ON THE NILPOTENCY INDEX OF
THE RADICAL OF A GROUP ALGEBRA. XI

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Let \( t(G) \) be the nilpotency index of the radical \( J(KG) \) of a group algebra \( KG \) of a finite \( p \)-solvable group \( G \) over a field \( K \) of characteristic \( p > 0 \). Then it is well known by D. A. R. Wallace [7] that

\[
p^e \geq t(G) \geq e(p - 1) + 1,
\]

where \( p^e \) is the order of a Sylow \( p \)-subgroup of \( G \).

H. Fukushima [1] characterized a group \( G \) of \( p \)-length 2 satisfying \( t(G) = e(p - 1) + 1 \), see also [4]. Unfortunately, his characterization holds under a condition such that the \( p^0 \)-part \( V = O^p_0(G)/O_p(G) \) of \( G \) is abelian.

In this paper, using Dickson near fields, we shall give an explicit example (see Example 1) such that a group \( G \) of \( p \)-length 2 has the non abelian \( p' \)-part \( V \) and satisfies \( t(G) = e(p - 1) + 1 \). This example will be new and have contributions in our research. Example 2 is also very interesting because quite different objects (see [3] and [5]) are unified on the ground of Dickson near fields.

Let \( H \) be a sharply 2-fold transitive group on \( \Delta = \{0, 1, \alpha, \beta, \ldots, \gamma\} \) (see [8, p. 22]). Let \( V = H_0 \) be a stabilizer of 0, and let \( U \) be the set consisting of the identity \( \epsilon \) and fixed point-free permutations in \( H \). Then \( U \) is an elementary abelian \( p \)-subgroup of \( H \) with the order \( p^s \) (see Lemma 1). Let \( \sigma \) be a permutation of order \( p \) on \( \Delta \) satisfying conditions

\[
\sigma H \sigma^{-1} \subseteq H, \quad \sigma^p = 1, \quad \sigma(0) = 0 \quad \text{and} \quad \sigma(1) = 1.
\]

Then it is easy to see \( \sigma U \sigma^{-1} \subseteq U \) and \( \sigma V \sigma^{-1} \subseteq V \). We set \( W = \langle \sigma \rangle \) and \( C_V(\sigma) = \{ v \in V \mid \sigma v = \nu \sigma \} \). Assume that there exists a normal subgroup \( T \) of \( WV \) contained in \( V \) such that \( V \) is a semi-direct product of \( T \) by \( C_V(\sigma) \). We set \( G = \langle W, T, U \rangle \).

Now, we present the well known results Lemmas 1 and 2 for completeness of this paper.

**Lemma 1.** \( U \) is a normal and elementary abelian \( p \)-subgroup of \( H \).

**Proof.** First we shall prove, for \( k \in \Delta^* = \Delta \setminus \{0\} \), there exists only one \( u_k \in U \) with \( u_k(0) = k \), equivalently, the following map \( \nu \) from \( U \) to \( \Delta \) is bijective:

\[
\nu: u \mapsto u(0).
\]

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For $\tau \in U \setminus \{\varepsilon\}$, there exists $\rho \in H_0$ with $\rho(\tau(0)) = k$ since $\tau(0) \neq 0$ and $H_0$ is transitive on $\Delta^*$. We set $u_k = \rho \tau \rho^{-1}$. Then $u_k \in U$ and $u_k(0) = k$. Thus $\nu$ is surjective. It follows from definition of $H$ and $U$ that
\[
U = H \setminus \bigcup_{a \in \Delta} (H_a \setminus \{\varepsilon\}), \quad (H_a \setminus \{\varepsilon\}) \cap (H_b \setminus \{\varepsilon\}) = \emptyset \text{ for } a \neq b.
\]
Using $|H| = |H_a||a^H| = |H_a||\Delta|$, where $a^H$ is an orbit of $a$, we can see $|U| = |\Delta|$. Hence $\nu$ is injective.

