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SEMIGROUPS OF LOCALLY LIPSCHITZ OPERATORS

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1. Introduction

In the previous paper [3] we characterized the continuous infinitesimal generators of semigroups of Lipschitz operators and applied the characterization to the Cauchy problem for a quasi-linear wave equation with dissipative term. The problem of characterizing the infinitesimal generator A of a semigroup of Lipschitz operators is closely related to the abstract Cauchy problem for A:

$$\text{(CP;}x\text{)} \qquad u'(t) = Au(t) \text{ for } t \geq 0, \quad \text{and} \quad u(0) = x.$$

In [3] the operator A was assumed to be continuous from a closed subset D of a real Banach space X into X satisfying a dissipative condition defined by means of a metric-like functional.

The purpose of this paper is to extend the previous result to the case where A is continuous and dissipative in a local sense with respect to a lower semicontinuous functional φ on X such that the domain of A is the effective domain of φ. In Section 2, we give the main result (Theorem 2.1) of this paper. The present situation is considered to treat systems of nonlinear partial differential equations and the lower semicontinuous functionals are supposed to be constructed according to the nature of the nonlinear systems. In fact, Section 5 presents how the result obtained here is applied to the Cauchy problem for Kirchhoff equation with real analytic initial data.

2. Main theorem

Let X be a real Banach space with norm $\|\cdot\|$, D a subset of X and φ a nonnegative, lower semicontinuous functional on X such that D is the effective domain of φ. For each $\alpha > 0$, the level set $\{x \in X; \varphi(x) \leq \alpha\}$ of φ is denoted by $D(\alpha)$. A nonnegative continuous function g on $[0, \infty)$ is called a *comparison function* if for each $\alpha \geq 0$ the Cauchy problem

$$w'(t) = g(w(t)) \text{ for } t \geq 0, \quad \text{and} \quad w(0) = \alpha$$

has a maximal solution $m(t; \alpha)$ on $[0, \infty)$. Such a comparison function was used in [5] to characterize quasi-contractive semigroups associated with semilinear evolution equations.

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Let us choose a comparison function g and introduce a class of semigroups of locally Lipschitz operators on D. A one-parameter family $\{S(t); t \geq 0\}$ of locally Lipschitz operators from D into itself is called a *semigroup of locally Lipschitz operators on D with respect to the functional φ* if it satisfies the following conditions:

(S1) $S(0)x = x$ for $x \in D$, and $S(t+s)x = S(t)S(s)x$ for $t, s \geq 0$ and $x \in D$.

(S2) For each $x \in D$, $S(t)x$ is continuous on $[0, \infty)$ in X.

(S3) For each $\tau > 0$ and $\alpha > 0$ there exists $L(\tau, \alpha) > 0$ such that

$$\|S(t)x - S(t)y\| \leq L(\tau, \alpha)\|x - y\| \text{ for } x, y \in D(\alpha) \text{ and } t \in [0, \tau].$$

(S4) $\varphi(S(t)x) \leq m(t; \varphi(x))$ for $x \in D$ and $t \geq 0$.

Let E be an open subset of X such that $D \subset E$. A nonnegative functional V on $E \times E$ satisfying the following conditions is employed to define a general type of dissipative condition.

(V1) There exists $L > 0$ such that

$$|V(x, y) - V(\hat{x}, \hat{y})| \leq L(\|x - \hat{x}\| + \|y - \hat{y}\|) \text{ for } (x, y), (\hat{x}, \hat{y}) \in E \times E.$$

(V2) For each $\alpha > 0$ there exist $C(\alpha) \geq c(\alpha) > 0$ such that

$$c(\alpha)\|x - y\| \leq V(x, y) \leq C(\alpha)\|x - y\| \text{ for } x, y \in D(\alpha).$$

Throughout this paper, we assume that an operator A from D into X satisfies the following three conditions:

(A1) For each $\alpha > 0$, the operator A is continuous on the level set $D(\alpha)$.

(A2) For each $\alpha > 0$ there exists $\omega(\alpha) \geq 0$ such that

$$D_+V(x, y)(Ax, Ay) \leq \omega(\alpha)V(x, y) \text{ for } x, y \in D(\alpha),$$

where the symbol D_+V is defined by

$$D_+V(x, y)(\xi, \eta) = \liminf_{h \downarrow 0}(V(x + h\xi, y + h\eta) - V(x, y))/h$$

for $(x, y) \in E \times E$ and $(\xi, \eta) \in X \times X$.

(A3) For each $x \in D$ and $\varepsilon > 0$ there exist $\delta \in (0, \varepsilon]$ and $x_\delta \in D$ such that $\|(x_\delta - x)/\delta - Ax\| \leq \varepsilon$ and $(\varphi(x_\delta) - \varphi(x))/\delta \leq g(\varphi(x)) + \varepsilon$.

Let J be an interval of the form $[0, \tau)$ or $[0, \tau]$. By a *solution to* (CP;x) *on* J we mean a differentiable function u from J into X such that $u(t) \in D$ for $t \in J$ and equation (CP;x) is satisfied for $t \in J$. The main result of this paper is given by

Theorem 2.1. *There exists a semigroup $\{S(t); t \geq 0\}$ of locally Lipschitz operators on D with respect to φ such that for each $x \in D$, $S(t)x$ is a unique solution to* (CP;x) *on* $[0, \infty)$.

By the following propositions, to prove Theorem 2.1 it suffices to show the existence of local solutions satisfying the growth condition for all $x \in D$. The proof will be divided into two parts. One is the construction of approximate solutions and the other is the convergence of approximate solutions for the Cauchy problem (CP;x). They will be discussed in Sections 3 and 4 respectively, by slightly modified methods in the previous paper [3].

Proposition 2.2. *For each $i = 1, 2$, let u_i be a solution to* (CP;x_i) *on $[0, \tau]$ such that $u_i(t) \in D(\alpha)$ for $t \in [0, \tau]$. Then we have $V(u_1(t), u_2(t)) \le e^{\omega(\alpha)t} V(x_1, x_2)$ for $t \in [0, \tau]$.*

Proposition 2.2 is shown similarly to the proof of [3, Proposition 1.1].

Proposition 2.3. *Assume that for each $x \in D$, there exists $\tau > 0$ such that the* (CP;x) *has a solution u on $[0, \tau]$ satisfying $\varphi(u(t)) \le m(t; \varphi(x))$ for $t \in [0, \tau]$. Then for each $x \in D$, the* (CP;x) *has a solution u on $[0, \infty)$ satisfying the growth condition $\varphi(u(t)) \le m(t; \varphi(x))$ for $t \ge 0$.*

Proof. Let $x \in D$. If $\overline{\tau}$ is defined by the supremum of $\tau > 0$ such that the (CP;x) has a solution u on $[0, \tau]$ satisfying $\varphi(u(t)) \le m(t; \varphi(x))$ for $t \in [0, \tau]$, then we have $\overline{\tau} > 0$ by assumption. By uniqueness (Proposition 2.2) there exists a solution u to (CP;x) on $[0, \overline{\tau})$ satisfying

$$\varphi(u(t)) \le m(t; \varphi(x)) \text{ for } t \in [0, \overline{\tau}). \tag{2.1}$$

If $\overline{\tau} = \infty$ then the proof is complete. Assume to the contrary that $\overline{\tau} < \infty$. Proposition 2.2 then shows that $V(u(t + h), u(t)) \le e^{\omega(\alpha)\overline{\tau}} V(u(h), x)$ for $t, t + h \in [0, \overline{\tau})$ and $h > 0$, where $\alpha = m(\overline{\tau}; \varphi(x))$. It follows that the limit $y := \lim_{t \uparrow \overline{\tau}} u(t)$ exists and is in $D(\alpha)$. By assumption, the (CP;y) has a solution w on $[0, \delta]$ satisfying

$$\varphi(w(t)) \le m(t; \varphi(y)) \text{ for } t \in [0, \delta]. \tag{2.2}$$

The function u can be extended to $[0, \overline{\tau} + \delta]$, by defining $u(t) = w(t - \overline{\tau})$ for $t \in [\overline{\tau}, \overline{\tau} + \delta]$. Clearly, it is a solution to (CP;x) on $[0, \overline{\tau} + \delta]$. By the lower semicontinuity of φ we have $\varphi(y) \le m(\overline{\tau}; \varphi(x))$ by (2.1). This inequality together with (2.2) implies $\varphi(u(t)) \le m(t; \varphi(x))$ for $t \in [\overline{\tau}, \overline{\tau} + \delta]$, since $m(t; \varphi(x)) = m(t - \overline{\tau}; m(\overline{\tau}; \varphi(x)))$ for $t \in [\overline{\tau}, \overline{\tau} + \delta]$. These facts together contradict the definition of $\overline{\tau}$. □

3. Construction of approximate solutions

In this section we discuss the construction of approximate solutions for the abstract Cauchy problem (CP;x).

