A REPRESENTATION OF RING HOMOMORPHISMS
ON UNITAL REGULAR COMMUTATIVE
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Abstract. We give a complete representation of a ring homomorphism
from a unital semisimple regular commutative Banach algebra into a
unital semisimple commutative Banach algebra, which need not be reg-
ular. As a corollary we give a sufficient condition in order that a ring
homomorphism is automatically linear or conjugate linear.

1. Introduction and results

Let $A$ and $B$ be two algebras. We say that a map $\rho : A \rightarrow B$ is a ring
homomorphism if $\rho$ preserves both addition and multiplication. That is,

$$
\rho(f + g) = \rho(f) + \rho(g),
\quad \rho(fg) = \rho(f)\rho(g)
$$

for every $f, g \in A$. Moreover if such $\rho$ preserves scalar multiplication, then
we say that $\rho$ is a homomorphism.

In this paper, $C(K)$ denotes the commutative Banach algebra of all
complex-valued continuous functions on a compact Hausdorff space $K$. We
say that a map $\rho : C(X) \rightarrow C(Y)$ is a $*$-ring homomorphism if $\rho$ is a ring
homomorphism which also preserves complex conjugate: $\rho(\overline{f}) = \overline{\rho(f)}$ for ever-

y $f \in C(X)$. Šemrl [6] made a study of $*$-ring homomorphisms on $C(X)$
into $C(Y)$ and remarked that the problem of a general description of all
ring homomorphisms on $C(X)$ into $C(Y)$ is much more difficult than the
problem of characterizing all $*$-ring homomorphisms. In fact, let $G$ be the
set of all surjective ring homomorphisms between the complex number field
$\mathbb{C}$. It is well-known that the cardinal number of $G$ is $2^c$ (cf. [1]). Here $c$
denotes the cardinal number of $\mathbb{C}$.

Let $A$ be a unital regular semisimple commutative Banach algebra and
$B$ a unital semisimple commutative Banach algebra, which need not be regular.
In this paper, we consider a ring homomorphism $\rho : A \rightarrow B$ and
give a representation of $\rho$; hence a description of a ring homomorphism
on $C(X)$ into $C(Y)$ is given. This is an answer to the Šemrl’s remark
above. As a corollary, we can show [5, Theorem 1] and a unital version
of [6, Theorem 5.2]. We also prove that an injective or a surjective ring

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homomorphism on $A$ to $B$ is linear or conjugate linear if the maximal ideal spaces of $A$ and $B$ are both infinite and if every constant function is mapped to a constant function.

Throughout this note, $A$ and $B$ denote a unital regular semisimple commutative Banach algebra and a unital semisimple commutative Banach algebra with the maximal ideal spaces $M_A$ and $M_B$, respectively. The units of $A$ and $B$ are denoted by the same symbol $e$. We simply write $f$ for the Gelfand transform of $f$. Before we state our main theorem, we need some terminologies.

**Definition 1.1.** Let $\rho: A \to B$ be a ring homomorphism. For each $y \in M_B$ we define the induced ring homomorphism $\rho_y: A \to \mathbb{C}$ and $\tilde{\rho}_y: \mathbb{C} \to \mathbb{C}$ as

$$\rho_y(f) = \rho(f)(y) \quad (f \in A),$$

$$\tilde{\rho}_y(z) = \rho(ze)(y) \quad (z \in \mathbb{C}).$$

Moreover, $q_y: A \to A/\ker \rho_y$ denotes the quotient map for every $y \in M_B$.

A decomposition of a topological space $T$ is a family $\{T_1, T_2, \ldots, T_n\}$ of finitely many subsets $T_1, T_2, \ldots, T_n \subset T$ with the following properties:

$$T = \bigcup_{j=1}^{k} T_j \quad \text{and} \quad T_j \cap T_k = \emptyset \text{ if } j \neq k.$$

Note that each $T_j$ need not be clopen.

Let $A$ be a commutative algebra with unit. Recall that $\mathcal{P}$ is a prime ideal of $A$ if $\mathcal{P}$ is a proper ideal which satisfies that $fg \in \mathcal{P}$ implies $f \in \mathcal{P}$ or $g \in \mathcal{P}$. Here and after the term ideal will mean algebra ideal. In particular, every maximal ideal is a prime ideal. By Lemma 2.2, we see that the kernel $\ker \rho_y$ of the map $\rho_y: A \to \mathbb{C}$ is a prime ideal if $\ker \rho_y \neq A$. Hence, the quotient algebra $A/\ker \rho_y$ is an integral domain. Therefore, we can define the quotient field $\mathcal{F}_y$ of $A/\ker \rho_y$ if $\ker \rho_y \neq A$.

Now we are in a position to state our results.

