0)}$ (Proposition 3.15), we have $\alpha\beta = \beta\alpha$. Hence, φ is a homomorphism. $\operatorname{Ker}\varphi = \{(1,1),(-1,\sigma)\} = \boldsymbol{Z}_2$. Since $(Spin(12))^{\sigma'}$ is connected and $\dim(\mathfrak{spin}(4) \oplus \mathfrak{spin}(8)) = 6$ (Lemma 3.12) $+28 = 34 = \dim((\mathfrak{spin}(12))^{\sigma'})$ (Lemma 3.16), φ is onto. Thus, we have the isomorphism

$$(Spin(4) \times Spin(8))/\boldsymbol{Z}_2 \cong (Spin(12))^{\sigma'}. \qquad \square$$

4. Group E_8

We use the same notation as in [2], [4] (however, some will rewritten). For example,

- C-Lie algebra ${\mathfrak{e}_8}^C = {\mathfrak{e}_7}^C \oplus \mathfrak{P}^C \oplus \mathfrak{P}^C \oplus C \oplus C \oplus C$ and C-linear transformations $\lambda, \widetilde{\lambda}$ of ${\mathfrak{e}_8}^C$,
- the groups ${E_8}^C = \{\alpha \in \mathrm{Iso}_C({\mathfrak{e}_8}^C) \mid \alpha[R_1,R_2] = [\alpha R_1, \alpha R_2]\}$ and $E_8 = ({E_8}^C)^{\tau\widetilde{\lambda}} = \{\alpha \in {E_8}^C \mid \tau\widetilde{\lambda}\alpha = \alpha\tau\widetilde{\lambda}\}$.

For $\alpha \in E_7$, the map $\widetilde{\alpha} \colon {\mathfrak{e}_8}^C \to {\mathfrak{e}_8}^C$ is defined by

$$\widetilde{\alpha}(\varPhi, P, Q, r, u, v) = (\alpha\varPhi\alpha^{-1}, \alpha P, \alpha Q, r, u, v).$$

Then, $\widetilde{\alpha} \in E_8$ and we identify α with $\widetilde{\alpha}$. The group E_8 contains E_7 as a subgroup by

$$E_7 = \{\widetilde{\alpha} \in E_8 \mid \alpha \in E_7\} = (E_8)_{(0,0,0,0,1,0)}.$$

We define a C-linear map $\widetilde{\kappa} \colon {\mathfrak{e}_8}^C \to {\mathfrak{e}_8}^C$ by

$$\widetilde{\kappa} = \mathrm{ad}(\kappa, 0, 0, -1, 0, 0) = \mathrm{ad}(\varPhi(-2E_1 \vee E_1, 0, 0, -1), 0, 0, -1, 0, 0),$$

and 14-dimensional C-vector spaces $\mathfrak{g}_{-2}$ and $\mathfrak{g}_2$ by

$$\begin{aligned}
\mathfrak{g}_{-2} &= \{R \in {\mathfrak{e}_8}^C \mid \widetilde{\kappa} R = -2R\} \\
&= \{(\varPhi(0, \zeta E_1, 0, 0), (\xi_1 E_1, \eta_2 E_2 + \eta_3 E_3 + F_1(y), \xi, 0), 0, 0, u, 0) \\
&\qquad \mid \zeta, \xi_1, \eta_i, \xi, u \in C,\ y \in \mathfrak{C}^C\}, \\
\mathfrak{g}_2 &= \{R \in {\mathfrak{e}_8}^C \mid \widetilde{\kappa} R = 2R\} \\
&= \{(\varPhi(0, 0, \zeta E_1, 0), 0, (\xi_2 E_1 + \xi_3 E_3 + F_1(x), \eta_1 E_1, 0, \eta), 0, 0, v) \\
&\qquad \mid \zeta, \xi_i, \eta_1, \eta, v \in C,\ x \in \mathfrak{C}^C\}.
\end{aligned}$$

Further, we define two C-linear maps $\widetilde{\mu}_1 \colon {\mathfrak{e}_8}^C \to {\mathfrak{e}_8}^C$ and $\delta \colon \mathfrak{g}_2 \to \mathfrak{g}_2$ by

$$\widetilde{\mu}_1(\varPhi, P, Q, r, u, v) = (\mu_1 \varPhi {\mu_1}^{-1}, i\mu_1 Q, i\mu_1 P, -r, v, u),$$

where

$$\mu_1(X, Y, \xi, \eta) = \left(\begin{pmatrix} i\eta & x_3 & \overline{x}_2 \\ \overline{x}_3 & i\eta_3 & -iy_1 \\ x_2 & -i\overline{y}_1 & i\eta_2 \end{pmatrix}, \begin{pmatrix} i\xi & y_3 & \overline{y}_2 \\ \overline{y}_3 & i\xi_3 & -ix_1 \\ y_2 & -i\overline{x}_1 & i\xi_2 \end{pmatrix}, i\eta_1, i\xi_1 \right),$$

and

$$\begin{aligned} &\delta(\varPhi(0,0,\zeta E_1,0),0,(\xi_2E_2+\xi_3E_3+F_1(x),\eta_1E_1,0,\eta),0,0,v) \\ &\quad = (\varPhi(0,0,-vE_1,0),0,(\xi_2E_2+\xi_3E_3+F_1(x),\eta_1E_1,0,\eta),0,0,-\zeta). \end{aligned}$$

In particular, the explicit form of the map $\widetilde{\mu}_1 \colon \mathfrak{g}_{-2} \to \mathfrak{g}_2$ is given by

$$\begin{aligned} &\widetilde{\mu}_1(\varPhi(0,\zeta E_1,0,0),(\xi_1E_1,\eta_2E_2+\eta_3E_3+F_1(y),\xi,0),0,0,u,0) \\ &\quad = (\varPhi(0,0,\zeta E_1,0),0,(-\eta_3E_2-\eta_2E_3+F_1(y),-\xi E_1,0,-\xi_1),0,0,u). \end{aligned}$$

The composition map $\delta\widetilde{\mu}_1 \colon \mathfrak{g}_{-2} \to \mathfrak{g}_2$ of $\widetilde{\mu}_1$ and $\delta\widetilde{\mu}_1$ is denoted by $\widetilde{\mu}_\delta$:

$$\begin{aligned} &\widetilde{\mu}_\delta(\varPhi(0,\zeta E_1,0,0),(\xi_1E_1,\eta_2E_2+\eta_3E_3+F_1(y),\xi,0),0,0,u,0) \\ &\quad = (\varPhi(0,0,-uE_1,0),0,(-\eta_3E_2-\eta_2E_3+F_1(y),-\xi E_1,0,-\xi_1),0,0,-\zeta). \end{aligned}$$

Now, we define the inner product $(R_1, R_2)_\mu$ in $\mathfrak{g}_{-2}$ by

$$(R_1, R_2)_\mu = \frac{1}{30} B_8(\widetilde{\mu}_\delta R_1, R_2),$$

where B_8 is the Killing form of ${\mathfrak{e}_8}^C$. The explicit form of $(R, R)_\mu$ is given by

$$(R, R)_\mu = -4\zeta u - \eta_2\eta_3 + \overline{y}y + \xi_1\xi$$

for $R = (\varPhi(0,\zeta E_1,0,0),(\xi_1E_1,\eta_2E_2+\eta_3E_3+F_1(y),\xi,0),0,0,u,0) \in \mathfrak{g}_{-2}$. Hereafter, we use the notation $(V^C)^{14}$ instead of $\mathfrak{g}_{-2}$.

