NOTE ON THE RELATION BETWEEN S-REDUCIBILITY, S-COREDUCTIBILITY AND STABLE HOMOTOPY TYPES OF SOME STUNTED LENS SPACES

Yasusuke KOTANI

1. Introduction

Let \( q \geq 2 \) and \( n \geq 0 \) be integers. Let \( L_{q}^{2n+1} = S^{2n+1}/(\mathbb{Z}/q) \) be the \((2n+1)\)-dimensional \( q \)-lens space and \( L_{q}^{2n} \) the \( 2n \)-skeleton of \( L_{q}^{2n+1} \) by the natural cell-decomposition. For \( k \geq 0 \), there is a natural inclusion \( L_{q}^{n+k} \subset L_{q}^{n+k} \), and so we get the \( q \)-stunted lens space \( L_{n+k}^{2n+1} = L_{q}^{n+k}/L_{q}^{n-1} \).

Two spaces \( X \) and \( Y \) are said to be of the same stable homotopy type if \( \Sigma^{u}X \) (the \( u \)-fold suspension of \( X \)) and \( \Sigma^{v}Y \) are of the same homotopy type for some non-negative integers \( u \) and \( v \).

A space \( X \) is reducible if there exists a map \( f: S^{n} \to X \) that induces an isomorphism
\[
f_*: \widetilde{H}_i(S^n; \mathbb{Z}) \xrightarrow{\cong} \widetilde{H}_i(X; \mathbb{Z}) \text{ for all } i \geq n,
\]
and is \( S \)-reducible if \( \Sigma^{u}X \) is reducible for some non-negative integer \( u \).

Dually, a space \( X \) is coreducible if there exists a map \( g: X \to S^{n} \) that induces an isomorphism
\[
g^*: \widetilde{H}^i(S^n; \mathbb{Z}) \xrightarrow{\cong} \widetilde{H}^i(X; \mathbb{Z}) \text{ for all } i \leq n,
\]
and is \( S \)-coreducible if \( \Sigma^{v}X \) is coreducible for some non-negative integer \( v \).

It is clear that \( S \)-reducibility and \( S \)-coreducibility are properties of the stable homotopy type, and that a space is \( S \)-reducible if and only if its \( S \)-dual, in the sense of Spanier and Whitehead [8], is \( S \)-coreducible.

By the integral homology and cohomology of \( q \)-stunted lens spaces, it follows that \( L_{2n+2k+\varepsilon}^{2n+1} \) is not \( S \)-reducible for \( \varepsilon = 0 \) and \( \delta < 2k \), and not \( S \)-coreducible for \( \delta = 1 \) and \( \varepsilon > 1 - 2k \).

The object of this paper is to determine a necessary and sufficient condition for two \( q \)-stunted lens spaces to be of the same stable homotopy type in case either of two spaces is \( S \)-reducible or \( S \)-coreducible.

Let \( \eta \) be the canonical complex line bundle over \( L_{q}^{2k+1} \) and denote simply by \( \eta \) its restriction to \( L_{q}^{2k} \). Let \( \overline{\eta} \) be the realification of \( \eta \). Let \( J(\overline{\eta} - 2) \) be the image of \( \overline{\eta} - 2 \in KO(L_{q}^{1}) \) by the \( J \)-homomorphism \( J: KO(L_{q}^{1}) \to \widetilde{J}(L_{q}^{1}) \).

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Then the order $h(l, q)$ of $J(\eta - 2) \in \tilde{J}(L_q^l)$ is completely determined in [3, Theorem 2.1].

Our main theorem is as follows.

**Theorem 1.1.** (i) Suppose that either the mod $q$ stunted lens space $L_{2n}^{2n+2k+\varepsilon}$ or $L_{2m}^{2m+2k+\varepsilon}$ is S-coreducible for $\varepsilon = 0$ or $1$. Then $L_{2n}^{2n+2k+\varepsilon}$ and $L_{2m}^{2m+2k+\varepsilon}$ are of the same stable homotopy type if and only if $n \equiv m \mod h(2k + \varepsilon, q)$.

(ii) Suppose that either the mod $q$ stunted lens space $L_{2n}^{2n+2k+1}$ or $L_{2m}^{2m+2k+1}$ is S-reducible for $\delta = 0$ or $1$. Then $L_{2n}^{2n+2k+1}$ and $L_{2m}^{2m+2k+1}$ are of the same stable homotopy type if and only if $n \equiv m \mod h(2k + 1 - \delta, q)$.

Susumu Kôno stated the above results in [6, (2.10)(2), (4)] without proof and without mentioning the relation to S-coreducibility and S-reducibility. So we state the relation to the S-coreducibility and S-reducibility, and give the complete proof.