**Theorem 1.1.** Let $\rho: A \to B$ be a ring homomorphism. Then there exist a decomposition $\{M_{-1}, M_0, M_1, M_m, M_p\}$ of $M_B$ and a continuous map $\Phi: M_B \setminus M_0 \to M_A$ with the following property:

For every $y \in M_m \cup M_p$ there exists a non-zero field homomorphism $\tau_y: \mathcal{F}_y \to \mathbb{C}$ such that

$$\rho(f)(y) = \begin{cases} 
  f(\Phi(y)) & y \in M_{-1} \\
  0 & y \in M_0 \\
  f(\Phi(y)) & y \in M_1 \\
  \tau_y(f(\Phi(y))) & y \in M_m \\
  \tau_y(q_y(f)) & y \in M_p 
\end{cases}$$
for every \( f \in A \).

Moreover, if \( \rho \) is surjective then the map \( \Phi \) is an injection defined on \( M_B \) into \( M_A \).

**Corollary 1.2.** Let \( \rho: A \to B \) be an injective or a surjective ring homomorphism satisfying \( \rho(\mathbb{C}e) \subset \mathbb{C}e \). If \( M_A \) and \( M_B \) are both infinite, then \( \rho \) is linear or conjugate linear.

Recall that a subset \( S \) of \( \mathbb{C}(X) \) is separating if for each \( x, y \in X \) with \( x \neq y \) there corresponds an \( f \in S \) so that \( f(x) \neq f(y) \). We say that \( S \) vanishes nowhere if for every \( x \in X \) there exists a function \( g \) of \( S \) such that \( g(x) \neq 0 \).

**Corollary 1.3** (cf. Molnar, [5]). Let \( \rho: \mathbb{C}(X) \to \mathbb{C}(Y) \) be a ring homomorphism whose range contains a separating subalgebra of \( \mathbb{C}(Y) \). If the range \( \rho(\mathbb{C}(X)) \) vanishes nowhere, then \( \rho \) is surjective.

**Corollary 1.4** (Šemrl, [6]). Let \( \rho: \mathbb{C}(X) \to \mathbb{C}(Y) \) be a \( * \)-ring homomorphism. Then there exist a clopen decomposition \( \{Y_{-1}, Y_0, Y_1\} \) of \( Y \) and a continuous map \( \Phi: Y_{-1} \cup Y_1 \to X \) such that

\[
\rho(f)(y) = \begin{cases} 
0 & y \in Y_0 \\
 f(\Phi(y)) & y \in Y_1 \\
 f(\Phi(y)) & y \in Y_1
\end{cases}
\]

for every \( f \in \mathbb{C}(X) \).

2. **Lemmas**

Let \( \tau: \mathbb{C} \to \mathbb{C} \) be a ring homomorphism. We simply say that \( \tau \) is a ring homomorphism on \( \mathbb{C} \). For example, \( \tau(z) = 0 \) \( (z \in \mathbb{C}) \), \( \tau(z) = z \) \( (z \in \mathbb{C}) \) and \( \tau(z) = \overline{z} \) \( (z \in \mathbb{C}) \) are ring homomorphisms on \( \mathbb{C} \); we call them trivial ring homomorphisms.

**Proposition 2.1.** Let \( \tau \) be a ring homomorphism on \( \mathbb{C} \). Then the following conditions are equivalent.

\begin{enumerate}
  \item \( \tau \) is trivial.
  \item There exist \( m_0, L_0 > 0 \) such that \( |z| < m_0 \) implies \( |\tau(z)| \leq L_0 \).
  \item \( \tau \) is continuous at \( 0 \).
  \item \( \tau \) is continuous at every point of \( \mathbb{C} \).
  \item \( \tau \) preserves complex conjugate.
\end{enumerate}
Proof. (i) ⇒ (ii) It is obvious.

(ii) ⇒ (iii) It is enough to consider the case where \( \tau \) is non-zero. Then by a simple calculation, we see that \( \tau(r) = r \) for every \( r \in \mathbb{Q} \), the rational number field of real numbers. For every \( \varepsilon > 0 \) fix an \( r_0 \in \mathbb{Q} \) with \( L_0 < r_0 \varepsilon \). If \( |z| < m_0/r_0 \) then we have \( |\tau(r_0z)| \leq L_0 \) by hypothesis. Since \( \tau \) fixes every rational number, we obtain \( |\tau(z)| \leq L_0/r_0 < \varepsilon \) if \( |z| < m_0/r_0 \). Thus \( \tau \) is continuous at 0.

(iii) ⇒ (iv) Let \( \{z_n\} \) be a sequence converging to \( z \). Since \( \tau \) is continuous at 0, we see that \( \tau(z_n - z) \to 0 \) as \( n \to \infty \). Hence \( \tau(z_n) \) converges to \( \tau(z) \).

(iv) ⇒ (v) We consider the case where \( \tau \) is non-zero. Then \( \tau(r) = r \) for every \( r \in \mathbb{Q} \). Since \( \tau \) is continuous, we have that \( \tau(t) = t \) for every \( t \in \mathbb{R} \), the real number field. We also have that \( \tau(i) = \pm i \) since \( \tau(-1) = -1 \). This implies that \( \tau(z) = \overline{\tau(z)} \) for every \( z \in \mathbb{C} \).