We define $\boldsymbol{R}$-vector spaces V^{14}, V^{13} and $(V')^{12}$ respectively by

$$\begin{aligned} V^{14} &= \{R \in (V^C)^{14} \mid \widetilde{\mu}_\delta\tau\widetilde{\lambda}R = -R\} \\ &= \{R = (\varPhi(0,\zeta E_1,0,0),(\xi E_1,\eta E_2-\tau\eta E_3+F_1(y),\tau\xi,0),0,0,-\tau\zeta,0) \\ &\qquad \mid \zeta,\xi,\eta \in C,\ y \in \mathfrak{C}\} \end{aligned}$$

with the norm

$$(R, R)_\mu = \frac{1}{30} B_8(\widetilde{\mu}_\delta R, R) = 4(\tau\zeta)\zeta + (\tau\eta)\eta + \overline{y}y + (\tau\xi)\xi,$$

$$\begin{aligned} V^{13} &= \{R \in V^{14} \mid (R, (\varPhi_1,0,0,0,1,0))_\mu = 0\} \\ &= \{R = (\varPhi(0,\zeta E_1,0,0),(\xi E_1,\eta E_2-\tau\eta E_3+F_1(y),\tau\xi,0),0,0,-\zeta,0) \\ &\qquad \mid \zeta \in \boldsymbol{R},\ \xi,\eta \in C,\ y \in \mathfrak{C}\} \end{aligned}$$

with the norm

$$(R, R)_\mu = \frac{1}{30} B_8(\widetilde{\mu}_\delta R, R) = 4\zeta^2 + (\tau\eta)\eta + \overline{y}y + (\tau\xi)\xi,$$

$$
\begin{aligned}
(V')^{12} &= \{R \in V^{13} \mid (R, (\varPhi_1, 0, 0, 0, -1, 0))_\mu = 0\} \\
&= \{R = (0, (\xi E_1, \eta E_2 - \tau\eta E_3 + F_1(y), \tau\xi, 0), 0, 0, 0, 0) \\
&\qquad \mid \xi, \eta \in C,\ y \in \mathfrak{C}\}
\end{aligned}
$$

with the norm

$$
(R, R)_\mu = \frac{1}{30} B_8(\widetilde{\mu}_\delta R, R) = (\tau\eta)\eta + \overline{y}y + (\tau\xi)\xi,
$$

where $\varPhi_1 = \varPhi(0, E_1, 0, 0)$. We use the notation $(V')^{12}$ to distinguish from the $\boldsymbol{R}$-vector space V^{12} defined in Section 3. The space $(V')^{12}$ above can be identified with the $\boldsymbol{R}$-vector space

$$
\begin{aligned}
&\{P \in \mathfrak{P}^C \mid \kappa P = -P,\ \mu\tau\lambda P = -P\} \\
=&\{P = (\xi E_1, \eta E_2 - \tau\eta E_3 + F_1(y), \tau\xi, 0) \in \mathfrak{P}^C \mid \xi, \eta \in C,\ y \in \mathfrak{C}\}
\end{aligned}
$$

with the norm

$$
(P, P)_\mu = -\frac{1}{2}(\mu P, \lambda P) = (\tau\eta)\eta + \overline{y}y + (\tau\xi)\xi.
$$

Now, we define a subgroup G_{14} of ${E_8}^C$ by

$$
G_{14} = \{\alpha \in {E_8}^C \mid \widetilde{\kappa}\alpha = \alpha\widetilde{\kappa},\ \widetilde{\mu}_\delta\alpha R = \alpha\widetilde{\mu}_\delta R,\ R \in (V^C)^{14}\}.
$$

Lemma 4.1. *The Lie algebra $\mathfrak{g}_{14}$ of the group G_{14} is given by*

$$
\begin{aligned}
\mathfrak{g}_{14} = \{&R \in {\mathfrak{e}_8}^C \\
&\mid \widetilde{\kappa}(\operatorname{ad} R) = (\operatorname{ad} R)\widetilde{\kappa},\ (\widetilde{\mu}_\delta(\operatorname{ad} R))R' = ((\operatorname{ad} R)\widetilde{\mu}_\delta)R',\ R' \in (V^C)^{14}\} \\
= \Biggl\{ &\Biggl(\varPhi\Biggl(D + \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & d_1 \\ 0 & -\overline{d}_1 & 0 \end{pmatrix}^{\sim} + \begin{pmatrix} \tau_1 & 0 & 0 \\ 0 & \tau_2 & t_1 \\ 0 & \overline{t}_1 & \tau_3 \end{pmatrix}^{\sim}, \begin{pmatrix} 0 & 0 & 0 \\ 0 & \alpha_2 & a_1 \\ 0 & \overline{a}_1 & \alpha_3 \end{pmatrix}, \\
&\begin{pmatrix} 0 & 0 & 0 \\ 0 & \beta_2 & b_1 \\ 0 & \overline{b}_1 & \beta_3 \end{pmatrix}, \nu \Biggr), \Biggl(\begin{pmatrix} 0 & 0 & 0 \\ 0 & \rho_2 & p_1 \\ 0 & \overline{p}_1 & \rho_3 \end{pmatrix}, \begin{pmatrix} \rho_1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, 0, \rho \Biggr), \\
&\Biggl(\begin{pmatrix} \zeta_1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 0 & \zeta_2 & z_1 \\ 0 & \overline{z}_1 & \zeta_3 \end{pmatrix}, \zeta, 0 \Biggr), r, 0, 0 \Biggr) \\
&\Biggl|\ D \in \mathfrak{so}(8)^C,\ \tau_i, \alpha_i, \beta_i, \nu, \rho_i, \rho, \zeta_i, \zeta, r \in C,\ \tau_1 + \tau_2 + \tau_3 = 0, \\
&d_1, t_1, a_1, b_1, p_1, z_1 \in \mathfrak{C}^C,\ \tau_1 + \frac{2}{3}\nu + 2r = 0 \Biggr\}.
\end{aligned}
$$

In particular, we have

$$
\dim_C(\mathfrak{g}_{14}) = 28 + 63 = 91.
$$

Proposition 4.2. $G_{14} \cong Spin(14, C)$.

Proof. Let $SO(14, C) = SO((V^{14})^C)$. Then, we have $G_{14}/\boldsymbol{Z}_2 \cong SO(14, C)$, $\boldsymbol{Z}_2 = \{1, \sigma\}$. Therefore, G_{14} is isomorphic to $Spin(14, C)$ as a double covering group of $SO(14, C)$. (In detail, see [2]). □

We define subgroups ${G_{14}}^{\mathrm{com}}$, ${G_{13}}^{\mathrm{com}}$ and ${G_{12}}^{\mathrm{com}}$ of the group E_8 by

$$\begin{aligned}
{G_{14}}^{\mathrm{com}} &= \{\alpha \in G_{14} \mid \tau\widetilde{\lambda}\alpha = \alpha\tau\widetilde{\lambda}\},\\
{G_{13}}^{\mathrm{com}} &= \{\alpha \in {G_{14}}^{\mathrm{com}} \mid \alpha(\Phi_1, 0, 0, 0, 1, 0) = (\Phi_1, 0, 0, 0, 1, 0)\},\\
{G_{12}}^{\mathrm{com}} &= \{\alpha \in {G_{13}}^{\mathrm{com}} \mid \alpha(\Phi_1, 0, 0, 0, -1, 0) = (\Phi_1, 0, 0, 0, -1, 0)\},
\end{aligned}$$

respectively.