The following two lemmas are proved in the same way as the verification of [3, Lemmas 2.1 and 2.2]. The symbol $B[z_0, r]$ stands for the closed ball with center z_0 and radius r.

Lemma 3.1. *Let $\alpha > 0$ and $z_0 \in D(\alpha)$. Assume $r > 0$ and $M > 0$ to be chosen such that $\|Ax\| \leq M$ for $x \in B[z_0, r] \cap D(\alpha)$. Let $\varepsilon > 0$ and $\sigma \in (0, r/(M+\varepsilon)]$. If a sequence $\{(s_i, z_i)\}_{i=0}^n$ in $[0,\sigma] \times D(\alpha)$ satisfies*

$$0 = s_0 < s_1 < \cdots < s_n \leq \sigma, \tag{3.1}$$

$$\|z_{i-1} + (s_i - s_{i-1})Az_{i-1} - z_i\| \leq \varepsilon(s_i - s_{i-1}) \text{ for } i = 1, 2, \ldots, n, \tag{3.2}$$

then we have $\|z_i - z_j\| \leq (M+\varepsilon)(s_i - s_j)$ for $0 \leq j \leq i \leq n$, and $\|Az_i\| \leq M$ for $0 \leq i \leq n$.

Lemma 3.2. *Let $\alpha > 0$ and $z_0 \in D(\alpha)$. Assume $r > 0$, $M > 0$ and $\eta > 0$ to be chosen such that $\|Ax\| \leq M$ and $\|Ax - Az_0\| \leq \eta$ for $x \in B[z_0, r] \cap D(\alpha)$. Let $\varepsilon > 0$ and $\sigma \in (0, r/(M+\varepsilon)]$. Then the following assertions hold*:

(i) *If a sequence $\{(s_i, z_i)\}_{i=0}^n$ in $[0,\sigma] \times D(\alpha)$ satisfies* (3.1) *and* (3.2), *then*

$$\|z_0 + s_n A z_0 - z_n\| \leq (\varepsilon + \eta)s_n.$$

(ii) *If a sequence $\{(s_i, z_i)\}_{i=0}^\infty$ in $[0,\sigma) \times D(\alpha)$ satisfies*

$$0 = s_0 < s_1 < \cdots < s_i < \cdots < \sigma, \quad \text{and} \quad \lim_{i\to\infty} s_i = \sigma, \tag{3.3}$$

$$\|z_{i-1} + (s_i - s_{i-1})Az_{i-1} - z_i\| \leq \varepsilon(s_i - s_{i-1}) \text{ for } i = 1, 2, \ldots, \tag{3.4}$$

then the limit $z = \lim_{i\to\infty} z_i$ exists and is in $D(\alpha)$, and

$$\|z_0 + \sigma A z_0 - z\| \leq (\varepsilon + \eta)\sigma.$$

To construct approximate solutions, we use the nonextensible maximal solution $m_\varepsilon(t;\alpha)$ to the Cauchy problem $w'(t) = g(w(t)) + \varepsilon$ for $t \geq 0$, and $w(0) = \alpha$, where $\varepsilon > 0$ and $\alpha \geq 0$. Let $[0, \tau_\varepsilon(\alpha))$ denote the maximal interval of existence of maximal solution $m_\varepsilon(t;\alpha)$. Then it is known that the following assertions hold:

(m_1) If $\beta \geq \alpha \geq 0$ then $\tau_\varepsilon(\beta) \leq \tau_\varepsilon(\alpha)$ and $m_\varepsilon(t;\alpha) \leq m_\varepsilon(t;\beta)$ for $t \in [0, \tau_\varepsilon(\beta))$.

(m_2) If $s \in [0, \tau_\varepsilon(\alpha))$ then $\tau_\varepsilon(m_\varepsilon(s;\alpha)) = \tau_\varepsilon(\alpha) - s$ and $m_\varepsilon(t+s;\alpha) = m_\varepsilon(t; m_\varepsilon(s;\alpha))$ for $t \in [0, \tau_\varepsilon(\alpha) - s)$.

(m_3) $\lim_{\varepsilon\downarrow 0} \tau_\varepsilon(\alpha) = \infty$ and $\lim_{\varepsilon\downarrow 0} m_\varepsilon(t;\alpha) = m(t;\alpha)$ uniformly on every compact subinterval of $[0,\infty)$.

Lemma 3.3. *For each $\varepsilon > 0$ and $x \in D$ there exist $\delta \in (0, \varepsilon]$ and $x_\delta \in D$ such that $\|(x_\delta - x)/\delta - Ax\| \leq \varepsilon$ and $\varphi(x_\delta) \leq m_\varepsilon(\delta; \varphi(x))$.*

Proof. Let $x \in D$ and $\varepsilon > 0$, and consider the function r on $[0,\infty)$ defined by

$$r(t) = \varphi(x) + t(g(\varphi(x)) + \varepsilon/2)$$

for $t \geq 0$. Then we have $r(0) = m_\varepsilon(0;\varphi(x))$ and $r'(0) < m_\varepsilon'(0;\varphi(x))$, and so there exists $\tau \in (0, \tau_\varepsilon(\varphi(x)))$ such that $r(t) \leq m_\varepsilon(t;\varphi(x))$ for $t \in [0,\tau]$.

By (A3) there exist a sequence $\{x_n\}$ in D and a null sequence $\{\delta_n\}$ of positive numbers such that $\|(x_n - x)/\delta_n - Ax\| \leq 1/n$ and $(\varphi(x_n) - \varphi(x))/\delta_n \leq g(\varphi(x)) + 1/n$ for $n \geq 1$. Choose an integer n such that $1/n \leq \varepsilon/2$ and $\delta_n \leq (\tau \wedge \varepsilon)$. Then the pair (x_n, δ_n) is the desired one because $\varphi(x_n) \leq r(\delta_n)$. □

Proposition 3.4. *Let $\alpha > 0$ and $z_0 \in D(\alpha)$. Assume $r > 0$, $M > 0$, $\eta > 0$ and $\varepsilon \in (0, 1]$ to be chosen such that $\|Ax\| \leq M$ and $\|Ax - Az_0\| \leq \eta$ for $x \in B[z_0, r] \cap D(\alpha)$, $r/(M+1) < \tau_\varepsilon(\varphi(z_0))$ and $m_\varepsilon(r/(M+1); \varphi(z_0)) \leq \alpha$. Set $\tau = r/(M+1)$. Then for each $\sigma \in (0, \tau]$ there exists $y_0 \in D(\alpha)$ such that $\|z_0 + \sigma Az_0 - y_0\| \leq (\eta + \varepsilon)\sigma$ and $\varphi(y_0) \leq m_\varepsilon(\sigma; \varphi(z_0))$.*

Proof. Let $\sigma \in (0, \tau]$. We begin by proving the existence of a sequence $\{(s_i, z_i)\}_{i=1}^\infty$ in $[0, \sigma) \times D(\alpha)$ satisfying (3.3), (3.4) and

$$\varphi(z_i) \leq m_\varepsilon(s_i - s_{i-1}; \varphi(z_{i-1})) \text{ for } i = 1, 2, \ldots. \tag{3.5}$$

To do this, let $k \geq 1$ and assume that a sequence $\{(s_i, z_i)\}_{i=0}^{k-1}$ in $[0, \sigma) \times D(\alpha)$ is chosen so that (3.3) through (3.5) are satisfied for $0 \leq i \leq k-1$. Then we define $\overline{h}_k$ by the supremum of all $h \geq 0$ such that $h < \sigma - s_{k-1}$ and there exists $x_h \in D$ satisfying $\|x_h - z_{k-1} - hAz_{k-1}\| \leq \varepsilon h$ and $\varphi(x_h) \leq m_\varepsilon(h; \varphi(z_{k-1}))$. By Lemma 3.3 we have $\overline{h}_k > 0$. A positive number h_k can be chosen so that $\overline{h}_k/2 < h_k < \sigma - s_{k-1}$ and there exists $z_k \in D$ satisfying $\|z_k - z_{k-1} - h_k Az_{k-1}\| \leq \varepsilon h_k$ and $\varphi(z_k) \leq m_\varepsilon(h_k; \varphi(z_{k-1}))$. If we define $s_k = s_{k-1} + h_k$ then $s_{k-1} < s_k < \sigma$, and (3.4) and (3.5) are satisfied with $i = k$. Notice that $z_k \in D(\alpha)$ by (3.5).