(v) ⇒ (i). By hypothesis, we have \( \tau(\mathbb{R}) \subseteq \mathbb{R} \), and hence \( \tau(x + h^2) - \tau(x) = \{\tau(h)\}^2 \geq 0 \) for every \( x, h \in \mathbb{R} \). It follows that \( \tau(x) \geq \tau(y) \) for \( x, y \in \mathbb{R} \) with \( x \geq y \). If \( \tau \) is non-zero, then \( \tau \) fixes all \( r \in \mathbb{Q} \). Therefore, we obtain \( \tau(x) = x \) for \( x \in \mathbb{R} \), so that \( \tau \) is trivial. \( \square \)

As remarked in the previous section, there exist non-trivial ring homomorphisms on \( \mathbb{C} \). By Proposition 2.1, non-trivial ring homomorphisms are discontinuous at each point of \( \mathbb{C} \). Moreover a non-trivial ring homomorphism \( \tau \) on \( \mathbb{C} \) has the following property:

For every pair \( m, L > 0 \) there exists a \( z \in \mathbb{C} \) such that \( |z| < m \) but \( |\tau(z)| > L \).

It is well-known that the kernels of non-zero complex homomorphisms on a unital commutative Banach algebra are maximal ideals. Let \( \mathbb{N} \) be the space of all natural numbers and \( K_0 = \{0\} \cup \{1/n; n \in \mathbb{N}\} \) with its usual topology. Šemrl showed the existence of a non-zero complex ring homomorphism \( \varphi \) on \( C(K_0) \) whose kernel ker \( \varphi \) is not a maximal ideal of \( C(K_0) \) ([6, Example 5.4]). We show that the kernel ker \( \phi \) of a non-zero complex ring homomorphism \( \phi \) on \( A \) is a prime ideal that is contained in a unique maximal ideal. De Marco and Orsatti [4] gave a characterization of a commutative ring with unit of which each prime ideal containing the Jacobson radical is contained in a unique maximal ideal.

**Lemma 2.2.** Let \( \phi: A \to \mathbb{C} \) be a non-zero ring homomorphism. Then the kernel ker \( \phi \) is a prime ideal which is contained in a unique maximal ideal of \( A \).

**Proof.** As a first step, we show that ker \( \phi \) is an ideal of \( A \). Since \( \phi \) preserves both addition and multiplication, it is enough to show that \( zf \) belongs to ker \( \phi \) for every \( z \in \mathbb{C} \) and \( f \in \text{ker} \phi \). Note that \( \phi(1) = 1 \) since \( \phi \) is non-zero.
Therefore, we have
\[ \phi(zf) = \phi(zf)\phi(e) = \phi(f)\phi(ze) = 0 \]
for every \( z \in \mathbb{C} \) and \( f \in \ker \phi \). Hence \( \ker \phi \) is an ideal of \( A \). It is now obvious that \( \ker \phi \) is a prime ideal.

Since \( \ker \phi \) is a proper ideal, there corresponds an \( x_0 \in M_A \) such that \( \ker \phi \subset \{ f \in A; f(x_0) = 0 \} \). We show that \( \{ f \in A; f(x_0) = 0 \} \) is the unique maximal ideal containing \( \ker \phi \). To this end, assume to the contrary that there exists an \( x_1 \in M_A \) such that \( x_0 \neq x_1 \) and \( \ker \phi \subset \{ f \in A; f(x_1) = 0 \} \). Let \( V_j \) be an open neighborhood of \( x_j \) for \( j = 0, 1 \) so that \( V_0 \cap V_1 = \emptyset \). Since \( A \) is regular, there corresponds an \( f_j \in A \) such that
\[ f_j(x_j) = 1 \quad \text{and} \quad f_j(A / \ker \phi) = 0 \quad (j = 0, 1). \]
Then \( f_0f_1 = 0 \) on \( M_A \). Since \( \ker \phi \) is a prime ideal, \( f_0 \) or \( f_1 \) belongs to \( \ker \phi \). This is a contradiction since \( f_j(x_j) = 1 \) for \( j = 0, 1 \). Hence \( \ker \phi \) is contained in the unique maximal ideal \( \{ f \in A; f(x_0) = 0 \} \).

**Lemma 2.3.** Let \( \phi: A \rightarrow \mathbb{C} \) be a non-zero ring homomorphism and \( q: A \rightarrow A/\ker \phi \) the quotient map. Then \( \phi \) is of the form \( \phi = \tau \circ q \) for some non-zero field homomorphism \( \tau \) on the quotient field \( F \) of \( A/\ker \phi \). If, in addition, \( \ker \phi \) is a maximal ideal of \( A \), then we may consider \( \tau \) a non-zero ring homomorphism on \( \mathbb{C} \) and \( q \in M_A \).