Lemma 4.3. *$\alpha \in (E_7)^{\kappa,\mu} = Spin(12)$ satisfies*

$$\alpha\Phi(0, E_1, 0, 0)\alpha^{-1} = \Phi(0, E_1, 0, 0) \text{ and } \alpha\Phi(0, 0, E_1, 0)\alpha^{-1} = \Phi(0, 0, E_1, 0).$$

Proof. We consider an 11-dimensional sphere $(S')^{11}$ by

$$\begin{aligned}
(S')^{11} &= \{P' \in (V')^{12} \mid (P, P)_\mu = 1\}\\
&= \{P' = (\xi E_1, \eta E_2 - \tau\eta E_3 + F_1(y), \tau\xi, 0)\\
&\qquad \mid \xi, \eta \in C,\ y \in \mathfrak{C},\ (\tau\eta)\eta + \overline{y}y + (\tau\xi)\xi = 1\}.
\end{aligned}$$

Since the group $Spin(12)$ acts on $(S')^{11}$, we can put

$$\alpha(E_1, 0, 1, 0) = (\xi E_1, \eta E_2 - \tau\eta E_3 + F_1(y), \tau\xi, 0) \in (S')^{11}.$$

Now, since $1/2\Phi(0, E_1, 0, 0) = (E_1, 0, 1, 0) \times (E_1, 0, 1, 0)$, we have

$$\begin{aligned}
&1/2\alpha\Phi(0, E_1, 0, 0)\alpha^{-1}\\
&= \alpha((E_1, 0, 1, 0) \times (E_1, 0, 1, 0))\alpha^{-1}\\
&= \alpha(E_1, 0, 1, 0) \times \alpha(E_1, 0, 1, 0)\\
&= (\xi E_1, \eta E_2 - \tau\eta E_3 + F_1(y), \tau\xi, 0) \times (\xi E_1, \eta E_2 - \tau\eta E_3 + F_1(y), \tau\xi, 0)\\
&= 1/2\Phi(0, ((\tau\eta)\eta + \overline{y}y + (\tau\xi)\xi)E_1, 0, 0).
\end{aligned}$$

Since $\alpha(E_1, 0, 1, 0) \in (S')^{11}$, we have $(\tau\eta)\eta + \overline{y}y + (\tau\xi)\xi = 1$. Thus, we obtain $\alpha(E_1, 0, 1, 0) \times \alpha(E_1, 0, 1, 0) = 1/2\Phi(0, E_1, 0, 0)$, that is, $\alpha\Phi(0, E_1, 0, 0)\alpha^{-1} = \Phi(0, E_1, 0, 0)$. Since $\alpha \in Spin(12) \subset E_7$ satisfies $\alpha\tau\lambda = \tau\lambda\alpha$, we have also $\alpha\Phi(0, 0, E_1, 0)\alpha^{-1} = \Phi(0, 0, E_1, 0)$. □

Proposition 4.4. ${G_{12}}^{\mathrm{com}} = Spin(12)$.

Proof. Now, let $\alpha \in {G_{12}}^{\mathrm{com}}$. From

$$\alpha(\Phi_1, 0, 0, 0, 1, 0) = (\Phi_1, 0, 0, 0, 1, 0),\ \alpha(\Phi_1, 0, 0, 0, -1, 0) = (\Phi_1, 0, 0, 0, -1, 0),$$

we have $\alpha(0, 0, 0, 0, 1, 0) = (0, 0, 0, 0, 1, 0)$. Hence, since $\alpha \in {G_{12}}^{\mathrm{com}} \subset E_8$, we see that $\alpha \in E_7$. We first show that $\kappa\alpha = \alpha\kappa$. Since ${G_{12}}^{\mathrm{com}} \subset E_7$, it suffices to consider the actions on $\mathfrak{P}^C$. Since $\alpha \in {G_{12}}^{\mathrm{com}}$ satisfies $\widetilde{\kappa}\alpha = \alpha\widetilde{\kappa}$, from

$$\widetilde{\kappa}\alpha P = \kappa\alpha P - \alpha P \text{ and } \alpha\widetilde{\kappa}P = \alpha\kappa P - \alpha P,\ \ P \in \mathfrak{P}^C,$$

we have $\kappa\alpha = \alpha\kappa$. Next, we show that $\mu\alpha = \alpha\mu$. Again, from

$$\alpha(\varPhi_1, 0, 0, 0, 1, 0) = (\varPhi_1, 0, 0, 0, 1, 0),\ \ \alpha(\varPhi_1, 0, 0, 0, -1, 0) = (\varPhi_1, 0, 0, 0, -1, 0),$$

we have $\alpha(\varPhi_1, 0, 0, 0, 0, 0) = (\varPhi_1, 0, 0, 0, 0, 0)$. Hence, since $\alpha \in E_7$, we have $\alpha\varPhi_1\alpha^{-1} = \varPhi_1$, that is, $\alpha\varPhi(0, E_1, 0, 0)\alpha^{-1} = \varPhi(0, E_1, 0, 0)$. Consequently

$$\begin{aligned}
\alpha(\varPhi(0, 0, E_1, 0), 0, 0, 0, 0, 1) &= \alpha(-\widetilde{\mu}_\delta(\varPhi(0, E_1, 0, 0), 0, 0, 0, 1, 0)) \\
&= -\widetilde{\mu}_\delta\alpha(\varPhi(0, E_1, 0, 0), 0, 0, 0, 1, 0) \\
&= -\widetilde{\mu}_\delta(\varPhi(0, E_1, 0, 0), 0, 0, 0, 1, 0) \\
&= (\varPhi(0, 0, E_1, 0), 0, 0, 0, 0, 1).
\end{aligned}$$

Similarly, we have

$$\alpha(\varPhi(0, 0, E_1, 0), 0, 0, 0, 0, -1) = (\varPhi(0, 0, E_1, 0), 0, 0, 0, 0, -1).$$

Hence, we have

$$\alpha(\varPhi(0, 0, E_1, 0), 0, 0, 0, 0, 0) = (\varPhi(0, 0, E_1, 0), 0, 0, 0, 0, 0).$$

Moreover, from $\alpha \in E_7$, we have $\alpha\varPhi(0, 0, E_1, 0)\alpha^{-1} = \varPhi(0, 0, E_1, 0)$. Hence, put together with $\alpha\varPhi(0, E_1, 0, 0)\alpha^{-1} = \varPhi(0, E_1, 0, 0)$, we have $\alpha\varPhi(0, E_1, E_1, 0)\alpha^{-1} = \varPhi(0, E_1, E_1, 0)$, that is, $\alpha\mu\alpha^{-1} = \mu$. Thus, we have $\mu\alpha = \alpha\mu$. Therefore, $\alpha \in (E_7)^{\kappa,\mu} = Spin(12)$.

Conversely, let $\alpha \in Spin(12)$. For $R \in {\mathfrak{e}_8}^C$,

$$\begin{aligned}
\widetilde{\kappa}\alpha R &= [(\kappa, 0, 0, -1, 0, 0), (\alpha\varPhi\alpha^{-1}, \alpha P, \alpha Q, r, u, v)] \\
&= ([\kappa, \alpha\varPhi\alpha^{-1}], \kappa\alpha P - \alpha P, \kappa\alpha Q + \alpha Q, 0, -2u, 2v)
\end{aligned}$$

and

$$\begin{aligned}
\alpha\widetilde{\kappa}R &= \alpha[((\kappa, 0, 0, -1, 0, 0), (\varPhi, P, Q, r, u, v)] \\
&= [\alpha(\kappa, 0, 0, -1, 0, 0), \alpha(\varPhi, P, Q, r, u, v)] \\
&= ([\alpha\kappa\alpha^{-1}, \alpha\varPhi\alpha^{-1}], \alpha\kappa\alpha^{-1}(\alpha P) - \alpha P, \alpha\kappa\alpha^{-1}(\alpha Q) + \alpha Q, 0, -2u, 2v).
\end{aligned}$$

From $\kappa\alpha = \alpha\kappa$, we have $[\alpha\kappa\alpha^{-1}, \alpha\varPhi\alpha^{-1}] = [\kappa, \alpha\varPhi\alpha^{-1}]$. Thus, we have $\widetilde{\kappa}\alpha R = \alpha\widetilde{\kappa}R$, that is, $\widetilde{\kappa}\alpha = \alpha\widetilde{\kappa}$. Next, from $\mu\alpha = \alpha\mu$ and Lemma 4.3, we have

$$\mu_1(\alpha\varPhi_1\alpha^{-1}){\mu_1}^{-1} = \alpha(\mu_1\varPhi_1{\mu_1}^{-1})\alpha^{-1} = \alpha\varPhi(0, 0, E_1, 0)\alpha^{-1} = \varPhi(0, 0, E_1, 0).$$