It remains to prove that $\lim_{i\to\infty} s_i = \sigma$. For this purpose, assume to the contrary that $\overline{s} = \lim_{i\to\infty} s_i < \sigma$. Since $z_i \in D(\alpha)$ for $i = 0, 1, \ldots,$ we deduce from Lemma 3.1 that the limit $z := \lim_{i\to\infty} z_i$ exists and is in $D(\alpha)$. Lemma 3.3 ensures the existence of a number $h > 0$ such that $h < \sigma - \overline{s}$ and there exists $x_h \in D$ satisfying $\|(x_h - z)/h - Az\| \leq \varepsilon/2$ and

$$\varphi(x_h) \leq m_{\varepsilon/2}(h; \varphi(z)). \tag{3.6}$$

Now, we set $\gamma_i = \overline{s} + h - s_{i-1}$ for $i \geq 1$. Since $\overline{h}_i < 2h_i = 2(s_i - s_{i-1}) \to 0$, $z_i \to z$ and $\gamma_i \to h$ as $i \to \infty$, there exists an integer $i_0 \geq 1$ such that $\overline{h}_i < \gamma_i < \sigma - s_{i-1}$ and $\|(x_h - z_{i-1})/\gamma_i - Az_{i-1}\| \leq \varepsilon$ for all $i \geq i_0$. Moreover, we have by (3.5), $\varphi(z_i) \leq m_\varepsilon(s_i - s_j; \varphi(z_j))$ for $i \geq j \geq 0$. By taking the limit as $i \to \infty$, the lower semicontinuity of φ implies $\varphi(z) \leq m_\varepsilon(\overline{s} - s_j; \varphi(z_j))$ for $j \geq 0$. This inequality and (3.6) together imply $\varphi(x_h) \leq m_\varepsilon(\gamma_i; \varphi(z_{i-1}))$ for $i \geq 1$, which is impossible by the definition of $\overline{h}_i$. Hence $\lim_{i\to\infty} s_i = \sigma$.

Since a sequence $\{(s_i, z_i)\}_{i=1}^\infty$ in $[0, \sigma) \times D(\alpha)$ satisfying (3.3) and (3.4) is shown to exist, we see by Lemma 3.2 that the limit $y_0 := \lim_{i\to\infty} z_i \in D(\alpha)$ exists and satisfies $\|z_0 + \sigma Az_0 - y_0\| \leq (\varepsilon + \eta)\sigma$. The desired inequality $\varphi(y_0) \leq m_\varepsilon(\sigma; \varphi(z_0))$ follows from (3.5) and the lower semicontinuity of φ. □

The existence of a forward difference approximate solution for (CP;x) is established by

Proposition 3.5. *Let $\alpha > 0$ and $x_0 \in D(\alpha)$. Assume $r > 0$, $M > 0$ and $\varepsilon \in (0,1]$ to be chosen such that $\|Ax\| \le M$ for $x \in B[x_0, r] \cap D(\alpha)$, $r/(M+1) < \tau_\varepsilon(\varphi(x_0))$ and $m_\varepsilon(r/(M+1); \varphi(x_0)) \le \alpha$. Let $\tau \in (0, r/(M+1)]$. Then there exists a sequence $\{(t_j, x_j)\}_{j=0}^{\infty}$ in $[0,\tau) \times D(\alpha)$ such that*

(i) $0 = t_0 < t_1 < \cdots < t_j < \cdots < \tau$,
(ii) $t_j - t_{j-1} \le \varepsilon$ *for* $j = 1, 2, \ldots,$
(iii) $\|x_{j-1} + (t_j - t_{j-1})Ax_{j-1} - x_j\| \le (\varepsilon/4)(t_j - t_{j-1})$ *for* $j = 1, 2, \ldots,$
(iv) *if* $x \in B[x_{j-1}, (M+1)(t_j - t_{j-1})] \cap D(\alpha)$, *then*

$$\|Ax - Ax_{j-1}\| \le \varepsilon/8 \text{ for } j = 1, 2, \ldots,$$

(v) $\varphi(x_j) \le m_\varepsilon(t_j - t_{j-1}; \varphi(x_{j-1}))$ *for* $j = 1, 2, \ldots,$
(vi) $\lim_{j\to\infty} t_j = \tau$.

Proof. Let i be a positive integer and assume that a sequence $\{(t_j, x_j)\}_{j=0}^{i-1}$ is chosen so that conditions (i) through (v) are satisfied for $0 \le j \le i-1$. If we define $\overline{h}_i$ by the supremum of all $h \in [0, \varepsilon]$ such that $h < \tau - t_{i-1}$ and $\|Ax - Ax_{i-1}\| \le \varepsilon/8$ for $x \in B[x_{i-1}, (M+1)h] \cap D(\alpha)$, then we have $\overline{h}_i > 0$ by the continuity of A on $D(\alpha)$. Let us choose $h_i \in (0, \varepsilon]$ so that $\overline{h}_i/2 < h_i < \tau - t_{i-1}$ and

$$\|Ax - Ax_{i-1}\| \le \varepsilon/8$$

for $x \in B[x_{i-1}, (M+1)h_i] \cap D(\alpha)$. If we put $t_i = t_{i-1} + h_i$, then conditions (i), (ii) and (iv) hold for $j = i$.

We need to show that conditions (iii) and (v) are satisfied with $j = i$. By Lemma 3.1 we have $\|x_{i-1} - x_0\| \le (M+1)t_{i-1}$, which implies $B[x_{i-1}, (M+1)h_i] \subset B[x_0, r]$. It follows that $\|Ax\| \le M$ for $x \in B[x_{i-1}, (M+1)h_i] \cap D(\alpha)$. Since $\varphi(x_{i-1}) \le m_\varepsilon(t_{i-1}; \varphi(x_0))$ by condition (v), we have by (m_1) and (m_2)

$$\tau_{\varepsilon/8}(\varphi(x_{i-1})) \ge \tau_\varepsilon(m_\varepsilon(t_{i-1}; \varphi(x_0))) = \tau_\varepsilon(\varphi(x_0)) - t_{i-1} > h_i,$$
$$m_{\varepsilon/8}(h_i; \varphi(x_{i-1})) \le m_\varepsilon(h_i; m_\varepsilon(t_{i-1}; \varphi(x_0))) \le m_\varepsilon(\tau; \varphi(x_0)) \le \alpha.$$

Applying Proposition 3.4 with $z_0 = x_{i-1}$, $r = (M+1)h_i$ and $\eta = \varepsilon/8$, one finds $x_i \in D(\alpha)$ satisfying condition (iii) and (v) with $j = i$.

The proof will be complete if condition (vi) is checked. By using the continuity of A on $D(\alpha)$ into X and the closedness of the set $D(\alpha)$, condition (vi) is verified by the same argument as in the proof of [3, Proposition 2.5]. $\square$

4. Convergence of approximate solutions and proof of main theorem

The following plays an important role in comparing between two approximate solutions to (CP;x).