Assume $\eta \tau$ has a fixed point $\ell$ for $\eta, \tau \in U$. Then we may assume $\ell = 0$ since $H$ is transitive on $\Delta$ and $\rho U \rho^{-1} = U$ for $\rho \in H$. Thus $\tau = \eta^{-1}$ follows from $\eta^{-1} \in U$, $\tau(0) = \eta^{-1}(0)$ and the above observation. This means $\eta \tau \in U$. Hence $U$ is a normal subgroup of $H$ because $\rho U \rho^{-1} = U$ for all $\rho \in H$.

Now, we shall show $U$ is elementary abelian. Let $p$ be a prime factor of $|U|$ and let $\tau$ be an element of order $p$ in the center of a Sylow $p$-subgroup of $U$. We set $\eta \in U \setminus \{\varepsilon\}$. Then there exists $\rho \in H_0$ with $\rho(\tau(0)) = \eta(0)$. Thus $\rho \tau \rho^{-1} = \eta$ follows from $\rho \tau \rho^{-1} \in U$ and $\rho \tau \rho^{-1}(0) = \eta(0)$. Thus the order of every element in $U$ is $p$ or $1$ and so $\eta$ is in the center of a $p$-group $U$. Thus $U$ is elementary abelian. \[\Box\]

The next shows $\Delta$ is a near field of characteristic $p$.

**Lemma 2.** $\Delta$ is a near field of characteristic $p$ and $\sigma$ is an automorphism of $\Delta$.

**Proof.** First, we shall prove that $\Delta$ is a near field. We can set a structure of a near field in a set $\Delta$ by the following method. It follows from Lemma 1 that there exists only one $u_a \in U$ with $u_a(0) = a$ for $a \in \Delta$. It is easy to see that for $a \in \Delta^* = \Delta \setminus \{0\}$, there exists only one $v_a \in V = H_0$ with $v_a(1) = a$. It is clear from definition that $u_0 = v_1 = \varepsilon$.

We define the sum and the product of elements $a, b$ in $\Delta$ by using the above $v_a$ and $u_b$:
\[
a + b := u_b(a), \quad ab := v_a(b) \text{ for } a \neq 0 \text{ and } 0b := 0.
\]
First we shall prove the next equations:
\[
u_u v_b = v_a v_b, \quad v_a u_b = u_a + a, \quad v_a u_b v_a^{-1} = u_a v_b.
\]
These follow from
\[
u_u v_b(0) = u_a(b) = b + a = u_b + a(0), 
\]
\[
u_a v_b(1) = v_a(b) = ab = v_a(1), \quad v_a u_b v_a^{-1}(0) = v_a u_b(0) = v_a(b) = ab = u_a(0).
\]
Next we shall prove the next equations from the first equation and the commutativity of $U$:

\[ a + (b + c) = u_{b+c}(a) = u_c u_b(a) = u_c(a + b) = (a + b) + c, \]
\[ a + b = u_{a+b}(0) = u_b u_a(0) = u_a u_b(0) = u_a(b) = b + a, \]
\[ a + 0 = a = u_a(0) = a, \]
\[ a + u_a^{-1}(0) = u_a^{-1}(0) + a = u_a(u_a^{-1}(0)) = \varepsilon(0) = 0. \]

We shall prove the next equations from the second equation for $a, b \in \Delta^*$. For $a = 0$ or $b = 0$, it is easy to prove our equations:

\[ a(bc) = v_a(bc) = v_a(v_b(c)) = v_a v_b(c) = v_{ab}(c) = (ab)c, \]
\[ a1 = v_a(1) = a = \varepsilon(a) = v_1(a) = 1a, \]
\[ av_a^{-1}(1) = v_a(v_a^{-1}(1)) = \varepsilon(1) = 1. \]