Hence, for $R=(\zeta\varPhi_1,P,0,0,u,0)\in(V^C)^{14}$,

$$\begin{aligned}\widetilde{\mu}_\delta\alpha R&=\widetilde{\mu}_\delta(\zeta\alpha\varPhi_1\alpha^{-1},\alpha P,0,0,u,0)\\&=(\varPhi(0,0,-uE_1,0),0,i\mu_1\alpha P,0,0,-\zeta)\end{aligned}$$

and

$$\begin{aligned}\alpha\widetilde{\mu}_\delta R&=\alpha(\varPhi(0,0,-uE_1,0),0,i\mu_1P,0,0,-\zeta)\\&=(\alpha\varPhi(0,0,-uE_1,0)\alpha^{-1},0,i\alpha\mu_1P,0,0,-\zeta)\\&=(\varPhi(0,0,-uE_1,0),0,i\alpha\mu_1P,0,0,-\zeta).\end{aligned}$$

Hence, from $\mu\alpha=\alpha\mu$, we have $\widetilde{\mu}_\delta\alpha R=\alpha\widetilde{\mu}_\delta R$, $R\in(V^C)^{14}$. From Lemma 4.3, we have $\alpha(\varPhi_1,0,0,0,0,0)=(\varPhi_1,0,0,0,0,0)$. Moreover, since $\alpha\in E_7$, we have

$$\alpha(0,0,0,0,1,0)=(0,0,0,0,1,0),\ \ \alpha(0,0,0,0,-1,0)=(0,0,0,0,-1,0).$$

Hence, we have

$$\alpha(\varPhi_1,0,0,0,1,0)=(\varPhi_1,0,0,0,1,0),\ \ \alpha(\varPhi_1,0,0,0,-1,0)=(\varPhi_1,0,0,0,-1,0).$$

Therefore, $\alpha\in {G_{12}}^{\rm com}$. Thus, the proof of the proposition is completed. □

Lemma 4.5. *The Lie algebras ${\mathfrak{g}_{14}}^{\rm com}$ and ${\mathfrak{g}_{13}}^{\rm com}$ of the groups ${G_{14}}^{\rm com}$ and ${G_{13}}^{\rm com}$ are given respectively by*

$$\begin{aligned}{\mathfrak{g}_{14}}^{\rm com}&=\{R\in\mathfrak{g}_{14}\mid\tau\widetilde{\lambda}(\mathrm{ad}\,R)=(\mathrm{ad}\,R)\tau\widetilde{\lambda}\}\\
&=\left\{\left(\varPhi\left(D+\begin{pmatrix}0&0&0\\0&0&d_1\\0&-\overline{d}_1&0\end{pmatrix}^{\sim}+i\begin{pmatrix}\epsilon_1&0&0\\0&\epsilon_2&t_1\\0&\overline{t}_1&\epsilon_3\end{pmatrix}^{\sim},\begin{pmatrix}0&0&0\\0&\rho_2&a_1\\0&\overline{a}_1&\rho_3\end{pmatrix},\right.\right.\right.\\
&\qquad\left.-\tau\begin{pmatrix}0&0&0\\0&\rho_2&a_1\\0&\overline{a}_1&\rho_3\end{pmatrix},\nu\right),\left(\begin{pmatrix}0&0&0\\0&\zeta_2&z_1\\0&\overline{z}_1&\zeta_3\end{pmatrix},\begin{pmatrix}\zeta_1&0&0\\0&0&0\\0&0&0\end{pmatrix},0,\zeta\right),\\
&\qquad\left.-\tau\lambda\left(\begin{pmatrix}0&0&0\\0&\zeta_2&z_1\\0&\overline{z}_1&\zeta_3\end{pmatrix},\begin{pmatrix}\zeta_1&0&0\\0&0&0\\0&0&0\end{pmatrix},0,\zeta\right),r,0,0\right)\\
&\qquad\left|\ D\in\mathfrak{so}(8),\ \epsilon_i\in\boldsymbol{R},\ \rho_i,\zeta_i,\zeta\in C,\ \nu,r\in i\boldsymbol{R},\ \epsilon_1+\epsilon_2+\epsilon_3=0,\right.\\
&\qquad\left.i\epsilon_1+\frac{2}{3}\nu+2r=0,\ d_1,t_1\in\mathfrak{C},\ a_1,z_1\in\mathfrak{C}^C\right\},\end{aligned}$$

$$\begin{aligned}
{\mathfrak{g}_{13}}^{\mathrm{com}} &= \{R \in {\mathfrak{g}_{14}}^{\mathrm{com}} \mid (\operatorname{ad} R)(\varPhi_1, 0,0,0,1,0) = 0\} \\
&= \left\{ \left(\varPhi\left(D + \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & d_1 \\ 0 & -\overline{d}_1 & 0 \end{pmatrix}^{\sim} + i \begin{pmatrix} \epsilon_1 & 0 & 0 \\ 0 & \epsilon_2 & t_1 \\ 0 & \overline{t}_1 & \epsilon_3 \end{pmatrix}^{\sim}, \begin{pmatrix} 0 & 0 & 0 \\ 0 & \rho_2 & a_1 \\ 0 & \overline{a}_1 & \rho_3 \end{pmatrix}, \right.\right.\right. \\
&\quad \left. - \tau \begin{pmatrix} 0 & 0 & 0 \\ 0 & \rho_2 & a_1 \\ 0 & \overline{a}_1 & \rho_3 \end{pmatrix}, \nu \right), \left(\begin{pmatrix} 0 & 0 & 0 \\ 0 & \zeta_2 & z_1 \\ 0 & \overline{z}_1 & -\tau\zeta_2 \end{pmatrix}, \begin{pmatrix} \zeta_1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, 0, \tau\zeta_1 \right), \\
&\quad \left. - \tau\lambda \left(\begin{pmatrix} 0 & 0 & 0 \\ 0 & \zeta_2 & z_1 \\ 0 & \overline{z}_1 & -\tau\zeta_2 \end{pmatrix}, \begin{pmatrix} \zeta_1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, 0, \tau\zeta_1 \right), 0, 0, 0 \right) \\
&\qquad \left| \; D \in \mathfrak{so}(8),\ \epsilon_i \in \boldsymbol{R},\ \rho_i, \zeta_i \in C,\ \nu \in i\boldsymbol{R},\ \epsilon_1 + \epsilon_2 + \epsilon_3 = 0, \right. \\
&\qquad \left. i\epsilon_1 + \frac{2}{3}\nu = 0,\ d_1, t_1, z_1 \in \mathfrak{C},\ a_1 \in \mathfrak{C}^C \right\}.
\end{aligned}$$

In particular, we have

$$\dim({\mathfrak{g}_{14}}^{\mathrm{com}}) = 28 + 63 = 91, \quad \dim({\mathfrak{g}_{13}}^{\mathrm{com}}) = 28 + 50 = 78.$$

Lemma 4.6. (1) *For $a \in \mathfrak{C}$, we define a C-linear transformation $\epsilon_{13}(a)$ of ${\mathfrak{e}_8}^C$ by*

$$\epsilon_{13}(a) = \exp(\operatorname{ad}(0, (F_1(a), 0,0,0), (0, F_1(a), 0, 0), 0, 0, 0)).$$

Then, $\epsilon_{13}(a) \in {G_{13}}^{\mathrm{com}}$ (Lemma 4.5). The action of $\epsilon_{13}(a)$ on V^{13} is given by

$$\begin{aligned}
&\epsilon_{13}(a)(\varPhi(0, \zeta E_1, 0, 0), (\xi E_1, \eta E_2 - \tau\eta E_3 + F_1(y), \tau\xi, 0), 0, 0, -\zeta, 0) \\
&= (\varPhi(0, \zeta' E_1, 0, 0), (\xi' E_1, \eta' E_2 - \tau\eta' E_3 + F_1(y'), \tau\xi', 0), 0, 0, -\zeta', 0),
\end{aligned}$$

$$\begin{cases}
\zeta' &= \zeta \cos|a| - \dfrac{(a, y)}{2|a|} \sin|a|, \\
\xi' &= \xi, \\
\eta' &= \eta, \\
y' &= y + \dfrac{2\zeta a}{|a|} \sin|a| - \dfrac{2(a, y)a}{|a|^2} \sin^2 \dfrac{|a|}{2}.
\end{cases}$$