**Proof.** Note that the quotient field \( F \) of \( A/\ker \phi \) is well-defined since \( \ker \phi \) is a prime ideal of \( A \), by Lemma 2.2. We define the map \( \tau: F \rightarrow \mathbb{C} \) by
\[
(\#) \quad \tau([f]/[g]) = \frac{\rho(f)}{\rho(g)} \quad ([f]/[g] \in F).
\]
Here \([f] \in A/\ker \phi \) denotes the equivalence class of \( f \in A \) with respect to \( \ker \phi \). Then \( \tau \) is a well-defined non-zero field homomorphism on \( F \). If we identify \([f]\) with \([f]/[e]\), it is obvious that \( \phi \) is of the form \( \phi = \tau \circ q \).

Moreover if \( \ker \phi \) is a maximal ideal of \( A \), then the quotient algebra \( A/\ker \phi \) is isometrically isomorphic to \( \mathbb{C} \). Thus, we may identify \( A/\ker \phi \) with the quotient field \( F \) of \( A/\ker \phi \). Let \( I \) be the isomorphism on \( A/\ker \phi \) onto \( \mathbb{C} \). Then \( \tau \circ I^{-1} \) is a ring homomorphism on \( \mathbb{C} \) and \( I \circ q \) a non-zero complex homomorphism on \( A \) with \( \rho = \tau \circ q = (\tau \circ I^{-1}) \circ (I \circ q) \). This completes the proof. \(

**Definition 2.1.** Let \( \rho: A \rightarrow B \) be a ring homomorphism. Put \( M_0 = \{ y \in M_B; \ker \rho_y = A \} \). We define the subsets \( M_{B(m)} \) and \( M_{B(p)} \) of \( M_B \setminus M_0 \) as
\[
M_{B(m)} = \{ y \in M_B \setminus M_0; \ker \rho_y \text{ is a maximal ideal of } A \},
\]
\[
M_{B(p)} = \{ y \in M_B \setminus M_0; \ker \rho_y \text{ is not a maximal ideal of } A \}.
\]
Let $M_{-1}$, $M_1$, $M_{m,-1}$ and $M_{m,1}$ be as follows:

\[
M_{-1} = \{y \in M_B(m); \tilde{\rho}_y(z) = z \ (z \in \mathbb{C})\}, \\
M_1 = \{y \in M_B(m); \tilde{\rho}_y(z) = z \ (z \in \mathbb{C})\}, \\
M_{m,-1} = \{y \in M_B(m); \tilde{\rho}_y \text{ is non-trivial and } \tilde{\rho}_y(i) = -i\}, \\
M_{m,1} = \{y \in M_B(m); \tilde{\rho}_y \text{ is non-trivial and } \tilde{\rho}_y(i) = i\}.
\]

The subsets $M_{p,-1}$ and $M_{p,1}$ of $M_B(p)$ are defined by

\[
M_{p,-1} = \{y \in M_B(p); \tilde{\rho}_y(i) = -i\}, \\
M_{p,1} = \{y \in M_B(p); \tilde{\rho}_y(i) = i\}.
\]

Then we write $M_{d,j} = M_{m,j} \cup M_{p,j} (j = -1, 1)$ and $M_d = M_{d,-1} \cup M_{d,1}$.

Note that $\tilde{\rho}_y$ is a non-trivial ring homomorphism on $\mathbb{C}$ for every $y \in M_d$. For if $\tilde{\rho}_y$ is trivial then

\[
\rho_y(zf) = \tilde{\rho}_y(z)\rho_y(f) \ (z \in \mathbb{C}, f \in A)
\]

implies that ker $\rho_y$ is maximal for every $y \in M_B \setminus M_0$. By definition, the subsets $M_{-1}$, $M_0$, $M_1$ and $M_d$ of $M_B$ are mutually disjoint and $M_B = M_{-1} \cup M_0 \cup M_1 \cup M_d$. Hence, $\{M_{-1}, M_0, M_1, M_d\}$ above is a decomposition of $M_B$. We call $\{M_{-1}, M_0, M_1, M_d\}$ the decomposition of $M_B$ with respect to $\rho$.

Until the end of this section, $\rho: A \to B$ denotes a ring homomorphism and $\{M_{-1}, M_0, M_1, M_d\}$ the decomposition of $M_B$ with respect to $\rho$.

**Lemma 2.4.** The sets $M_0$, $M_{-1} \cup M_{d,-1}$ and $M_1 \cup M_{d,1}$ are clopen in $M_B$. Also $M_{-1}$ and $M_1$ are both closed in $M_B$.

**Proof.** By definition, it is easy to see that

\[
M_0 = \{y \in M_B; \tilde{\rho}_y(i) = 0\}, \\
M_{-1} \cup M_{d,-1} = \{y \in M_B; \tilde{\rho}_y(i) = -i\}, \\
M_1 \cup M_{d,1} = \{y \in M_B; \tilde{\rho}_y(i) = i\}.
\]

Therefore, $M_0$, $M_{-1} \cup M_{d,-1}$ and $M_1 \cup M_{d,1}$ are clopen since the function $\rho(ie)$ is continuous on $M_B$.