For $a \in \Delta^*, v_a^{-1}(1) \neq 0$ follows from $v_a(0) = 0 \neq 1$ and we can see $v_{v_a^{-1}(1)} = v_a^{-1}$ by $v_{v_a^{-1}(1)}(1) = v_a^{-1}(1)$. Thus we have

\[ v_a^{-1}(1)a = v_{v_a^{-1}(1)}(a) = v_a^{-1}(a) = v_a^{-1}(v_a(1)) = 1. \]

The next equation follows from the third equation:

\[ a(b + c) = v_a(b + c) = v_a(u_c(b)) = v_a u_c v_a^{-1}(v_a(b)) = u_{ac}(ab) = ab + ac. \]

Thus $\Delta$ is a near field by our definition of the sum and the product. Moreover $\Delta$ is of characteristic $p$ because $u_{p^1} = u_1^p = \varepsilon = u_0$.

Next we shall show $\sigma$ is an automorphism of $\Delta$. It is easy to see from the definitions of $U$ and $V$ that

\[ \sigma U \sigma^{-1} \subseteq U \quad \text{and} \quad \sigma V \sigma^{-1} \subseteq V. \]

It follows from the definitions of $u_a$ and $v_a$ that

\[ \sigma u_a \sigma^{-1} = u_{\sigma(a)} \quad \text{and} \quad \sigma v_b \sigma^{-1} = v_{\sigma(b)} \]

by equations

\[ \sigma u_a \sigma^{-1}(0) = \sigma u_a(0) = \sigma(a) = u_{\sigma(a)}(0) \]

and

\[ \sigma v_b \sigma^{-1}(1) = \sigma v_b(1) = \sigma(b) = v_{\sigma(b)}(1). \]

Since $\sigma$ is a permutation on $\Delta$, it follows from the next equations that $\sigma$ is an automorphism of $\Delta$:

\[ u_{\sigma(a+b)} = \sigma u_{a+b} \sigma^{-1} = \sigma u_a \sigma^{-1} \sigma u_b \sigma^{-1} = u_{\sigma(a)} u_{\sigma(b)} = u_{\sigma(a)+\sigma(b)} \]

and

\[ v_{\sigma(ab)} = \sigma v_{ab} \sigma^{-1} = \sigma v_a \sigma^{-1} \sigma v_b \sigma^{-1} = v_{\sigma(a)} v_{\sigma(b)} = v_{\sigma(a) \sigma(b)}. \]
We can see from Lemma 2 and the classification of finite near fields (see [9]) that $\Delta$ is a Dickson near field because $\Delta$ has an automorphism of order $p$ where $p$ is the characteristic of $\Delta$.

**Lemma 3.** $WT$ is a Frobenius group with kernel $T$ and complement $W$.

*Proof.* We note $W \cap V = \{\varepsilon\}$ since $\sigma(1) = 1$. Let $x = \sigma^k v$ be an element of $WT \setminus W$, where $v \in T$, and let $x^{-1} \sigma^t = \sigma^t \neq \varepsilon$ be an element of $x^{-1} W x \cap W$. Then we may assume $s = 1$ because the order of $\sigma$ is $p$. Thus $x^{-1} W x \cap W$ contains $v^{-1} \sigma v = \sigma^t$. The element $\sigma^{t-1} = v^{-1}. \sigma v \sigma^{-1}$ is contained in $W \cap V = \{\varepsilon\}$. Hence $\sigma v = v \sigma$. Thus $v \in C_V(\sigma) \cap T = \{\varepsilon\}$ and $x = \sigma^k v = \sigma^k$ is contained in $W$. Therefore we have

$$x^{-1} W x \cap W = \{\varepsilon\} \text{ for } x \in WT \setminus W. \quad \square$$