(2) *For $t \in \boldsymbol{R}$, we define a C-linear transformation $\theta_{13}(t)$ of ${\mathfrak{e}_8}^C$ by*

$$\theta_{13}(t) = \exp(\operatorname{ad}(0, (0, -tE_1, 0, -t), (tE_1, 0, t, 0), 0, 0, 0)).$$

Then, $\theta_{13}(t) \in {G_{13}}^{\rm com}$ (Lemma 4.5). The action of $\theta_{13}(t)$ on V^{13} is given by

$$\begin{aligned}
&\theta_{13}(t)(\varPhi(0,\zeta E_1,0,0),(\xi E_1,\eta E_2-\tau\eta E_3+F_1(y),\tau\xi,0),0,0,-\zeta,0)\\
&=(\varPhi(0,\zeta' E_1,0,0),(\xi' E_1,\eta' E_2-\tau\eta' E_3+F_1(y'),\tau\xi',0),0,0,-\zeta',0),
\end{aligned}$$

$$\begin{cases}
\zeta' &= \zeta\cos t-\dfrac{1}{4}(\tau\xi+\xi)\sin t,\\
\xi' &= \dfrac{1}{2}(\xi-\tau\xi)+\dfrac{1}{2}(\xi+\tau\xi)\cos t+2\zeta\sin t,\\
\eta' &= \eta,\\
y' &= y.
\end{cases}$$

Lemma 4.7. ${G_{13}}^{\rm com}/{G_{12}}^{\rm com} \simeq S^{12}$.
In particular, ${G_{13}}^{\rm com}$ is connected.

Proof. Let $S^{12}=\{R\in V^{13} \mid (R,R)_\mu=1\}$. The group ${G_{13}}^{\rm com}$ acts on $(S^C)^{12}$. We shall show that this action is transitive. To prove this, it suffices to show that any $R\in S^{12}$ can be transformed to $1/2(\varPhi_1,0,0,0,-1,0)\in S^{12}$. Now, for a given

$$R=(\varPhi(0,\zeta E_1,0,0),(\xi E_1,\eta E_2-\tau\eta E_3+F_1(y),\tau\xi,0),0,0,-\zeta,0)\in S^{12},$$

choose $a\in\mathfrak{C}$ such that $|a|=\pi/2$, $(a,y)=0$. Operate $\epsilon_{13}(a)\in {G_{13}}^{\rm com}$ (Lemma 4.6 (1)) on R. Then, we have

$$\epsilon_{13}(a)R=(0,(\xi E_1,\eta E_2-\tau\eta E_3+F_1(y'),\tau\xi,0),0,0,0,0)=R_1\in(S')^{11}\subset S^{12},$$

where $(S')^{11}=\{R\in(V')^{12}\mid(R,R)_\mu=1\}$. Here, since the group $Spin(12)$ ($\subset {G_{13}}^{\rm com}$) acts transitively on $S^{11}=\{P\in V^{12}\mid(P,P)_\mu=1\}$, there exists $\beta\in Spin(12)$ such that $\beta P=(0,E_1,0,1)$ for any $P\in S^{11}$. Hence, we have

$$\begin{aligned}
\beta R_1 &= \beta(0,P',0,0,0,0)=(0,\beta P',0,0,0,0)\\
&=(0,\beta\mu P,0,0,0,0)=(0,\mu\beta P,0,0,0,0)\\
&=(0,\mu(0,E_1,0,1),0,0,0,0)=(0,(E_1,0,1,0),0,0,0,0)\\
&=R_2\in(S')^{11},
\end{aligned}$$

where $P\in S^{11}$.
Finally, operate $\theta_{13}(-\pi/2)\in {G_{13}}^{\rm com}$ (Lemma 4.6 (2)) on R_2. Then, we have

$$\theta_{13}(-\pi/2)R_2=\frac{1}{2}(\varPhi_1,0,0,0,-1,0).$$

This shows the transitivity. The isotropy subgroup at $1/2(\varPhi_1,0,0,0,-1,0)$ of ${G_{13}}^{\rm com}$ is obviously ${G_{12}}^{\rm com}$. Thus, we have the homeomorphism

$${G_{13}}^{\rm com}/{G_{12}}^{\rm com}\simeq S^{12}. \qquad \square$$

Proposition 4.8. ${G_{13}}^{\mathrm{com}} \cong \mathit{Spin}(13)$.

Proof. Since the group ${G_{13}}^{\mathrm{com}}$ is connected (Lemma 4.7), we can define a homomorphism $\pi\colon {G_{13}}^{\mathrm{com}} \to SO(13) = SO(V^{13})$ by

$$\pi(\alpha) = \alpha|V^{13}.$$

$\mathrm{Ker}\,\pi = \{1, \sigma\} = \boldsymbol{Z}_2$. Since $\dim({\mathfrak{g}_{13}}^{\mathrm{com}}) = 78$ (Lemma 4.5) $= \dim(\mathfrak{so}(13))$, π is onto. Hence, ${G_{13}}^{\mathrm{com}}/\boldsymbol{Z}_2 \cong SO(13)$. Therefore, ${G_{13}}^{\mathrm{com}}$ is isomorphic to $\mathit{Spin}(13)$ as a double covering group of $SO(13) = SO(V^{13})$. $\square$

Proposition 4.9. ${G_{14}}^{\mathrm{com}} \cong \mathit{Spin}(14)$.

Proof. Since the group ${G_{14}}^{\mathrm{com}}$ acts on V^{14} and ${G_{14}}^{\mathrm{com}}$ is connected (Proposition 4.2), we can define a homomorphism $\pi\colon {G_{14}}^{\mathrm{com}} \to SO(14) = SO(V^{14})$ by

$$\pi(\alpha) = \alpha|V^{14}.$$

$\mathrm{Ker}\,\pi = \{1, \sigma\} = \boldsymbol{Z}_2$. Since $\dim({\mathfrak{g}_{14}}^{\mathrm{com}}) = 91$ (Lemma 4.5) $= \dim(\mathfrak{so}(14))$, π is onto. Hence, ${G_{14}}^{\mathrm{com}}/\boldsymbol{Z}_2 \cong SO(14)$. Therefore, ${G_{14}}^{\mathrm{com}}$ is isomorphic to $\mathit{Spin}(14)$ as a double covering group of $SO(14) = SO(V^{14})$. $\square$

Now, we shall consider the following group

$$((\mathit{Spin}(13))^{\sigma'})_{(0,F_1(y),0,0)^-} = \left\{ \alpha \in (\mathit{Spin}(13))^{\sigma'} \;\middle|\; \begin{array}{l} \alpha(0,(0,F_1(y),0,0),0,0,0,0) \\ = (0,(0,F_1(y),0,0),0,0,0,0) \end{array} \text{ for all } y \in \mathfrak{C} \right\}.$$

Lemma 4.10. *The Lie algebra* $((\mathfrak{spin}(13))^{\sigma'})_{(0,F_1(y),0,0)^-}$ *of the group* $((\mathit{Spin}(13))^{\sigma'})_{(0,F_1(y),0,0)^-}$ *is given by*

$$\begin{aligned}
&((\mathfrak{spin}(13))^{\sigma'})_{(0,F_1(y),0,0)^-} \\
&= \{R \in (\mathfrak{spin}(13))^{\sigma'} \mid (\mathrm{ad}\,R)(0,(0,F_1(y),0,0),0,0,0,0) = 0\} \\
&= \left\{ \left(\Phi\left(i \begin{pmatrix} \epsilon_1 & 0 & 0 \\ 0 & \epsilon_2 & 0 \\ 0 & 0 & \epsilon_3 \end{pmatrix}^{\sim}, \begin{pmatrix} 0 & 0 & 0 \\ 0 & \rho_2 & 0 \\ 0 & 0 & \rho_3 \end{pmatrix}, -\tau \begin{pmatrix} 0 & 0 & 0 \\ 0 & \rho_2 & 0 \\ 0 & 0 & \rho_3 \end{pmatrix}, \nu \right), \right.\right. \\
&\quad \left(\begin{pmatrix} 0 & 0 & 0 \\ 0 & \zeta_2 & 0 \\ 0 & 0 & -\tau\zeta_2 \end{pmatrix}, \begin{pmatrix} \zeta_1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, 0, \tau\zeta_1 \right), \\
&\quad - \tau\lambda\left(\begin{pmatrix} 0 & 0 & 0 \\ 0 & \zeta_2 & 0 \\ 0 & 0 & -\tau\zeta_2 \end{pmatrix}, \begin{pmatrix} \zeta_1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, 0, \tau\zeta_1 \right), 0, 0, 0 \Bigg) \\
&\quad \left.\left.\;\middle|\; \epsilon_i \in \boldsymbol{R},\; \rho_i, \zeta_i \in C,\; \nu \in i\boldsymbol{R},\; \epsilon_1 + \epsilon_2 + \epsilon_3 = 0,\; i\epsilon_1 + \frac{2}{3}\nu = 0 \right.\right\},
\end{aligned}$$