Proposition 4.1. *Let $\alpha > 0$ and $(z_0, \hat{z}_0) \in D(\alpha) \times D(\alpha)$. Assume $r > 0$, $M > 0$, $\eta, \hat{\eta} \in (0,1)$ and $\varepsilon, \hat{\varepsilon} \in (0, 1/2)$ to be chosen such that*

$$\begin{gathered}
\|Ax\| \le M \quad \text{and} \quad \|Ax - Az_0\| \le \eta/4 \quad \text{for } x \in B[z_0, r] \cap D(\alpha);\\
\|A\hat{x}\| \le M \quad \text{and} \quad \|A\hat{x} - A\hat{z}_0\| \le \hat{\eta}/4 \quad \text{for } \hat{x} \in B[\hat{z}_0, r] \cap D(\alpha);\\
r/(M+1) < \tau_\varepsilon(\varphi(z_0)) \quad \text{and} \quad r/(M+1) < \tau_{\hat{\varepsilon}}(\varphi(\hat{z}_0));\\
m_\varepsilon(r/(M+1); \varphi(z_0)) \le \alpha \quad \text{and} \quad m_{\hat{\varepsilon}}(r/(M+1); \varphi(\hat{z}_0)) \le \alpha.
\end{gathered}$$

Let $\sigma \in (0, r/(M+1)]$. Then there exists a pair $(y_0, \hat{y}_0) \in D(\alpha) \times D(\alpha)$ such that

$$\|z_0 + \sigma A z_0 - y_0\| \le (\eta + \varepsilon)\sigma \quad \text{and} \quad \|\hat{z}_0 + \sigma A\hat{z}_0 - \hat{y}_0\| \le (\hat{\eta} + \hat{\varepsilon})\sigma; \tag{4.1}$$

$$V(y_0, \hat{y}_0) \le \exp(\omega(\alpha)\sigma)(V(z_0, \hat{z}_0) + L(\eta + \hat{\eta} + \varepsilon + \hat{\varepsilon})\sigma); \tag{4.2}$$

$$\varphi(y_0) \le m_\varepsilon(\sigma; \varphi(z_0)) \quad \text{and} \quad \varphi(\hat{y}_0) \le m_{\hat{\varepsilon}}(\sigma; \varphi(\hat{z}_0)). \tag{4.3}$$

Proof. We first show that there exist a sequence $\{s_j\}_{j=0}^\infty$ in $[0, \sigma)$ and a sequence $\{(z_j, \hat{z}_j)\}_{j=0}^\infty$ in $D(\alpha) \times D(\alpha)$ such that

$$0 = s_0 < s_1 < \cdots < s_j < \cdots < \sigma \quad \text{and} \quad \lim_{j \to \infty} s_j = \sigma,$$

$$\begin{gathered}
\|z_{j-1} + (s_j - s_{j-1})Az_{j-1} - z_j\| \le (\eta/2 + \varepsilon)(s_j - s_{j-1}),\\
\|\hat{z}_{j-1} + (s_j - s_{j-1})A\hat{z}_{j-1} - \hat{z}_j\| \le (\hat{\eta}/2 + \hat{\varepsilon})(s_j - s_{j-1}),
\end{gathered} \tag{4.4}$$

$$\begin{gathered}
\varphi(z_j) \le m_\varepsilon(s_j - s_{j-1}; \varphi(z_{j-1})),\\
\varphi(\hat{z}_j) \le m_{\hat{\varepsilon}}(s_j - s_{j-1}; \varphi(\hat{z}_{j-1})),
\end{gathered} \tag{4.5}$$

$$\begin{aligned}
&(V(z_j, \hat{z}_j) - V(z_{j-1}, \hat{z}_{j-1}))/(s_j - s_{j-1})\\
&\qquad \le \omega(\alpha)V(z_{j-1}, \hat{z}_{j-1}) + L(\eta + \hat{\eta} + \varepsilon + \hat{\varepsilon})
\end{aligned} \tag{4.6}$$

for $j = 1, 2, \ldots$. To do this, let $i \ge 1$ and assume that a sequence $\{s_j\}_{j=0}^{i-1}$ in $[0, \sigma)$ and a sequence $\{(z_j, \hat{z}_j)\}_{j=0}^{i-1}$ in $D(\alpha) \times D(\alpha)$ are chosen so that the desired conditions are satisfied for $0 \le j \le i-1$. Let us consider the number $\overline{h}_i$ defined by the supremum of all $h \ge 0$ such that $h < \sigma - s_{i-1}$ and

$$\begin{aligned}
&V(z_{i-1} + hAz_{i-1}, \hat{z}_{i-1} + hA\hat{z}_{i-1}) - V(z_{i-1}, \hat{z}_{i-1})\\
&\qquad \le (\omega(\alpha)V(z_{i-1}, \hat{z}_{i-1}) + (L/4)(\eta + \hat{\eta}))h.
\end{aligned}$$

Then we have $\overline{h}_i > 0$ by condition (A2), so that one finds a number $h_i > 0$ satisfying $\overline{h}_i/2 < h_i < \sigma - s_{i-1}$ and

$$(V(z_{i-1} + h_i A z_{i-1}, \hat{z}_{i-1} + h_i A \hat{z}_{i-1}) - V(z_{i-1}, \hat{z}_{i-1}))/h_i \\ \leq \omega(\alpha) V(z_{i-1}, \hat{z}_{i-1}) + (L/4)(\eta + \hat{\eta}).$$

Set $s_i = s_{i-1} + h_i$. Then it is obvious that $s_{i-1} < s_i < \sigma \leq r/(M+1)$.

To show the existence of the desired pair $(z_i, \hat{z}_i)$, we verify that all assumptions of Proposition 3.4 are satisfied with z_0, r and η replaced by z_{i-1}, $(M+1)h_i$ and $\eta/2$. Since $\eta/2 + \varepsilon < 1$ we apply Lemma 3.1 to obtain $\|z_{i-1} - z_0\| \leq (M + \eta/2 + \varepsilon)s_{i-1}$. This inequality implies that $B[z_{i-1}, (M+1)h_i] \subset B[z_0, r]$, by the choice of σ. It follows that $\|Ax\| \leq M$ and $\|Ax - Az_0\| \leq \eta/4$ for $x \in B[z_{i-1}, (M+1)h_i] \cap D(\alpha)$. By the second inequality we have $\|Ax - Az_{i-1}\| \leq \eta/2$ for $x \in B[z_{i-1}, (M+1)h_i] \cap D(\alpha)$. Since $\varphi(z_{i-1}) \leq m_\varepsilon(s_{i-1}; \varphi(z_0))$ we have by (m_1) and (m_2)

$$\tau_\varepsilon(\varphi(z_{i-1})) \geq \tau_\varepsilon(m_\varepsilon(s_{i-1}; \varphi(z_0))) = \tau_\varepsilon(\varphi(z_0)) - s_{i-1} > h_i, \\ m_\varepsilon(h_i; \varphi(z_{i-1})) \leq m_\varepsilon(s_{i-1} + h_i; \varphi(z_0)) \leq \alpha.$$

Proposition 3.4 then ensures the existence of an element $z_i \in D(\alpha)$ such that $\|z_{i-1} + h_i A z_{i-1} - z_i\| \leq (\eta/2 + \varepsilon)h_i$ and $\varphi(z_i) \leq m_\varepsilon(h_i; \varphi(z_{i-1}))$. It is shown similarly that there exists $\hat{z}_i \in D(\alpha)$ satisfying the inequalities (4.4) and (4.5). The desired inequality (4.6) with $j = i$ and the fact that $\lim_{i\to\infty} s_i = \sigma$ are obtained similarly to the proof of [3, Proposition 3.1].