Next, we show that $M_1$ is closed in $M_B$. For every $y \in M_{d,1}$ we can find a $z_0 \in \mathbb{C}$ such that $\tilde{\rho}_y(z_0) \neq z_0$ since $\tilde{\rho}_y$ is non-trivial. Put

\[
V = \{w \in M_B; |\rho(z_0e)(w) - \rho(z_0e)(y)| < |z_0 - \tilde{\rho}_y(z_0)|/2\}.
\]

Then $V$ is an open neighborhood of $y$ with $V \cap M_1 = \emptyset$. Since $M_1 \cup M_{d,1}$ is clopen, this implies that $M_1$ is closed. In a way similar to the above, we see that $M_{-1}$ is closed and the proof is omitted. \hfill $\Box$
**Definition 2.2.** By Lemma 2.2, for every \( y \in M_B \setminus M_0 \) there exists a unique \( x \in M_A \) such that \( \ker \rho_y \subset \{ f \in A; f(x) = 0 \} \). We denote the correspondence defined on \( M_B \setminus M_0 \) into \( M_A \) as \( \Phi \); That is, \( \ker \rho_y \) is contained in the unique maximal ideal \( \{ f \in A; f(\Phi(y)) = 0 \} \) for every \( y \in M_B \setminus M_0 \). We call \( \Phi \) the representing map for \( \rho \).

**Lemma 2.5.** Let \( r \in \mathbb{Q} \), \( G \) open in \( M_A \) and \( \Phi \) the representing map for \( \rho \). Suppose that \( h \in A \) satisfies \( h(G) = r \) then \( \rho_y(h) = r \) for every \( y \in \Phi^{-1}(G) \).

**Proof.** Put \( h_r = h - re \in A \) and fix \( y \in \Phi^{-1}(G) \). Since \( A \) is regular, there exists a function \( g \in A \) such that \( g(\Phi(y)) = 1 \) and \( g(M_A \setminus G) = 0 \). Then \( gh_r = 0 \) on \( M_A \). Since \( \ker \rho_y \) is a prime ideal, \( g \) or \( h_r \) belongs to \( \ker \rho_y \). On the other hand, \( g \) does not belong to \( \{ f \in A; f(\Phi(y)) = 0 \} \) since \( g(\Phi(y)) = 1 \). So we conclude that \( h_r \in \ker \rho_y \). Therefore we have \( \rho_y(h) = r \) for every \( y \in \Phi^{-1}(G) \).

**Lemma 2.6.** Let \( \Phi \) be the representing map for \( \rho \). Then the range \( \Phi(M_d) \) is at most finite.

**Proof.** Assume to the contrary that \( \Phi(M_d) \) has a countable subset \( \{ x_n \}_{n=1}^{\infty} \) such that \( x_j \neq x_k \) if \( j \neq k \). Without loss of generality, we may assume that each \( x_j \) is an isolated point of \( \{ x_n \}_{n=1}^{\infty} \). By definition, for every \( n \in \mathbb{N} \) there exists a \( y_n \in M_d \) such that \( x_n = \Phi(y_n) \). By induction, we can find an open neighborhood \( U_j \) of \( x_j \) with

\[
(\bar{U}_j \setminus \{ x_j \}) \cap \{ x_n \}_{n=1}^{\infty} = \emptyset \quad \text{and} \quad \bar{U}_{j+1} \subset M_A \setminus \bigcup_{k=1}^{j} \bar{U}_k
\]

for every \( j \in \mathbb{N} \). Here \( \bar{U}_j \) denotes the closure of \( U_j \) in \( M_A \). Let \( V_j \) be an open neighborhood of \( x_j \) so that \( \bar{V}_j \subset U_j \). Since \( A \) is regular, \( A \) is normal (cf. [2, Theorem 6.3 of Chapter I]). That is, there exists a \( g_j \in A \) such that \( g_j(\bar{V}_j) = 1 \) and \( g_j(M_A \setminus U_j) = 0 \). Since \( \tilde{\rho}_{y_j} \) is non-trivial, there corresponds a \( z_j \in \mathbb{C} \) so that

\[
|z_j| < (2^j \| g_j \|)^{-1} \quad \text{and} \quad |\tilde{\rho}_{y_j}(z_j)| > 2^j,
\]

by Proposition 2.1. Here \( \| \cdot \| \) denotes the Banach norm on \( A \). Put \( f_j = z_j g_j \in A \). Then \( \rho_y(f_j) = \tilde{\rho}_{y_j}(z_j) \rho_y(g_j) \) for every \( y \in M_B \). Therefore, by Lemma 2.5 we see that \( \rho_y(f_j) = \tilde{\rho}_{y_j}(z_j) \). Since \( \| f_j \| < 2^{-j} \), the series \( \sum_{n=1}^{\infty} f_n \) converges in \( A \), say \( f_0 \). Note that \( f_j = 0 \) on \( V_k \) if \( k \neq j \). Thus we see that \( f_0 = f_j \) on \( V_j \) for every \( j \in \mathbb{N} \). By Lemma 2.5, we obtain \( \rho_{y_j}(f_0 - f_j) = 0 \). Therefore,

\[
|\rho_{y_j}(f_0)| = |\rho_{y_j}(f_j)| = |\tilde{\rho}_{y_j}(z_j)| \geq 2^j \quad (j \in \mathbb{N} \).
\]

This is a contradiction since \( \rho(f_0) \) is bounded on \( M_B \). Hence we have proved that the range \( \Phi(M_d) \) is at most finite.