In particular, we have

$$\dim(((\mathfrak{spin}(13))^{\sigma'})_{(0,F_1(y),0,0)^-}) = 10.$$

Lemma 4.11. $((Spin(13))^{\sigma'})_{(0,F_1(y),0,0)^-}/Spin(4) \simeq S^4$.
In particular, $((Spin(13))^{\sigma'})_{(0,F_1(y),0,0)^-}$ *is connected.*

Proof. We define a 5-dimensional $\boldsymbol{R}$-vector spaces W^5 by

$$\begin{aligned} W^5 &= \{R \in V^{13} \mid \sigma' R = R\} \\ &= \{R = (\varPhi(0, \zeta E_1, 0, 0), (\xi E_1, \eta E_2 - \tau\eta E_3, \tau\xi, 0), 0, 0, -\zeta, 0) \\ &\qquad \mid \zeta \in \boldsymbol{R},\ \xi, \eta \in C\} \end{aligned}$$

with the norm

$$(R, R)_\mu = \frac{1}{30} B_8(\widetilde{\mu}_\delta R, R) = 4\zeta^2 + (\tau\eta)\eta + (\tau\xi)\xi.$$

Then, $S^4 = \{R \in W^5 \mid (R, R)_\mu = 1\}$ is a 4-dimensional sphere. The group $((Spin(13))^{\sigma'})_{(0,F_1(y),0,0)^-}$ acts on S^4. We shall show that this action is transitive. To prove this, it suffices to show that any $R \in S^4$ can be transformed to $1/2(\varPhi_1, 0, 0, 0, -1, 0) \in S^4$ under the action of $((Spin(13))^{\sigma'})_{(0,F_1(y),0,0)^-}$. Now, for a given

$$R = (\varPhi(0, \zeta E_1, 0, 0), (\xi E_1, \eta E_2 - \tau\eta E_3, \tau\xi, 0), 0, 0, -\zeta, 0) \in S^4,$$

choose $t \in \boldsymbol{R}$, $0 \le t < \pi$ such that $\tan t = \dfrac{4\zeta}{\xi + \tau\xi}$ (if $\xi + \tau\xi$=0, let $= \pi/2$). Operate $\theta_{13}(t) \in ((Spin(13))^{\sigma'})_{(0,F_1(y),0,0)^-}$ (Lemmas 4.6 (2), 4.10) on R. Then, we have

$$\theta_{13}(t)R = (0, (\xi' E_1, \eta E_2 - \tau\eta E_3, \tau\xi', 0), 0, 0, 0, 0) = R_1 \in S^3 \subset S^4.$$

Since the group $((Spin(12))^{\sigma'})_{(0,F_1(y),0,0)}$ ($\subset ((Spin(13))^{\sigma'})_{(0,F_1(y),0,0)^-}$) acts transitively on S^3 (Lemma 3.14), there exists $\beta \in ((Spin(12))^{\sigma'})_{(0,F_1(y),0,0)}$ such that

$$\beta R_1 = (0, (E_1, 0, 1, 0), 0, 0, 0, 0) = R_2 \in S^3.$$

Finally, operate $\theta_{13}(-\pi/2) \in ((Spin(13))^{\sigma'})_{(0,F_1(y),0,0)^-}$ on R_2. Then, we have

$$\theta_{13}(-\pi/2)R_2 = \frac{1}{2}(\varPhi_1, 0, 0, 0, -1, 0).$$

This shows the transitivity. The isotropy subgroup at $1/2(\varPhi_1, 0, 0, 0, -1, 0)$ of $((Spin(13))^{\sigma'})_{(0,F_1(y),0,0)^-}$ is $((Spin(12))^{\sigma'})_{(0,F_1(y),0,0)}$ (Lemma 4.7) $=$ $Spin(4)$. Thus, we have the homeomorphism

$$((Spin(13))^{\sigma'})_{(0,F_1(y),0,0)^-}/Spin(4) \simeq S^4. \qquad \square$$

Proposition 4.12. $((Spin(13))^{\sigma'})_{(0,F_1(y),0,0)^-} \cong Spin(5)$.

Proof. Since $((Spin(13))^{\sigma'})_{(0,F_1(y),0,0)^-}$ is connected (Lemma 4.11), we can define a homomorphism $\pi\colon ((Spin(13))^{\sigma'})_{(0,F_1(y),0,0)^-} \to SO(5) = SO(W^5)$ by

$$\pi(\alpha) = \alpha|W^5.$$

$\operatorname{Ker}\pi = \{1,\sigma\} = \boldsymbol{Z}_2$. Since $\dim(((\mathfrak{spin}(13))^{\sigma'})_{(0,F_1(y),0,0)^-}) = 10$ (Lemma 4.10) $= \dim(\mathfrak{so}(5))$, π is onto. Hence, $((Spin(13))^{\sigma'})_{(0,F_1(y),0,0)^-}/\boldsymbol{Z}_2 \cong SO(5)$. Therefore, $((Spin(13))^{\sigma'})_{(0,F_1(y),0,0)^-}$ is isomorphic to $Spin(5)$ as a double covering group of $SO(5)$. □

Lemma 4.13. *The Lie algebra $(\mathfrak{spin}(13))^{\sigma'}$ of the group $(Spin(13))^{\sigma'}$ is given by*

$$
\begin{aligned}
&(\mathfrak{spin}(13))^{\sigma'}\\
&= \left\{ \left(\varPhi\left(D + i\begin{pmatrix} \epsilon_1 & 0 & 0 \\ 0 & \epsilon_2 & 0 \\ 0 & 0 & \epsilon_3 \end{pmatrix}^{\sim}, \begin{pmatrix} 0 & 0 & 0 \\ 0 & \rho_2 & 0 \\ 0 & 0 & \rho_3 \end{pmatrix}, -\tau\begin{pmatrix} 0 & 0 & 0 \\ 0 & \rho_2 & 0 \\ 0 & 0 & \rho_3 \end{pmatrix}, \nu \right), \right.\right.\\
&\quad \left(\begin{pmatrix} 0 & 0 & 0 \\ 0 & \zeta_2 & 0 \\ 0 & 0 & -\tau\zeta_2 \end{pmatrix}, \begin{pmatrix} \zeta_1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, 0, \tau\zeta_1 \right),\\
&\quad -\tau\lambda\left(\begin{pmatrix} 0 & 0 & 0 \\ 0 & \zeta_2 & 0 \\ 0 & 0 & -\tau\zeta_2 \end{pmatrix}, \begin{pmatrix} \zeta_1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, 0, \tau\zeta_1 \right), 0, 0, 0 \Bigg)\\
&\qquad \Bigg|\; D \in \mathfrak{so}(8),\ \epsilon_i \in \boldsymbol{R},\ \rho_i, \zeta_i \in C,\ \nu \in i\boldsymbol{R},\\
&\qquad \left. \epsilon_1 + \epsilon_2 + \epsilon_3 = 0,\ i\epsilon_1 + \frac{2}{3}\nu = 0 \right\}.
\end{aligned}
$$

In particular, we have

$$\dim((\mathfrak{spin}(13))^{\sigma'}) = 28 + 10 = 38.$$

Now, we shall determine the group structure of $(Spin(13))^{\sigma'}$.