This paper is organized as follows. In Section 2, we recall the known results. A proof of the main theorem is given in Section 3. As a concluding remark, we consider the cases $q = 2^r (r \geq 1)$.

2. Preliminaries

Let $\alpha$ be a real vector bundle over a finite CW-complex $X$. Then the Thom complex $X^\alpha$ is defined to be the one-point compactification of $\alpha$. Let $J(\alpha) \in \tilde{J}(X)$ denote the stable fibre homotopy class of $\alpha$.

Now let us recall that the relations given in [1] between stable fibre homotopy classes and stable homotopy types or S-coreducibility of Thom complexes.

**Proposition 2.1** ([1, Proposition (2.6)]). Let $\alpha$ and $\beta$ be real vector bundles over $X$. Then $X^\alpha$ and $X^\beta$ are of the same stable homotopy type if $J(\alpha) = J(\beta)$.

**Proposition 2.2** ([1, Proposition (2.8)]). Let $\alpha$ be a real vector bundle over a connected space $X$. Then $X^\alpha$ is S-coreducible if and only if $J(\alpha) = 0$.

Let $\eta$ be the canonical complex line bundle over $L_q^{2k+1}$ and denote simply by $\eta$ its restriction to $L_q^{2k}$. Let $\bar{\eta}$ be the realification of $\eta$. Then there are natural homeomorphisms given in [5, Theorem 4.7, Corollary 4.8]:

$$L_{2n}^{2n+2k+\varepsilon} = (L_q^{2k+\varepsilon})^n \bar{\eta},$$

$$L_{2n+1}^{2n+2k+\varepsilon} = (L_q^{2k+\varepsilon})^n \bar{\eta}/S^{2n}$$
for $\varepsilon = 0$ or 1.

$S$-duality of mod $q$ stunted lens spaces is given in [6].

**Lemma 2.3** ([6, Lemma 2.9] and [7, Proposition 5]). Let $\delta, \varepsilon \in \{0, 1\}$. Suppose that

$$N \equiv 0 \mod \begin{cases} h(2k + 1 - \delta, q) & \text{if } \varepsilon = 1, \\ h(2k, q) & \text{if } \varepsilon = 0, \end{cases}$$

and $2N > 2n + 2k + \varepsilon + 1$. Then an $S$-dual of the mod $q$ stunted lens space $L_{2n+\delta}^{2n+2k+\varepsilon}$ is $L_{2N-2n-\delta-1}^{2N-2n-2k-\varepsilon-1}$.

3. **Proof of Main Theorem**


**Theorem 3.1** ([3, Theorem 1.5]). Two mod $q$ stunted lens spaces $L_{2n+\delta}^{2n+2k+\varepsilon}$ and $L_{2m+\delta}^{2m+2k+\varepsilon}$ for $\delta, \varepsilon \in \{0, 1\}$ are of the same stable homotopy type if $n \equiv m \mod h(2k + \varepsilon, q)$.

However, by making use of $S$-duality, we have

**Theorem 3.2.** Two mod $q$ stunted lens spaces $L_{2n+\delta}^{2n+2k+1}$ and $L_{2m+\delta}^{2m+2k+1}$ for $\delta = 0$ or 1 are of the same stable homotopy type if $n \equiv m \mod h(2k + 1 - \delta, q)$.

Theorem 3.2 is better than Theorem 3.1 for $\varepsilon = 1$ since in general

$$h(2k + 1 - \delta, q) \leq h(2k + 1, q).$$

**Proof of Theorem 3.2.** Since $n \equiv m \mod h(2k + 1 - \delta, q)$, we have

$$(N - n - k - 1)J(\overline{\eta} - 2) = (M - m - k - 1)J(\overline{\eta} - 2) \in \overline{J}(L_{q}^{2k+1-\delta})$$

for some integers $N$ and $M$ such that $N, M \equiv 0 \mod h(2k + 1 - \delta, q)$ and $N > n + k + 1, M > m + k + 1$. Then, by Proposition 2.1,

$$(L_{q}^{2k+1-\delta})^{(N - n - k - 1)\overline{\eta}} = L_{2N-2n-2k-2}^{2N-2n-\delta-1}$$

and

$$(L_{q}^{2k+1-\delta})^{(M - m - k - 1)\overline{\eta}} = L_{2M-2m-2k-2}^{2M-2m-\delta-1}$$

are of the same stable homotopy type.

On the other hand, by Lemma 2.3, $L_{2N-2n-\delta-1}^{2N-2n-2k-2}$ is an $S$-dual of $L_{2n+\delta}^{2n+2k+1}$. Thus, by the property of $S$-duality, $L_{2n+\delta}^{2n+2k+1}$ and $L_{2m+\delta}^{2m+2k+1}$ are of the same stable homotopy type.