Now, we apply (ii) of Lemma 3.2 to show that $y_0 = \lim_{j\to\infty} z_j$ and $\hat{y}_0 = \lim_{j\to\infty} \hat{z}_j$ exist and are in $D(\alpha)$ and that they satisfy (4.1). It is easily seen that the pair $(y_0, \hat{y}_0)$ is the desired one satisfying (4.2) and (4.3). □

Proposition 4.2. *Let $\alpha > 0$ and $x_0 \in D(\alpha)$. Assume $R > 0$, $M > 0$ and $\lambda, \mu \in (0, 1/2)$ to be chosen such that $\|Ax\| \leq M$ for $x \in B[x_0, R] \cap D(\alpha)$ and such that $\tau_\varepsilon(\varphi(x_0)) > R/(M+1)$ and $m_\varepsilon(R/(M+1); \varphi(x_0)) \leq \alpha$ for $\varepsilon = \lambda, \mu$. Let $\tau \in (0, R/(M+1)]$, and suppose that for each $\varepsilon = \lambda, \mu$, a sequence $\{(t_i^\varepsilon, x_i^\varepsilon)\}_{i=0}^\infty$ in $[0, \tau) \times D(\alpha)$ satisfies the following conditions:*

(i) $0 = t_0^\varepsilon < t_1^\varepsilon < \cdots < t_i^\varepsilon < \cdots < \tau$.
(ii) $t_i^\varepsilon - t_{i-1}^\varepsilon \leq \varepsilon$ *for* $i = 1, 2, \ldots$.
(iii) $\|x_{i-1}^\varepsilon + (t_i^\varepsilon - t_{i-1}^\varepsilon) A x_{i-1}^\varepsilon - x_i^\varepsilon\| \leq (\varepsilon/4)(t_i^\varepsilon - t_{i-1}^\varepsilon)$ *for* $i = 1, 2, \ldots,$ *where* $x_0^\varepsilon = x_0$.
(iv) *If* $x \in B[x_{i-1}^\varepsilon, (M+1)(t_i^\varepsilon - t_{i-1}^\varepsilon)] \cap D(\alpha)$, *then*

$$\|Ax - Ax_{i-1}^\varepsilon\| \leq \varepsilon/8 \text{ for } i = 1, 2, \ldots.$$

(v) $\varphi(x_i^\varepsilon) \leq m_\varepsilon(t_i^\varepsilon - t_{i-1}^\varepsilon; \varphi(x_{i-1}^\varepsilon))$ *for* $i = 1, 2, \ldots$.
(vi) $\lim_{i\to\infty} t_i^\varepsilon = \tau$.

Let $P = \{t_i^\lambda; i = 0,1,2,\dots\} \cup \{t_j^\mu; j = 0,1,2,\dots\}$. *Set* $s_0 = 0$ *and define* $s_k = \inf(P \setminus \{s_0, s_1, \dots, s_{k-1}\})$ *for* $k = 1,2,\dots$. *Then there exists a sequence* $\{(z_k^\lambda, z_k^\mu)\}_{k=0}^\infty$ *in* $D(\alpha) \times D(\alpha)$ *with three properties listed below.*

(a) *If* $s_k = t_i^\lambda$ *then* $z_k^\lambda = x_i^\lambda$; *and if* $s_k = t_j^\mu$ *then* $z_k^\mu = x_j^\mu$.

(b) *For each* $\varepsilon = \lambda, \mu$, *we have*

$$\sum_{j=q}^{k} \|z_{j-1}^\varepsilon + (s_j - s_{j-1})Az_{j-1}^\varepsilon - z_j^\varepsilon\| \le 2\varepsilon(s_k - s_{q-1}) + 3\varepsilon \sum_{t_i^\varepsilon \in \{s_q,\dots,s_k\}} (t_i^\varepsilon - t_{i-1}^\varepsilon)$$

for $1 \le q \le k$ *and* $k = 1,2,\dots$.

(c) *For each* $\varepsilon = \lambda, \mu$ *we have* $\varphi(z_k^\varepsilon) \le m_\varepsilon(s_k; \varphi(x_0))$ *for* $k = 0,1,2,\dots$.

(d) *For* $k = 0,1,2,\dots$ *we have*

$$V(z_k^\lambda, z_k^\mu) \le \exp(\omega(\alpha)s_k)(2L(\lambda+\mu)s_k + \eta_k(\lambda,\mu)).$$

Here the symbol $\eta_k(\lambda,\mu)$ *is defined by*

$$\eta_k(\lambda,\mu) = 3L\left(\lambda \sum_{t_i^\lambda \in \{s_1,\dots,s_k\}} (t_i^\lambda - t_{i-1}^\lambda) + \mu \sum_{t_j^\mu \in \{s_1,\dots,s_k\}} (t_j^\mu - t_{j-1}^\mu)\right).$$

Proof. Set $z_0^\varepsilon = x_0$ for each $\varepsilon = \lambda, \mu$. Let $l \ge 1$ and assume that a sequence $\{(z_k^\lambda, z_k^\mu)\}_{k=0}^{l-1}$ in $D(\alpha) \times D(\alpha)$ is defined so that conditions (a) through (d) are satisfied for $0 \le k \le l-1$. We shall apply Proposition 4.1 to find the desired pair (z_l^λ, z_l^μ) in $D(\alpha) \times D(\alpha)$. To do this, let i and j be positive integers such that $t_{i-1}^\lambda < s_l \le t_i^\lambda$ and $t_{j-1}^\mu < s_l \le t_j^\mu$. Then we begin by showing that

$$B[z_{l-1}^\lambda, (M+1)(s_l - s_{l-1})] \subset B[x_{i-1}^\lambda, (M+1)(t_i^\lambda - t_{i-1}^\lambda)], \tag{4.7}$$

$$B[x_{i-1}^\lambda, (M+1)(t_i^\lambda - t_{i-1}^\lambda)] \subset B[x_0, R]. \tag{4.8}$$

The set inclusion (4.8) follows from Lemma 3.1. By the definition of $\{s_l\}$ we have $t_{i-1}^\lambda \le s_{l-1} < s_l \le t_i^\lambda$ and $t_{i-1}^\lambda = s_p$ for some p, and then $x_{i-1}^\lambda = z_p^\lambda$ by hypothesis (a) of induction. Since the set $\{s_{p+1}, \dots, s_{l-1}\}$ has no points t_i^λ, hypothesis (b) of induction implies that

$$\|z_{j-1}^\lambda + (s_j - s_{j-1})Az_{j-1}^\lambda - z_j^\lambda\| \le 2\lambda(s_j - s_{j-1}) \tag{4.9}$$

for $j = p+1, \dots, l-1$. By (4.8) we have $\|Ax\| \le M$ for $x \in B[z_p^\lambda, (M+1)(t_i^\lambda - s_p)] \cap D(\alpha)$. It follows from Lemma 3.1 that $\|z_{l-1}^\lambda - z_p^\lambda\| \le (M+1)(s_{l-1} - s_p)$, which implies that (4.7) is true.

By (4.7) and (4.8) we have $\|Ax\| \le M$ and $\|Ax - Az_{l-1}^\lambda\| \le \lambda/4$ for $x \in B[z_{l-1}^\lambda, (M+1)(s_l - s_{l-1})] \cap D(\alpha)$. We apply the above argument again, with i replaced by j, to show that the above inequalities hold with λ replaced by μ. Since $\varphi(z^\varepsilon_{l-1}) \le m_\varepsilon(s_{l-1};\varphi(x_0))$ we have $\tau_\varepsilon(\varphi(z^\varepsilon_{l-1})) \ge \tau_\varepsilon(\varphi(x_0)) - s_{l-1} > s_l - s_{l-1}$ and $m_\varepsilon(s_l - s_{l-1};\varphi(z^\varepsilon_{l-1})) \le m_\varepsilon(s_l;\varphi(x_0)) \le \alpha$. We therefore deduce from Proposition 4.1 that there exists a pair (y^λ_l, y^μ_l) in $D(\alpha)\times D(\alpha)$ satisfying

$$\|z^\varepsilon_{l-1} + (s_l - s_{l-1})Az^\varepsilon_{l-1} - y^\varepsilon_l\| \le 2\varepsilon(s_l - s_{l-1}) \text{ for } \varepsilon = \lambda,\mu, \tag{4.10}$$

$$\begin{aligned} V(y^\lambda_l, y^\mu_l) \le \exp(\omega(\alpha)(s_l - s_{l-1}))(V(z^\lambda_{l-1}, z^\mu_{l-1}) \\ + 2L(\lambda+\mu)(s_l - s_{l-1})), \end{aligned} \tag{4.11}$$

$$\varphi(y^\varepsilon_l) \le m_\varepsilon(s_l - s_{l-1};\varphi(z^\varepsilon_{l-1})) \text{ for } \varepsilon = \lambda,\mu. \tag{4.12}$$