Theorem 4.14.

$$(Spin(13))^{\sigma'} \cong (Spin(5) \times Spin(8))/\boldsymbol{Z}_2,\ \boldsymbol{Z}_2 = \{(1,1),(-1,\sigma)\}.$$

Proof. Let $Spin(13) = {G_{13}}^{\mathrm{com}}$, $Spin(5) = ((Spin(13))^{\sigma'})_{(0,F_1(y),0,0)^-}$ and $Spin(8) = ((F_4)_{E_1})^{\sigma'} \subset ((E_6)_{E_1})^{\sigma'} \subset ((E_7)^{\kappa,\mu})^{\sigma'} \subset ({G_{13}}^{\mathrm{com}})^{\sigma'}$ (Theorem 1.2, Propositions 4.4, 4.8). Now, we define a map φ: $Spin(5) \times Spin(8) \to (Spin(13))^{\sigma'}$ by

$$\varphi(\alpha, \beta) = \alpha\beta.$$

Then, φ is well-defined: $\varphi(\alpha,\beta) \in (Spin(13))^{\sigma'}$. Since $[R_D, R_5] = 0$ for $R_D = (\varPhi(D,0,0,0),0,0,0,0,0) \in \mathfrak{spin}(8)$, $R_5 \in \mathfrak{spin}(5) = ((\mathfrak{spin}(13))^{\sigma'})_{(0,F_1(y),0,0)^-}$ (Proposition 4.12), we have $\alpha\beta = \beta\alpha$. Hence, φ is a homomorphism. $\operatorname{Ker}\varphi = \{(1,1),(-1,\sigma)\} = \boldsymbol{Z}_2$. Since $(Spin(13))^{\sigma'}$ is connected and $\dim(\mathfrak{spin}(5) \oplus \mathfrak{spin}(8)) = 10$ (Lemma 4.10) $+28 = 38 = \dim((\mathfrak{spin}(13))^{\sigma'})$ (Lemma 4.13), φ is onto. Thus, we have the isomorphism

$$(Spin(5) \times Spin(8))/\boldsymbol{Z}_2 \cong ((Spin(13))^{\sigma'}. \qquad \square$$

Now, we shall consider the following group

$$\begin{aligned}
&((Spin(14))^{\sigma'})_{(0,F_1(y),0,0)^-} \\
&= \left\{ \alpha \in (Spin(14))^{\sigma'} \;\middle|\; \begin{array}{l} \alpha(0,(0,F_1(y),0,0),0,0,0,0) \\ = (0,(0,F_1(y),0,0),0,0,0,0) \end{array} \text{ for all } y \in \mathfrak{C} \right\}.
\end{aligned}$$

Lemma 4.15. *The Lie algebra* $((\mathfrak{spin}(14))^{\sigma'})_{(0,F_1(y),0,0)^-}$ *of the group* $((Spin(14))^{\sigma'})_{(0,F_1(y),0,0)^-}$ *is given by*

$$\begin{aligned}
&((\mathfrak{spin}(14))^{\sigma'})_{(0,F_1(y),0,0)^-} \\
&= \{ R \in (\mathfrak{spin}(14))^{\sigma'} \mid (\operatorname{ad} R)(0,(0,F_1(y),0,0),0,0,0,0) = 0 \} \\
&= \left\{ \left(\varPhi\left(i \begin{pmatrix} \epsilon_1 & 0 & 0 \\ 0 & \epsilon_2 & 0 \\ 0 & 0 & \epsilon_3 \end{pmatrix}^{\sim}, \begin{pmatrix} 0 & 0 & 0 \\ 0 & \rho_2 & 0 \\ 0 & 0 & \rho_3 \end{pmatrix}, \right. \right. \right. \\
&\qquad \left. - \tau \begin{pmatrix} 0 & 0 & 0 \\ 0 & \rho_2 & 0 \\ 0 & 0 & \rho_3 \end{pmatrix}, \nu \right), \left(\begin{pmatrix} 0 & 0 & 0 \\ 0 & \zeta_2 & 0 \\ 0 & 0 & \zeta_3 \end{pmatrix}, \begin{pmatrix} \zeta_1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, 0, \zeta \right), \\
&\qquad \left. - \tau\lambda \left(\begin{pmatrix} 0 & 0 & 0 \\ 0 & \zeta_2 & 0 \\ 0 & 0 & \zeta_3 \end{pmatrix}, \begin{pmatrix} \zeta_1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, 0, \zeta \right), r, 0, 0 \right) \\
&\qquad\quad \left| \; \epsilon_i \in \boldsymbol{R},\ \rho_i, \zeta_i, \zeta \in C,\ \nu, r \in i\boldsymbol{R}, \right. \\
&\qquad\quad \left. \epsilon_1 + \epsilon_2 + \epsilon_3 = 0,\ i\epsilon_1 + \frac{2}{3}\nu + 2r = 0 \right\}.
\end{aligned}$$

In particular, we have

$$\dim(((\mathfrak{spin}(14))^{\sigma'})_{(0,F_1(y),0,0)^-}) = 15.$$

Lemma 4.16. *For $t \in \boldsymbol{R}$, we define a C-linear transformation $\theta_{14}(t)$ of ${\mathfrak{e}_8}^C$ by*

$$\theta_{14}(t) = \exp(\mathrm{ad}(0, (0, itE_1, 0, it), (itE_1, 0, it, 0), 0, 0, 0)).$$

Then, $\theta_{14}(t) \in ((Spin(14))^{\sigma'})_{(0,F_1(y),0,0)^-}$ (Lemma 4.15). The action of $\theta_{14}(t)$ on V^{14} is given by

$$\begin{aligned}&\theta_{14}(t)(\varPhi(0,\zeta E_1,0,0),(\xi E_1,\eta E_2-\tau\eta E_3+F_1(y),\tau\xi,0),0,0,-\tau\zeta,0)\\ &= (\varPhi(0,\zeta' E_1,0,0),(\xi' E_1,\eta' E_2-\tau\eta' E_3+F_1(y'),\tau\xi',0),0,0,-\tau\zeta',0),\end{aligned}$$

$$\begin{cases}\zeta' &= \dfrac{1}{2}(\zeta+\tau\zeta)+\dfrac{1}{2}(\zeta-\tau\zeta)\cos t-\dfrac{i}{4}(\xi+\tau\xi)\sin t,\\ \xi' &= \dfrac{1}{2}(\xi-\tau\xi)+\dfrac{1}{2}(\xi+\tau\xi)\cos t-i(\zeta-\tau\zeta)\sin t,\\ \eta' &= \eta,\\ y' &= y.\end{cases}$$

Lemma 4.17. *$((Spin(14))^{\sigma'})_{(0,F_1(y),0,0)^-}/Spin(5) \simeq S^5$.*
In particular, $((Spin(14))^{\sigma'})_{(0,F_1(y),0,0)^-}$ is connected.

Proof. We define a 6-dimensional $\boldsymbol{R}$-vector space W^6 by

$$\begin{aligned}W^6 &= \{R \in V^{14} \mid \sigma' R = R\}\\ &= \{R = (\varPhi(0,\zeta E_1,0,0),(\xi E_1,\eta E_2-\tau\eta E_3,\tau\xi,0),0,0,-\tau\zeta,0)\\ &\qquad \mid \zeta,\xi,\eta \in C\}\end{aligned}$$

with the norm

$$(R,R)_\mu = \frac{1}{30}B_8(\widetilde{\mu}_\delta R, R) = 4(\tau\zeta)\zeta + (\tau\eta)\eta + (\tau\xi)\xi.$$