\( \square \)
3. A proof of main result

Proof of Theorem 1.1. Let \( \{M_{-1}, M_0, M_1, M_d\} \) and \( \Phi \) be the decomposition of \( M_B \) with respect to \( \rho \) and the representing map for \( \rho \), respectively. For every \( y \in M_B \setminus M_0 \), let \( q_y : A \to A/\ker \rho_y \) denote the quotient map. Recall that \( M_{B(m)} \) is the set of all \( y \in M_B \) so that \( \ker \rho_y \) is a maximal ideal of \( A \). By Lemma 2.3, we can find a field homomorphism \( \tau_y \) on the quotient field \( \mathcal{F}_y \) of the integral domain \( A/\ker \rho_y \) into \( \mathbb{C} \) such that \( \rho_y = \tau_y \circ q_y \). If, in addition, \( y \in M_{B(m)} \), then we may consider that \( \tau_y \) is a ring homomorphism on \( \mathbb{C} \) and \( q_y \in M_A \). In this case, we therefore have \( \ker q_y = \ker \rho_y = \ker \Phi(y) \). Hence, we see that \( q_y = \Phi(y) \) for every \( y \in M_{B(m)} \). By the formula (3), we also have \( \tau_y = \tilde{\rho}_y \) for every \( y \in M_{B(m)} \). That is, \( \tau_y(z) = z \) if \( y \in M_{-1} \), \( \tau_y(z) = z \) if \( y \in M_1 \) and \( \tau_y \) is non-trivial if \( y \in M_{m,-1} \cup M_{m,1} \). Therefore, we have

\[
\rho(f)(y) = \begin{cases} 
0 & y \in M_0 \\
\tau_y(f(\Phi(y))) & y \in M_{B(m)} \\
\tau_y(q_y(f)) & y \in M_{B(p)} \\
\frac{f(\Phi(y))}{\tau_y(q_y(f))} & y \in M_{m,-1} \\
\tau_y(q_y(f)) & y \in M_{m,1} \\
\frac{f(\Phi(y))}{\tau_y(q_y(f))} & y \in M_{p,-1} \\
\frac{f(\Phi(y))}{\tau_y(q_y(f))} & y \in M_{p,1}
\end{cases}
\]

for every \( f \in A \).

By Lemma 2.6, we may put \( \Phi(M_d) = \{x_1, x_2, \ldots, x_m\} \). Then we see that the set \( M_d(x_j) = \{y \in M_d; \Phi(y) = x_j\} \) is open in \( M_B \) for \( j = 1, 2, \ldots, m \). Indeed, assume to the contrary that \( M_d(x_j) \) is not open. Then there exist a \( y_j \in M_d(x_j) \) and a net \( \{y_\alpha\} \) in \( M_B \setminus M_d(x_j) \) such that \( y_\alpha \) converges to \( y_j \). Since \( M_{-1} \cup M_0 \cup M_1 \) is closed in \( M_B \) by Lemma 2.4, we see that \( M_d \) is an open subset of \( M_B \). Therefore, without loss of generality we may assume \( \{y_\alpha\} \subset M_d \setminus M_d(x_j) \). Fix open neighborhoods \( O_1, O_2 \) of \( x_j \) with \( \overline{O}_1 \subset O_2 \) and \( \overline{O}_2 \cap \Phi(M_d) = \{x_j\} \). Here, \( \overline{\tau} \) denotes the closure in \( M_A \). Since \( A \) is regular, we can find a function \( h_j \in A \) so that \( h_j(\overline{O}_1) = 1 \) and \( h_j(M_A \setminus O_2) = 0 \). By Lemma 2.5, we have that \( \rho_{y_\alpha}(h_j) = 1 \) and \( \rho_{y_\alpha}(h_j) = 0 \) for every \( \alpha \). This is a contradiction since \( \rho(h_j) \) is continuous on \( M_B \). Therefore, the set \( M_d(x_j) = \{y \in M_d; \Phi(y) = x_j\} \) is open in \( M_B \) for \( j = 1, 2, \ldots, m \).