Now, we define (z^λ_l, z^μ_l) in $D(\alpha)\times D(\alpha)$ by

$$z^\lambda_l = \begin{cases} y^\lambda_l & \text{if } s_l < t^\lambda_i, \\ x^\lambda_i & \text{if } s_l = t^\lambda_i, \end{cases} \quad \text{and} \quad z^\mu_l = \begin{cases} y^\mu_l & \text{for } s_l < t^\mu_j, \\ x^\mu_j & \text{for } s_l = t^\mu_j. \end{cases}$$

By hypothesis (c) of induction and (4.9) we have $\varphi(y^\varepsilon_l) \le m_\varepsilon(s_l;\varphi(x_0))$ for $\varepsilon = \lambda,\mu$. This together with condition (v) implies that condition (c) is satisfied with $k = l$.

We prove that (b) is true for $k = l$. Since the sequence $\{(s_j, z^\lambda_j)\}_{j=p+1}^{l-1}$ and (s_l, y^λ_l) satisfy (4.9) and (4.10) with $\varepsilon = \lambda$, we have

$$\|z^\lambda_p + (s_l - s_p)Az^\lambda_p - y^\lambda_l\| \le ((\lambda/8) + 2\lambda)(s_l - s_p)$$

by (i) of Lemma 3.2 with $z_0 = z^\lambda_p (= x^\lambda_{i-1})$, $r = (M+1)(t^\lambda_i - t^\lambda_{i-1})$, $\eta = \lambda/8$ and $\varepsilon = 2\lambda$. If $s_l = t^\lambda_i$, then the above inequality combined with condition (iii) implies $\|y^\lambda_l - x^\lambda_i\| \le 3\lambda(t^\lambda_i - t^\lambda_{i-1})$. It is thus shown that

$$\|z^\lambda_l - y^\lambda_l\| \le 3\lambda \sum_{t^\lambda_i = s_l} (t^\lambda_i - t^\lambda_{i-1}) \quad \text{and} \quad \|z^\mu_l - y^\mu_l\| \le 3\mu \sum_{t^\mu_j = s_l} (t^\mu_j - t^\mu_{j-1}). \tag{4.13}$$

Combining this inequality and (4.10), and adding the resultant inequality to the inequality (b) with $k = l-1$ we obtain the desired property (b) with $k = l$.

The proof is completed by induction on k, since condition (d) is shown to be true for $k = l$, by substituting the inequality (d) with $k = l-1$ into the inequality

$$\begin{aligned} V(z^\lambda_l, z^\mu_l) &\le \exp(\omega(\alpha)(s_l - s_{l-1}))(V(z^\lambda_{l-1}, z^\mu_{l-1}) + 2L(\lambda+\mu)(s_l - s_{l-1})) \\ &\quad + 3L\left(\lambda \sum_{t^\lambda_i = s_l} (t^\lambda_i - t^\lambda_{i-1}) + \mu \sum_{t^\mu_j = s_l} (t^\mu_j - t^\mu_{j-1})\right) \end{aligned}$$

which is obtained by (4.11) and (4.13). □

Proof of Theorem 2.1. Let $x_0 \in D$ and choose $\alpha > 0$ such that $\varphi(x_0) < \alpha$. By condition (A1) there exist $r > 0$ and $M > 0$ such that $\|Ax\| \leq M$ for $x \in B[x_0, r] \cap D(\alpha)$. Let us choose $a > 0$ so that $m(a; \varphi(x_0)) < \alpha$, and put $R = r \wedge ((M+1)a)$. Then we have $\|Ax\| \leq M$ for $x \in B[x_0, R] \cap D(\alpha)$ and $m(R/(M+1); \varphi(x_0)) < \alpha$. By property ($\mathrm{m}_3$), there exists $\varepsilon_0 \in (0, 1/2)$ such that $\tau_\varepsilon(\varphi(x_0)) > R/(M+1)$ and $m_\varepsilon(R/(M+1); \varphi(x_0)) \leq \alpha$ for $\varepsilon \in (0, \varepsilon_0]$.

Let $\tau \in (0, R/(M+1)]$. Proposition 3.5 asserts that for each $\varepsilon \in (0, \varepsilon_0]$, there exists a sequence $\{(t_i^\varepsilon, x_i^\varepsilon)\}_{i=0}^\infty$ in $[0, \tau) \times D(\alpha)$ satisfying conditions (i) through (vi) in Proposition 4.2. Let us define a step function $u^\varepsilon \colon [0, \tau) \to X$ by $u^\varepsilon(t) = x_i^\varepsilon$ for $t \in [t_i^\varepsilon, t_{i+1}^\varepsilon)$ and $i = 0, 1, 2, \ldots$. Then we want to show that there exists a function u on $[0, \tau]$ such that

$$\sup_{t \in [0, \tau)} \|u^\varepsilon(t) - u(t)\| \to 0 \tag{4.14}$$

as $\varepsilon \downarrow 0$. For this purpose, let $\lambda, \mu \in (0, \varepsilon_0]$ and $\{s_k\}_{k=0}^\infty$ the sequence defined as in Proposition 4.2. Then there exists a sequence $\{(z_k^\lambda, z_k^\mu)\}_{k=0}^\infty$ in $D(\alpha) \times D(\alpha)$ satisfying properties (a) through (d) of Proposition 4.2.

Let $t \in [0, \tau)$, and let $k \geq 1$ be an integer such that $t \in [s_{k-1}, s_k)$. If i and j are positive integers such that $t_{i-1}^\lambda \leq s_{k-1} < s_k \leq t_i^\lambda$ and $t_{j-1}^\mu \leq s_{k-1} < s_k \leq t_j^\mu$, then we have by (4.7), $\|z_{k-1}^\lambda - x_{i-1}^\lambda\| \leq (M+1)(t_i^\lambda - t_{i-1}^\lambda)$ and $\|z_{k-1}^\mu - x_{j-1}^\mu\| \leq (M+1)(t_j^\mu - t_{j-1}^\mu)$. Combining these estimates with (d) of Proposition 4.2 we find, by property (V1),

$$V(u^\lambda(t), u^\mu(t)) \leq 5L \exp(\omega(\alpha)\tau)(\lambda + \mu)\tau + L(M+1)(\lambda + \mu).$$

It follows from condition (V2) that the sequence $\{u^\varepsilon(t)\}$ converges to a function $u(t)$ uniformly on $[0, \tau)$ as $\varepsilon \downarrow 0$. Since $\|u(t) - u(s)\| \leq M\,|t - s|$ for $t, s \in [0, \tau)$ (by Lemma 3.1), there exists a continuous function u defined on $[0, \tau]$ satisfying (4.14). Finally, it is easily seen that u is a solution to (CP;x_0) on $[0, \tau]$, by condition (iii) of Proposition 4.2. $\square$

5. Application

We study the Cauchy problem for quasi-linear equation

$$\begin{cases} \partial_t u = \partial_x v \\ \partial_t v = \beta'(\|u\|^2)\partial_x u \end{cases} \tag{5.1}$$

with periodic boundary condition

$$u(x + 2\pi, t) = u(x, t), \quad v(x + 2\pi, t) = v(x, t) \tag{5.2}$$

for $(x, t) \in \mathbb{R} \times [0, \infty)$. Here $\beta \in C^2([0, \infty); \mathbb{R})$ is a convex function satisfying $\beta'(r) \geq c^2 > 0$ for $r \geq 0$ and $\beta(0) = 0$, and the symbol $\|u\|$ is defined by $\|u\|^2 = \int_0^{2\pi} |u(x)|^2\, dx$. Without loss of generality we may assume that $0 < c \le 1$.