$S$-coreducibility and $S$-reducibility of mod $q$ stunted lens spaces are stated as follows.
Proposition 3.3. (i) The mod $q$ stunted lens space $L_{2n}^{2n+2k+\varepsilon}$ is $S$-coreducible for $\varepsilon = 0$ or 1 if and only if $n \equiv 0 \mod h(2k + \varepsilon, q)$.

(ii) The mod $q$ stunted lens space $L_{2n+\delta}^{2n+2k+1}$ is $S$-reducible for $\delta = 0$ or 1 if and only if $n + k + 1 \equiv 0 \mod h(2k + 1 - \delta, q)$.

Proof. (i) By Proposition 2.2, $L_{2n}^{2n+2k+\varepsilon} = (L_q^{2k+\varepsilon})^{n\overline{\eta}}$ is $S$-coreducible if and only if

$$J(n\overline{\eta}) = nJ(\overline{\eta} - 2) = 0.$$ 

Hence we get $n \equiv 0 \mod h(2k + \varepsilon, q)$.

(ii) By Lemma 2.3, an $S$-dual of $L_{2n+\delta}^{2n+2k+1}$ is $L_{2n}^{2N-2n-\delta-1}$ for some integer $N$ such that $N \equiv 0 \mod h(2k + 1 - \delta, q)$ and $N > n + k + 1$. Since a space is $S$-reducible if and only if its $S$-dual is $S$-coreducible, by (i), $L_{2n+\delta}^{2n+2k+1}$ is $S$-reducible if and only if

$$N - n - k - 1 \equiv 0 \mod h(2k + 1 - \delta, q)$$

for some integer $N$ such that $N \equiv 0 \mod h(2k + 1 - \delta, q)$ and $N > n + k + 1$. Hence we get $n + k + 1 \equiv 0 \mod h(2k + 1 - \delta, q)$. \qed

Now we can prove main theorem.

**Proof of Theorem 1.1.** The sufficient condition follows immediately from Theorems 3.1 and 3.2.

Next, we consider the necessary condition. Since two spaces are of the same stable homotopy type, $L_{2n}^{2n+2k+\varepsilon}$ is $S$-coreducible if and only if $L_{2m}^{2m+2k+\varepsilon}$ is $S$-reducible. That is, by Proposition 3.3(i), we see that

$$n \equiv 0 \mod h(2k + \varepsilon, q) \iff m \equiv 0 \mod h(2k + \varepsilon, q).$$

Hence we get $n \equiv m \mod h(2k + \varepsilon, q)$.

Similarly, $L_{2n+\delta}^{2n+2k+1}$ is $S$-reducible if and only if $L_{2m+\delta}^{2m+2k+1}$ is $S$-reducible. That is, by Proposition 3.3(ii), we see that

$$n + k + 1 \equiv 0 \mod h(2k + 1 - \delta, q) \iff m + k + 1 \equiv 0 \mod h(2k + 1 - \delta, q).$$

Hence we get $n \equiv m \mod h(2k + 1 - \delta, q)$. \qed

**Concluding remarks**

For the cases $q = 2^r (r \geq 1)$, Theorem 1.1 is restated as follows.

**Theorem 1.1’.** (i) Suppose that $n \equiv 0 \mod h(2k + \varepsilon, 2^r)$. Then two mod $2^r$ stunted lens spaces $L_{2n}^{2n+2k+\varepsilon}$ and $L_{2m}^{2m+2k+\varepsilon}$ for $\varepsilon = 0$ or 1 are of the same stable homotopy type if and only if $n \equiv m \mod h(2k + \varepsilon, 2^r)$.

(ii) Suppose that $n + k + 1 \equiv 0 \mod h(2k + 1 - \delta, 2^r)$. Then two mod $2^r$ stunted lens spaces $L_{2n+\delta}^{2n+2k+1}$ and $L_{2m+\delta}^{2m+2k+1}$ for $\delta = 0$ or 1 are of the same stable homotopy type if and only if $n \equiv m \mod h(2k + 1 - \delta, 2^r)$. 
However, it is known that the above result holds under the weaker assumptions

(i) $2n \equiv 0 \mod h(2k + \varepsilon, q)$,
(ii) $2(n + k + 1) \equiv 0 \mod h(2k + 1 - \delta, q)$,

for the cases $q = 2$ ([2]), $q = 4$ ([9]) and $q = 8$ ([6]).

Susumu Kôno claims that for all the cases $q = 2^r (r \geq 1)$, above result holds under the weaker assumptions.

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REFERENCES


YASUSUKE KOTANI
THE GRADUATE SCHOOL OF NATURAL SCIENCE AND TECHNOLOGY
OKAYAMA UNIVERSITY
OKAYAMA 700-8530, JAPAN

e-mail address: kotani@math.okayama-u.ac.jp

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