Finally we show that the map \( \Phi \) on \( M_B \setminus M_0 \) into \( M_A \) is continuous. Indeed, we see that \( \Phi \) is continuous at each \( y_0 \in M_d \) since \( M_d(\Phi(y_0)) = \{y \in M_d; \Phi(y) = \Phi(y_0)\} \) is open as proved above. We show that \( \Phi \) is continuous on \( M_{-1} \cup M_1 \). Let \( y_1 \) be a point of \( M_1 \) and \( \{y_\beta\}_{\beta \in \Gamma} \) an arbitrary net in \( M_B \setminus M_0 \) converging to \( y_1 \). Since \( M_0 \cup M_{-1} \) is closed in \( M_B \), we see
that \( M_1 \cup M_d \) is an open subset of \( M_B \). Hence, without loss of generality we may assume \( \{y_\beta\}_{\beta \in \Gamma} \subset M_1 \cup M_d \). We assert that there exists a \( \beta_0 \in \Gamma \) such that \( y_\beta \in M_1 \cup \{y \in M_d; \Phi(y) = \Phi(y_1)\} \) for every \( \beta \in \Gamma \) with \( \beta \geq \beta_0 \).

In fact, let \( W_1 \) be an open neighborhood of \( \Phi(y_1) \) and \( W_2 \) an open subset containing \( \Phi(M_d) \setminus \{\Phi(y_1)\} \) so that \( \overline{W}_1 \cap \overline{W}_2 = \emptyset \). Then we can find a \( g_0 \in A \) such that \( g_0(\overline{W}_1) = 1 \) and \( g_0(\overline{W}_2) = 0 \). By Lemma 2.5, we see that \( \rho(y_0) = 1 \) and \( \rho(y_0) = 0 \) for every \( y \in \Phi^{-1}(W_2) \). By the continuity of \( \rho(y_0) \), there exists a \( \beta_0 \in \Gamma \) such that \( \beta \geq \beta_0 \) implies \( |\rho(y_0)(y_\beta) - 1| < 1/2 \).

That is, \( \Phi(y_\beta) \not\in \Phi(M_d) \setminus \{\Phi(y_1)\} \) if \( \beta \geq \beta_0 \). Therefore, we see that \( y_\beta \in M_1 \cup \{y \in M_d; \Phi(y) = \Phi(y_1)\} \) for every \( \beta \in \Gamma \) with \( \beta \geq \beta_0 \). Hence, if \( \beta \geq \beta_0 \) then we have

\[
f(\Phi(y_\beta)) = \begin{cases} 
\rho(f)(y_\beta) & y_\beta \in M_1 \\
f(\Phi(y_1)) & \Phi(y_\beta) = \Phi(y_1)
\end{cases}
\]

for every \( f \in A \). Consequently, \( \beta \geq \beta_0 \) implies that

\[
|f(\Phi(y_\beta)) - f(\Phi(y_1))| \leq |\rho(f)(y_\beta) - \rho(f)(y_1)|
\]

for every \( f \in A \). Thus \( \Phi(y_\beta) \) converges to \( \Phi(y_1) \). This implies that \( \Phi \) is continuous on \( M_1 \). In a way similar to the above, we can show that \( \Phi \) is continuous on \( M_{-1} \) and the proof is omitted. Thus, we have proved that the map \( \Phi \) is continuous on \( M_B \setminus M_0 \).

Suppose that \( \rho \) is surjective. Then \( M_0 \) is an empty set. Hence \( \Phi \) is the map defined on \( M_B \) into \( M_A \). We show that ker \( \rho_y = \{f \in A; f(\Phi(y)) = 0\} \). Recall that ker \( \rho_y \subset \{f \in A; f(\Phi(y)) = 0\} \). So it is enough to show that \( \rho_y(f) \neq 0 \) implies \( f(\Phi(y)) \neq 0 \). Let \( a \in A \) satisfy \( \rho_y(a) \neq 0 \). Since \( \rho_y(A) = \mathbb{C} \), there corresponds a \( b \in A \) such that \( \rho_y(a)\rho_y(b) = 1 \). Therefore, \( ab - e \) belongs to ker \( \rho_y \). We conclude that \( a(\Phi(y)) \neq 0 \) since \( (ab - e)(\Phi(y)) = 0 \). Thus, we have proved that ker \( \rho_y = \{f \in A; f(\Phi(y)) = 0\} \). Hence \( M_B = M_{-1} \cup M_1 \cup M_{m,-1} \cup M_{m,1} \).

Let \( w_1, w_2 \in M_B \) satisfy \( w_1 \neq w_2 \). Since \( \rho \) is surjective, there exists an \( a_0 \in A \) such that \( \rho(a_0)(w_1) = 1 \) and \( \rho(a_0)(w_2) = 0 \). By the formula for \( \rho \), it is easy to see that

\[
a_0(\Phi(w_1)) = 1 \quad \text{and} \quad a_0(\Phi(w_2)) = 0.
\]

Therefore, we have \( \Phi(w_1) \neq \Phi(w_2) \). This implies that \( \Phi \) is injective. \( \square \)

**Proof of Corollary 1.2.** Let \( \{M_{-1}, M_0, M_1, M_{d,-1}, M_{d,1}\} \) be the decomposition of \( M_B \) with respect to \( \rho \) and \( \Phi \) the representing map for \( \rho \). Since \( \rho(\mathbb{C}e) \subset \mathbb{C}e \), we have \( M_B = M_{-1} \cup M_{d,-1} \) or \( M_B = M_0 \) or \( M_B = M_1 \cup M_{d,1} \). It is enough to consider the case where \( M_B = M_{-1} \cup M_{d,-1} \) or \( M_B = M_1 \cup M_{d,1} \).