It is known ([2], [4] and [6]) that there exists a unique global solution (u, v) of the above problem for real analytic initial data. In this section we give an operator-theoretical approach to the above problem, by using Theorem 2.1.

Let $L^2_{2\pi}$ be the space of all measurable functions w such that $\|w\| < \infty$ and $w(x + 2\pi) = w(x)$ for $x \in \mathbb{R}$. This space is a Hilbert space equipped with inner product $\langle \cdot, \cdot \rangle$ and the associated norm $\|\cdot\|$. By $H^k_{2\pi}$ we denote the subspace of $L^2_{2\pi}$ consisting of w such that $\partial^j_x w \in L^2_{2\pi}$ for $1 \le j \le k$, and set $H^\infty_{2\pi} = \bigcap_{k=1}^\infty H^k_{2\pi}$.

Let X be the Hilbert space $L^2_{2\pi} \times L^2_{2\pi}$ with inner product $\langle (u, v), (w, z) \rangle = \langle u, w \rangle + \langle v, z \rangle$. Let $R > 0$ and $\delta > 0$, and define a subset D of X by the set of all elements $(u, v) \in H^\infty_{2\pi} \times H^\infty_{2\pi}$ such that $\beta(\|u\|^2) + \|v\|^2 \le R^2$ and $|u|^2 + |v|^2 < \infty$, where the functional $|w|$ on $H^\infty_{2\pi}$ is defined by

$$|w| = \left(\sum_{k=1}^\infty \frac{(2\delta)^{2k}}{(2k)!} \|\partial^k_x w\|^2 \right)^{1/2}.$$

It should be noticed by Lemma 6.1 that $w \in H^\infty_{2\pi}$ is real analytic if and only if it satisfies $|w| < \infty$ for some $\delta > 0$. (See also [1] and [4].) By considering the operator A from D into X defined by

$$A(u, v) = (\partial_x v, \beta'(\|u\|^2)\partial_x u) \text{ for } (u, v) \in D,$$

the problem is converted into the abstract Cauchy problem for A.

Let $E = \{(u, v) \in X; \beta(\|u\|^2) + \|v\|^2 < (2R)^2\}$. It is obvious that E is open in X such that $D \subset E$. To check condition (A2) we use the nonnegative functional V on $E \times E$ defined by

$$V((u, v), (w, z)) = \left(\beta'(\|u\|^2)\|u - w\|^2 + \|v - z\|^2\right)^{1/2}.$$

Since

$$\|u\| < 2R/c \quad \text{and} \quad \|v\| < 2R \quad \text{for } (u, v) \in E, \tag{5.3}$$

we have

$$c\|(u, v) - (w, z)\| \le V((u, v), (w, z)) \le \left(\beta'((2R/c)^2) \vee 1\right)^{1/2} \|(u, v) - (w, z)\|$$

for $(u, v), (w, z) \in E$, which means that condition (V2) is satisfied. To verify condition (V1), let $(u, v), (\hat{u}, \hat{v}), (w, z), (\hat{w}, \hat{z}) \in E$. By the triangle inequality we have

$$\begin{aligned}
&V((u,v),(w,z)) - V((\hat{u},\hat{v}),(\hat{w},\hat{z})) \\
&\quad \le \left(\beta'(\|u\|^2)\|u-\hat{u}\|^2 + \|v-\hat{v}\|^2\right)^{1/2} \\
&\qquad + \left(\beta'(\|u\|^2)\|w-\hat{w}\|^2 + \|z-\hat{z}\|^2\right)^{1/2} \\
&\qquad + \left|\sqrt{\beta'(\|u\|^2)} - \sqrt{\beta'(\|\hat{u}\|^2)}\right| \|\hat{u}-\hat{w}\|^2.
\end{aligned}$$

Since $\sqrt{\beta'(r)}$ is locally Lipschitz continuous on $[0,\infty)$ and $\beta'(r)$ is locally bounded on $[0,\infty)$, we see by (5.3) that the right-hand side is bounded by $L(\|(u,v)-(\hat{u},\hat{v})\| + \|(w,z)-(\hat{w},\hat{z})\|)$ for some positive constant L.

Let us employ the lower semicontinuous functional φ defined by $\varphi(u,v) = \beta'(\|u\|^2)\,|u|^2 + |v|^2$ for $(u,v)\in D$, and ∞ otherwise. Let $\alpha > 0$. If $\varphi(u,v) \le \alpha$ then we have $\|u\| \le R/c$, $\|\partial_x^k u\| \le ((2k)!\alpha)^{1/2}(c(2\delta)^k)$ and $\|\partial_x^k v\| \le ((2k)!\alpha)^{1/2}(2\delta)^k$ for $k = 1,2,\ldots$. The continuity of A from $D(\alpha)$ into X follows from Landau's inequality $\|\partial_x u\|^2 \le \|u\|\|\partial_x^2 u\|$ for $u \in L^2_{2\pi}$. Since

$$\begin{aligned}
&V((u,v),(w,z))D_+V((u,v),(w,z))(A(u,v),A(w,z)) \\
&\quad = \beta''(\|u\|^2)\langle u,\partial_x v\rangle\|u-w\|^2 + (\beta'(\|u\|^2)-\beta'(\|w\|^2))\langle \partial_x w, v-z\rangle
\end{aligned}$$

for $(u,v),(w,z) \in D(\alpha)$, condition (A2) is easily seen to be satisfied. To check condition (A3) we first show that for each $(u_0,v_0) \in D$ and $\lambda > 0$, there exists $(u_\lambda, v_\lambda) \in H^1_{2\pi} \times H^1_{2\pi}$ such that

$$u_\lambda - u_0 = \lambda \partial_x v_\lambda, \tag{5.4}$$

$$v_\lambda - v_0 = \lambda\beta'(\|u_\lambda\|^2)\partial_x u_\lambda. \tag{5.5}$$

To do this, let $\beta(\|u_0\|^2) + \|v_0\|^2 \le R^2$ and define $\tilde{\beta}(r) = \int_0^r \beta'(s \wedge (R/c)^2)ds$ for $r \ge 0$. It should be noticed that $\tilde{\beta}$ is convex. Let us define a sequence $\{(u_n,v_n)\} \in H^1_{2\pi} \times H^1_{2\pi}$ inductively by $u_n - u_0 = \lambda\partial_x v_n$ and $v_n - v_0 = \lambda\tilde{\beta}'(\|u_{n-1}\|^2)\partial_x u_n$ for $n = 1,2,\ldots$. Taking the inner products of the first equation and the second one with $\tilde{\beta}'(\|u_{n-1}\|^2)u_n$ and v_n respectively, we have

$$\tilde{\beta}'(\|u_{n-1}\|^2)(\|u_n\|^2 - \|u_0\|^2) + (\|v_n\|^2 - \|v_0\|^2) \le 0, \tag{5.6}$$

so that $c^2\|u_n\|^2 + \|v_n\|^2 \le \beta'((R/c)^2)\|u_0\|^2 + \|v_0\|^2$ for $n \ge 1$. By τ_h we denote the operator on $L^2_{2\pi}$ defined by $(\tau_h u)(x) = u(x+h)$ for $x \in \mathbb{R}$. Then we have $\tau_h u_n - \tau_h u_0 = \lambda\partial_x\tau_h v_n$ and $\tau_h v_n - \tau_h v_0 = \lambda\tilde{\beta}'(\|u_{n-1}\|^2)\partial_x\tau_h u_n$ for $n = 1,2,\ldots$. Applying the above argument again, with u_n, v_n replaced by $\tau_h u_n - u_n, \tau_n v_n - v_n$ we find

$$c^2\|\tau_h u_n - u_n\|^2 + \|\tau_h v_n - v_n\|^2 \le \beta'((R/c)^2)\|\tau_h u_0 - u_0\|^2 + \|\tau_h v_0 - v_0\|^2$$

for $n \geq 1$. It follows that the sequence $\{(u_n, v_n)\}$ has a convergent subsequence in X. The limit (u_λ, v_λ) satisfies $u_\lambda - u_0 = \lambda \partial_x v_\lambda$ and $v_\lambda - v_0 = \lambda \tilde{\beta}'(\|u_\lambda\|^2)\partial_x u_\lambda$. By an argument similar to the derivation of (5.6) and the convexity of $\tilde{\beta}$ we have $\tilde{\beta}(\|u_\lambda\|^2) + \|v_\lambda\|^2 \leq R^2$, which implies that $c^2\|u_\lambda\|^2 \leq R^2$ and then $\tilde{\beta}'(\|u_\lambda\|^2) = \beta'(\|u_\lambda\|^2)$. The desired claim is thus shown. The above argument also shows that

$$\beta(\|u_\lambda\|^2) + \|v_\lambda\|^2 \leq \beta(\|u_0\|^2) + \|v_0\|^2. \tag{5.7}$$