**Lemma 4.** $G = TC_G(\sigma) T$.

*Proof.* Clearly $TC_G(\sigma) T$ contains $T$ and $W$. Let $u_\delta$ be an arbitrary element of $U \setminus \{\varepsilon\}$, where $\delta$ is an arbitrary element in $\Delta^* = \Delta \setminus \{0\}$. Then $v_\delta = v_\gamma v_\lambda = v_{\gamma \lambda}$ where $v_\gamma \in T$ and $v_\lambda \in C_V(\sigma)$, namely, $\sigma(\lambda) = \lambda$. Thus $\delta = \gamma \lambda$ and so $u_\delta = v_\gamma u_\lambda v_{\gamma^{-1}} \in TC_G(\sigma) T$. It follows from $U \subset TC_G(\sigma) T$ that $G = TC_G(\sigma) T$. \quad \square

**Lemma 5.** $(J(K\hat{W}\hat{T}K)G)^n \subseteq J(KW)^n \hat{T}KG$, where $\hat{T} = \sum_{t \in T} t$.

*Proof.* Since $T$ is normal in $WV$ and $G = TC_G(\sigma) T$ by Lemma 4, we can see $s \sigma = \sigma s$ for every $s \in \hat{T}KG \hat{T} = \hat{T}KC_G(\sigma) \hat{T}$. Clearly the result holds for $n = 1$. Assume that the result holds for $n$. Then using the last assertion, we conclude that

$$(J(K\hat{W}\hat{T}K))^n+1 \subseteq J(KW)^n \hat{T}KGJ(KW) \hat{T}KG$$

$$= J(KW)^n \hat{T}KG \hat{T}J(KW)KG$$

$$\subseteq J(KW)^n+1 \hat{T}KG. \quad \square$$

**Theorem.** Let $S$ be a subgroup of $V$ containing $T$ and let $p^{n+1}$ be the order of a Sylow $p$-subgroup $WU$ of $M = \langle S, W, U \rangle$. Then $t(M) = (s+1)(p-1)+1$.

*Proof.* Let $v$ be an arbitrary element of $S$. Then $v = tc$ where $t \in T$ and $c \in C_V(\sigma)$. Hence we have

$$v \sigma v^{-1} = t c \sigma c^{-1} t^{-1} = t \sigma t^{-1} \in G = \langle T, W, U \rangle.$$ 

Noting $T$ is normal in $V$, we have that $G$ is a normal in $M$ and the index $[M : G]$ is relatively prime to $p$. Therefore we obtain $t(M) = t(G)$ and it is enough to prove in case $M = G$. Since the radical $J(KG)$ contains the kernel $J(KU)KG$ of the natural homomorphism $\nu$ of the group algebra $KG$ onto $K(G/U)$, it follows that $\nu(J(KG)) = \nu(J(KW) \hat{T})$ by Lemma 3 and
where \( m \) and \( u \) on \( \Theta \).

Theorem 1]). by [9, Satz 18] or [6, Theorem 5] because \( \tau \). Let \( D \) be a finite field of order \( q^{pn} \) and let \( D = D_{q^{pn}} \) be a finite Dickson near field defined by the automorphism \( \tau : x \rightarrow x^{q^p} \) of \( F \). Then an automorphism \( \sigma : x \rightarrow x^{q^n} \) of \( F \) is also of \( D \) by [9, Satz 18] or [6, Theorem 5] because \( p^{rn} = q^n \equiv 1 \mod n \) (see also [6, Theorem 1]).

Example 1. Let \((q, n)\) be a Dickson pair where \( p \) is a prime and \( q = p^r \) for a positive integer \( r \). Then \((q^p, n)\) is also a Dickson pair because \( q^p \equiv -1 \mod 4 \) if and only if \( q \equiv -1 \mod 4 \). Let \( F = F_{q^{pn}} \) be a finite field of order \( q^{pn} \) and let \( D = D_{q^{pn}} \) be a finite Dickson near field defined by the automorphism \( \tau : x \rightarrow x^{q^p} \) of \( F \). Then an automorphism \( \sigma : x \rightarrow x^{q^n} \) of \( F \) is also of \( D \).