Suppose that \( M_B = M_1 \cup M_{d,1} \). First, we show that \( M_1 \neq \emptyset \). Suppose not. Then \( M_B = M_{d,1} \). If \( \rho \) is surjective, the map \( \Phi \) is injective by Theorem 1.1. Since \( \Phi(M_{d,1}) \) is finite by Lemma 2.6, so is \( M_{d,1} = M_B \). This is a
contradiction. Therefore, $M_1 \neq \emptyset$ if $\rho$ is surjective. Consider the case where $\rho$ is injective. Since $M_A$ is infinite, there exists an $x_0 \in M_A \setminus \Phi(M_{d,1})$. We can find an open subset $V$ of $M_A$ so that $\Phi(M_{d,1}) \subset V$ and $x_0 \notin \overline{V}$. Since $A$ is regular, there corresponds an $f_0 \in A$ such that $f_0(x_0) = 1$ and $f_0(\overline{V}) = 0$. By Lemma 2.5 we see that $\rho_y(f_0) = 0$ for every $y \in M_{d,1} = MB$. Since $f_0$ is not identically zero, this contradicts that $\rho$ is injective. Consequently, we have that $M_1 \neq \emptyset$.

Now we show that $M_B = M_1$. Suppose that there exists a $y_1 \in M_{d,1}$. Since $\tilde{\rho}_{y_1}$ is non-trivial, we can find a $z_1 \in \mathbb{C}$ such that $\tilde{\rho}_{y_1}(z_1) \neq z_1$. Note that $\tilde{\rho}_{y}(z_1) = z_1$ for every $y \in M_1$. This is a contradiction since $\rho(\mathbb{C}e) \subset \mathbb{C}e$. Therefore, we have proved that $M_B = M_1$ if $M_B = M_1 \cup M_{d,1}$. In a way similar to the above, we see that $M_B = M_{-1}$ if $M_B = M_{-1} \cup M_{d,-1}$. Hence, $\rho$ is linear or conjugate linear. \hfill \Box

**Proof of Corollary 1.3.** Let $\{Y_{-1}, Y_0, Y_1, Y_d\}$ be the decomposition of $Y$ with respect to $\rho$ and $\Phi$ the representing map for $\rho$. Since the range $\rho(C(X))$ vanishes nowhere, we see that $Y_0$ is an empty set. Since $\rho(C(X))$ contains a separating subalgebra, in a way similar to the proof of Theorem 1.1, we can prove that ker $\rho_y$ is a maximal ideal for every $y \in Y$ and that $\Phi: Y \to X$ is injective. Hence, $Y$ is homeomorphic to the range $\Phi(Y)$. Let $\varphi: \Phi(Y) \to Y$ be the homeomorphism defined by

$$\varphi(x) = \Phi^{-1}(x) \quad (x \in \Phi(Y)).$$

Note that

$$\rho(f)(y) = \begin{cases} f(\Phi(y)) & y \in Y_{-1} \\ f(\Phi(y)) & y \in Y_1 \\ \tau_y f(\Phi(y)) & y \in Y_d \end{cases}$$

for every $f \in C(X)$. Here $\tau_y$ denotes a non-trivial ring homomorphism on $\mathbb{C}$. We define the continuous function $h: \Phi(Y) \to \mathbb{C}$ by

$$h(x) = \begin{cases} g(\varphi(x)) & x \in \Phi(Y_{-1}) \\ g(\varphi(x)) & x \in \Phi(Y_1) \\ \tau_{\varphi(x)}^{-1}(g(\varphi(x))) & x \in \Phi(Y_d) \end{cases}$$

for each $g \in C(Y)$. Since $\Phi(Y_{-1}), \Phi(Y_1)$ and $\Phi(Y_d)$ are disjoint closed subsets of the compact Hausdorff space $X$, there exists an $\tilde{h}$ of $C(X)$ such that $\tilde{h}|_{\Phi(Y)} = h$. Then it is easy to see that $\rho(\tilde{h}) = g$. Hence $\rho$ is surjective. \hfill \Box

**Proof of Corollary 1.4.** Let $\{Y_{-1}, Y_0, Y_1, Y_d\}$ be the decomposition of $Y$ with respect to $\rho$ and $\Phi$ the representing map for $\rho$. Since $\rho$ preserves complex conjugate, by Proposition 2.1 we have that $\tilde{\rho}_{y}$ is trivial for every $y \in Y$. 
Therefore, $Y_d$ is an empty set. By Lemma 2.4, we see that $Y_{-1}$, $Y_0$ and $Y_1$ are all clopen. This completes the proof. □

References


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