Now, let $(u_0, v_0) \in D$ and let $(u_\lambda, v_\lambda) \in H^1_{2\pi} \times H^1_{2\pi}$ satisfy (5.4) and (5.5). Then it is easily seen that $(u_\lambda, v_\lambda) \in H^\infty_{2\pi} \times H^\infty_{2\pi}$ and

$$\partial_x^k u_\lambda - \partial_\lambda^k u_0 = \lambda \partial_x^k \partial_x v_\lambda, \quad \partial_x^k v_\lambda - \partial_x^k v_0 = \lambda \beta'(\|u_\lambda\|^2)\partial_x^k \partial_x u_\lambda$$

for $k = 1, 2, \ldots$. Similarly to the derivation of (5.6) we have

$$\beta'(\|u_\lambda\|^2)\|\partial_x^k u_\lambda\|^2 + \|\partial_x^k v_\lambda\|^2 \leq \beta'(\|u_\lambda\|^2)\|\partial_x^k u_0\|^2 + \|\partial_x^k v_0\|^2 \tag{5.8}$$

for $k = 1, 2, \ldots$. This together with (5.7) implies that $(u_\lambda, v_\lambda) \in D$ and

$$\varphi(u_\lambda, v_\lambda) - \varphi(u_0, v_0) \leq (\beta'(\|u_\lambda\|^2) - \beta'(\|u_0\|^2)) \, |u_0|^2$$

for $\lambda > 0$. Since the sequences $\{\partial_x^2 u_\lambda\}$, $\{\partial_x v_\lambda\}$ and $\{\partial_x^2 v_\lambda\}$ are bounded in $L^2_{2\pi}$ as $\lambda \downarrow 0$ (by (5.7) and (5.8)), we have $u_\lambda \to u_0$, $\partial_x u_\lambda \to \partial_x u_0$ and $\partial_x v_\lambda \to \partial_x v_0$ in $L^2_{2\pi}$ as $\lambda \downarrow 0$. It follows that

$$\lim_{\lambda \downarrow 0} \|((u_\lambda, v_\lambda) - (u_0, v_0))/\lambda - A(u_0, v_0)\| = 0$$

(by (5.4) and (5.5)) and

$$\limsup_{\lambda \downarrow 0} (\varphi(u_\lambda, v_\lambda) - \varphi(u_0, v_0))/\lambda \leq 2\beta''(\|u_0\|^2)\langle u_0, \partial_x v_0 \rangle \, |u_0|^2 .$$

By the definition of φ and (5.3), the right-hand side is estimated by $a\|\partial_x v_0\|\varphi(u_0, v_0)$, where a denotes various positive constants depending only on R and c. Let $\chi(r) = (e^{2\delta r} + e^{-2\delta r})/2 - 1$ for $r \geq 0$. Lemma 6.2 then asserts that condition (A3) is satisfied with $g(r) = ar\chi^{-1}(r)$ for $r \geq 0$. Since $g(0) = 0$, $g(r) > 0$ for $r > 0$ and $\int_1^\infty \frac{1}{g(r)} dr = \infty$, we see that g is a comparison function. By using Theorem 2.1 the following theorem can be obtained.

Theorem 5.1. *For each* $(u_0, v_0) \in D$ *there exists a unique pair*

$$(u(\cdot, t), v(\cdot, t)) \in C^1\big([0, \infty); L^2_{2\pi} \times L^2_{2\pi}\big)$$

satisfying $(u(\cdot, t), v(\cdot, t)) \in D$ *for* $t \geq 0$, *such that* (5.1) *and* (5.2) *are satisfied.*

6. Appendix

Lemma 6.1. *Let $\delta > 0$ and $|w|$ the functional on $H^\infty_{2\pi}$ defined by*

$$|w|^2 = \sum_{k=1}^\infty \frac{(2\delta)^{2k}}{(2k)!}\|\partial_x^k w\|^2.$$

Then we have

$$\sup_{k\ge 1}\frac{\delta^k}{k!}\|\partial_x^k w\| \le |w| \le \sup_{k\ge 1}\frac{(2\delta)^k}{k!}\|\partial_x^k w\|$$

for $w \in H^\infty_{2\pi}$.

Proof. Let $w \in H^\infty_{2\pi}$. First, assume that $|w| < \infty$ and let $k = 1, 2, \ldots$. Since

$$\|\partial_x^k w\|^2 \le \sum_{j=1}^\infty \frac{(2k)!}{(2\delta)^{2k}}\frac{(2\delta)^{2j}}{(2j)!}\|\partial_x^j w\|^2$$

and $(2k)! = 2k\cdot(2k-1)\cdots 2\cdot 1 \le (2^k k!)^2$, we have $\|\partial_x^k w\| \le (k!/\delta^k)\,|w|$. Next, assume that $K := \sup_{k\ge 1}\frac{(2\delta)^k}{k!}\|\partial_x^k w\| < \infty$. Then we have

$$((2\delta)^{2k}/(2k)!)\|\partial_x^k w\|^2 \le K^2((k!)^2/(2k)!)$$

for $k = 1, 2, \ldots$. The desired inequality $|w| \le K$ follows from the inequality that $(k!)^2/(2k)! = (2^k k!)^2/(4^k(2k)!) \le (1/2)^k$ for $k = 1, 2, \ldots$. □

Lemma 6.2. *Let $|w|$ be the functional in Lemma 6.1 and $\chi(r) = (e^{2\delta r} + e^{-2\delta r})/2 - 1$ for $r \ge 0$. Then we have $\|\partial_x w\| \le (\|w\| \vee 1)\chi^{-1}(|w|^2)$ for $w \in H^\infty_{2\pi}$ with $|w| < \infty$.*

Proof. Let $w \in H^\infty_{2\pi}$ and $|w| < \infty$, and define $c_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} w(x)e^{-inx}dx$ for $n = 0, \pm 1, \pm 2, \ldots$. Then it is well-known that

$$\|\partial_x^k w\|^2 = 2\pi \sum_{n=-\infty}^{\infty} |n|^{2k}\,|c_n|^2\,. \tag{6.1}$$

We employ the function $f(r) = \chi(\sqrt{r})$ for $r \ge 0$. Since $f(r) = \sum_{k=1}^\infty \frac{(2\delta)^{2k}}{(2k)!} r^k$ for $r \ge 0$, we have $f(0) = 0$ and $f''(r) \ge 0$ for $r \ge 0$.

Now, let $K = \|w\| \vee 1$. Since $\sum_{n=-\infty}^\infty 2\pi\,|c_n|^2/K^2 \le 1$ and

$$(\|\partial_x w\|/K)^2 = \sum_{n=-\infty}^\infty (2\pi\,|c_n|^2/K^2)\,|n|^2 + \left(1 - \sum_{n=-\infty}^\infty (2\pi\,|c_n|^2/K^2)\right)0,$$

we have by the convexity of f and the fact that $f(0) = 0$,

$$f((\|\partial_x w\|/K)^2) \le \frac{2\pi}{K^2}\sum_{n=-\infty}^\infty f(|n|^2)\,|c_n|^2 = (|w|/K)^2.$$

Here we have used the identity (6.1) to obtain the last equality. The desired inequality follows readily from the above inequality and the definition of f. □

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