Let \( \omega \) be a generator of the multiplicative group \( F^* \) and we set \( a = \omega^n \), \( b = \omega \) in \( F^* \). Then the multiplicative group \( D^* \) of \( D \) has the structure

\[ D^* = \langle a, b \mid a^m = 1, \ b^n = a^t, \ bab^{-1} = a^{q^p} \rangle, \]

where \( m = \frac{q^{pn} - 1}{n} \), \( t = \frac{m}{q^p - 1} \). Here we use the usual symbol as the product in \( D \) for simplicity. Do not confuse with the product in \( F \). We consider some permutations on \( D \):

\[ u_c : x \rightarrow x + c \text{ for } c \in D, \quad v_c : x \rightarrow cx \text{ for } c \in D^*. \]

Then we have some relations

\[ u_c u_d = u_{d+c}, \quad v_c v_d = v_{cd}, \quad v_c u_d v_c^{-1} = u_{cd}, \quad \sigma u_c \sigma^{-1} = u_{\sigma(c)}, \quad \sigma v_c \sigma^{-1} = v_{\sigma(c)} \]

on \( u_c, v_c, \sigma \). We set

\[ U = \{ u_c \mid c \in D \}, \quad V = \{ v_c \mid c \in D^* \}, \quad W = \langle \sigma \rangle \]

and

\[ T = \{ v_c \in V \mid c \in \langle a^{\frac{q^n - 1}{n}} \rangle \}. \]

It is easy to see that \( UV \) is sharply 2-fold transitive on \( D \), \( T \) is normal in \( WV \) and the order of \( T \) is \( \frac{q^{mn} - 1}{q^n - 1} \) because products of \( a \) and \( x \) in \( D \) are the same in \( F \). On the other hand, the set \( C_V(\sigma) \) is equal to \( F_{q^n}^* \) as a set and the order of \( C_V(\sigma) \) is \( q^n - 1 \). Since \( \frac{q^{mn} - 1}{q^n - 1} \) and \( q^n - 1 \) are relatively prime, we have \( V = C_V(\sigma) T, \ C_V(\sigma) \cap T = \{ e \} \). Let \( S \) be a subgroup of \( V \) containing \( T \) and \( M = \langle S, W, U \rangle \). Then \( t(M) = (rpm + 1)(p - 1) + 1 \) by Theorem, where \( rpm^{n+1} \) is the order of a Sylow \( p \)-subgroup \( WU \) of \( M \).
If we put $D = F$ for the extreme case $n = 1$, we have the same example as in [3].

**Example 2.** If $(q, n) \neq (3, 2)$ and $p$ is not a divisor of $r$, then $D_{qn}$ has no automorphisms of order $p$, and so we consider $D_{qn}$. But $D_{3^2}$ has an automorphism $\sigma$ of order 3 and we can consider the affine group $G = \langle \sigma, V, U \rangle$ over $D_{3^2}$ where $D_{3^2}$ is a Dickson near fields defined by an automorphism $x \rightarrow x^3$ of $F_{3^2} = F_3[x]/(x^2 + 1) = \{ s + ti \mid i^2 = -1, s, t \in F_3 \}$, $\sigma$ is defined by $\sigma(s + ti) = s + t + ti$, and the permutation group $U, V$ are defined as in Example 1. This group $G$ is isomorphic to $Qd(3)$, namely, a group defined by semi-direct product of $F_3^{(2)}$ by $SL(2, 3)$ using the natural action, where $F_3^{(2)}$ is 2-dimensional vector space over $F_3$ and $SL(2, 3)$ is the special linear group over $F_3^{(2)}$. In this case $3^3$ is the order of a Sylow 3-subgroup of $G$ and it is known form [5] that $t(G) = 9 > 7 = 3(3 - 1) + 1$.

This observation is very interesting because quite different objects (see [3] and [5]) are unified on the ground of Dickson near fields.

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**References**


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