Then, $S^5 = \{R \in W^6 \mid (R,R)_\mu = 1\}$ is a 5-dimensional sphere. The group $((Spin(14))^{\sigma'})_{(0,F_1(y),0,0)^-}$ acts on S^5. We shall show that this action is transitive. To prove this, it suffices to show that any $R \in S^5$ can be transformed to $1/2(i\varPhi_1,0,0,0,i,0) \in S^5$ under the action of $((Spin(14))^{\sigma'})_{(0,F_1(y),0,0)^-}$. Now, for a given

$$R = (\varPhi(0,\zeta E_1,0,0),(\xi E_1,\eta E_2-\tau\eta E_3,\tau\xi,0),0,0,-\tau\zeta,0) \in S^5,$$

choose $t \in \boldsymbol{R}$, $0 \le t < \pi$ such that $\tan t = -\dfrac{2i(\zeta-\tau\zeta)}{\xi+\tau\xi}$ (if $\xi+\tau\xi = 0$, let $t = \pi/2$). Operate $\theta_{14}(t) \in ((Spin(14))^{\sigma'})_{(0,F_1(y),0,0)^-}$ (Lemmas 4.15, 4.16) on R. Then, we have

$$\begin{aligned}\theta_{14}(t)R &= (\varPhi(0,(\zeta' E_1,0,0),(\xi' E_1,\eta E_2-\tau\eta E_3,\tau\xi',0),0,0,-\zeta',0)\\ &= R_1\in S^4\subset S^5.\end{aligned}$$

Since the group $((Spin(13))^{\sigma'})_{(0,F_1(y),0,0)^-}$ $(\subset ((Spin(14))^{\sigma'})_{(0,F_1(y),0,0)^-})$ acts transitively on S^4 (Lemma 4.11), there exists $\beta\in((Spin(13))^{\sigma'})_{(0,F_1(y),0,0)^-}$ such that

$$\beta R_1=\frac{1}{2}(\varPhi_1,0,0,0,-1,0)=R_2\in S^3.$$

Moreover, operate $\theta_{14}(\pi/2)$ and $\alpha(\pi/4)$ (Lemma 3.13) in order,

$$\theta_{14}(\pi/2)R_2=(0,(-iE_1,0,i,0),0,0,0,0)=R_3,$$

and

$$\alpha(\pi/4)R_3=(0,(E_1,0,1,0),0,0,0,0)=R_4.$$

Finally, operate $\theta_{14}(-\pi/2)\in((Spin(14))^{\sigma'})_{(0,F_1(y),0,0)^-}$ on R_4. Then, we have

$$\theta_{14}(-\pi/2)R_4=\frac{1}{2}(i\varPhi_1,0,0,0,i,0).$$

This shows the transitivity. The isotropy subgroup at $1/2(i\varPhi_1,0,0,0,i,0)$ of $((Spin(14))^{\sigma'})_{(0,F_1(y),0,0)^-}$ is $((Spin(13))^{\sigma'})_{(0,F_1(y),0,0)^-}$ (Proposition 4.8) $=Spin(5)$. Thus, we have the homeomorphism

$$((Spin(14))^{\sigma'})_{(0,F_1(y),0,0)^-}/Spin(5)\simeq S^5.$$ □

Proposition 4.18. $((Spin(14))^{\sigma'})_{(0,F_1(y),0,0)^-}\cong Spin(6)$.

Proof. Since $((Spin(14))^{\sigma'})_{(0,F_1(y),0,0)^-}$ is connected (Lemma 4.17), we can define a homomorphism $\pi\colon ((Spin(14))^{\sigma'})_{(0,F_1(y),0,0)^-}\to SO(6)=SO(W^6)$ by

$$\pi(\alpha)=\alpha|W^6.$$

$\operatorname{Ker}\pi=\{1,\sigma\}=\boldsymbol{Z}_2$. Since $\dim(((\mathfrak{spin}(14))^{\sigma'})_{(0,F_1(y),0,0)^-})=15$ (Lemma 4.15) $=\dim(\mathfrak{so}(6))$, π is onto. Hence, $((Spin(14))^{\sigma'})_{(0,F_1(y),0,0)^-}/\boldsymbol{Z}_2\cong SO(6)$. Therefore, $((Spin(14))^{\sigma'})_{(0,F_1(y),0,0)^-}$ is isomorphic to $Spin(5)$ as a double covering group of $SO(6)$. □

Lemma 4.19. *The Lie algebra* $(\mathfrak{spin}(14))^{\sigma'}$ *of the group* $((Spin(14))^{\sigma'}$ *is given by*

$$
\begin{aligned}
&(\mathfrak{spin}(14))^{\sigma'} \\
&= \left\{ \left(\Phi\left(D + i \begin{pmatrix} \epsilon_1 & 0 & 0 \\ 0 & \epsilon_2 & 0 \\ 0 & 0 & \epsilon_3 \end{pmatrix}^{\sim}, \begin{pmatrix} 0 & 0 & 0 \\ 0 & \rho_2 & 0 \\ 0 & 0 & \rho_3 \end{pmatrix}, \right.\right.\right. \\
&\quad - \tau \left.\begin{pmatrix} 0 & 0 & 0 \\ 0 & \rho_2 & 0 \\ 0 & 0 & \rho_3 \end{pmatrix}, \nu \right), \left(\begin{pmatrix} 0 & 0 & 0 \\ 0 & \zeta_2 & 0 \\ 0 & 0 & \zeta_3 \end{pmatrix}, \begin{pmatrix} \zeta_1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, 0, \zeta \right), \\
&\quad \left. - \tau\lambda \left(\begin{pmatrix} 0 & 0 & 0 \\ 0 & \zeta_2 & 0 \\ 0 & 0 & \zeta_3 \end{pmatrix}, \begin{pmatrix} \zeta_1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, 0, \zeta \right), r, 0, 0 \right) \\
&\quad \left| \; D \in \mathfrak{so}(8),\ \epsilon_i \in \boldsymbol{R},\ \rho_i, \zeta_i, \zeta \in C,\ \nu \in i\boldsymbol{R}, \right. \\
&\quad \left. \epsilon_1 + \epsilon_2 + \epsilon_3 = 0,\ i\epsilon_1 + \frac{2}{3}\nu + 2r = 0 \right\}.
\end{aligned}
$$

In particular, we have

$$\dim((\mathfrak{spin}(14))^{\sigma'}) = 28 + 15 = 43.$$

Now, we shall determine the group structure of $(Spin(14))^{\sigma'}$.

Theorem 4.20.

$$(Spin(14))^{\sigma'} \cong (Spin(6) \times Spin(8))/\boldsymbol{Z}_2,\ \boldsymbol{Z}_2 = \{(1,1), (-1,\sigma)\}.$$

Proof. Let $Spin(14) = {G_{14}}^{\mathrm{com}}$, $Spin(6) = ((Spin(14))^{\sigma'})_{(0,F_1(y),0,0)^-}$ and $Spin(8) = ((F_4)_{E_1})^{\sigma'} \subset ((E_6)_{E_1})^{\sigma'} \subset ((E_7)^{\kappa,\mu})^{\sigma'} \subset ({G_{13}}^{\mathrm{com}})^{\sigma'} \subset ({G_{14}}^{\mathrm{com}})^{\sigma'}$ (Theorem 1.2, Propositions 4.8, 4.9). Now, we define a map φ: $Spin(6) \times Spin(8) \to (Spin(14))^{\sigma'}$ by

$$\varphi(\alpha, \beta) = \alpha\beta.$$

Then, φ is well-defined: $\varphi(\alpha,\beta) \in (Spin(14))^{\sigma'}$. Since $[R_D, R_6] = 0$ for $R_D = (\Phi(D,0,0,0),0,0,0,0,0) \in \mathfrak{spin}(8)$, $R_6 \in \mathfrak{spin}(6) = ((\mathfrak{spin}(14))^{\sigma'})_{(0,F_1(y),0,0)^-}$ (Proposition 4.18), we have $\alpha\beta = \beta\alpha$. Hence, φ is a homomorphism. $\operatorname{Ker}\varphi = \{(1,1),(-1,\sigma)\} = \boldsymbol{Z}_2$. Since $(Spin(14))^{\sigma'}$ is connected and $\dim(\mathfrak{spin}(6) \oplus \mathfrak{spin}(8)) = 15$ (Lemma 4.15) $+28 = 43 = \dim((\mathfrak{spin}(14))^{\sigma'})$ (Lemma 4.19), φ is onto. Thus, we have the isomorphism

$$(Spin(6) \times Spin(8))/\boldsymbol{Z}_2 \cong ((Spin(14))^{\sigma'}. \qquad \square$$

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