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LADDER INDEX OF GROUPS

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1. Stability

In 1969, Shelah distinguished stable and unstable theory in [S]. He introduced these notions in order to study the number of non-isomorphic models of cardinality κ for any uncountable κ.

Let T be a first order stable theory in a language L. A theory T is said to be unstable if there are some L-formula $\varphi(\bar{x},\bar{y})$, a model A of T and $\bar{a}_i \in A$ such that

$$\forall i,j<\omega, \quad A \models \varphi(\bar{a}_i,\bar{a}_j) \iff i<j.$$

T is stable if it is not unstable. Also, we call the structure stable or unstable if the theory $\mathrm{Th}(A)$ is stable or unstable respectively.

By this definition, it is clear that every finite structure is stable. In the rest of this note, we suppose every model of a theory T is infinite.

Theorem 1. *Let A be a stable structure.*

(a) *For any $\bar{a} \in A$, $(A,\bar{a})$ is also stable.*
(b) *If a structure B is interpretable in A, then B is stable.*

Let κ be an infinite cardinal. A theory T is said to be κ-stable if for any model A of T, and any subset X of A with $|X| \leq \kappa$, $|S_1(X;A)| \leq \kappa$, where $S_1(X;A)$ is a set of all complete 1-types over X realized by A. A structure A is κ-stable if $\mathrm{Th}(A)$ is. Then the following hold.

Theorem 2. *The following are equivalent.*

(a) *T is stable.*
(b) *For at least one infinite cardinal κ, T is κ-stable.*

Lemma 3. *Let A be an L-structure, κ be an infinite cardinal and $X \subset A$ be a set of power κ. If $|S_n(X;A)| > |X|$ for some integer n, then A is not κ-stable.*

2. Ladder index

Let T be a complete theory in a language L. Let $\varphi(\bar{x},\bar{y})$ be an L-formula with free variables $\bar{x}$ and $\bar{y}$. An n-ladder for φ is a sequence $(\bar{a}_0,\cdots,\bar{a}_{n-1};\bar{b}_0,\cdots,\bar{b}_{n-1})$ of tuple in some model A of T, such that

$$\forall i,j<n, \quad A \models \varphi(\bar{a}_i,\bar{b}_j) \iff i \leq j.$$

φ is said to be a stable formula if φ has no n-ladder for some n. φ is unstable, otherwise.

Lemma 4. *A theory T is unstable if and only if there is an unstable L-formula for T.*

It is well known that every module is stable. In this section, we show that stable groups satisfy some descending chain condition. First of all, we call a structure *group-like* if the restriction of A to some language is a group. The restriction is said to be a group structure of A. A stable group is a stable group-like structure. It may be generalized in some sense later.

Lemma 5 (Baldwin-Saxl)**.** *Let L be a language and A be a stable group as an L-structure. Let G be a group structure in A. Let $\varphi(x, \bar{y})$ be an L-formula. Let $\mathcal{S}$ be a set of all definable subgroups by the formula of the form $\varphi(\bar{b}, A)$ for some $\bar{b} \in A$. Let $\bigcap \mathcal{S}$ be a collection of all intersections of arbitrary many elements of $\mathcal{S}$. Then,*

(a) *There is an integer n such that any element of $\bigcap \mathcal{S}$ is an intersection of at most n many elements of $\mathcal{S}$.*

(b) *There is an integer m such that there is no descending chain of more than m many elements of $\bigcap \mathcal{S}$ by inclusion.*

Definition 6. For a given formula φ, the ladder index of φ is the least number n such that φ has no n-ladder.

In this note, we consider the ladder index for the commutativity formula $xy = yx$. The ladder index of a group G for the commutativity formula is denoted by $\ell(G)$.

Note. For any ladder $(a_0, a_1, \cdots, a_n; b_0, b_1, \cdots, b_n)$ in a group G if we replace a_0 and b_n by any central elements of G, the new sequence is also a ladder.

For any subset X of a group G, the centralizer $C_G(X)$ is a group with elements which commute with all elements of X. Hence $C_G(X) = \bigcap_{g \in X} C_G(g)$. By model theoretic notation, $C_G(g) = \varphi(A, g)$, where $\varphi(x, y)$ is $xy = yx$. By Baldwin-Saxl Lemma, stable groups satisfy the descending chain condition (dcc) on centralizers. A group with the minimal condition (equivalently dcc) on centralizers is said to be an M_C-group.

Lemma 7. *For any group G and A a subset of G, $C_G(C_G(C_G(A))) = C_G(A)$.*

Lemma 8. *The maximal condition and the minimal condition on centralizers are equivalent.*

3. Finite gap number

In this section we study the property of ladder index for a commutativity formula. In group theory, there is a notion (e.g. in [LR]) as follows.

Definition 9. A group G has a finite central gap number, or shortly finite gap number if for any subgroups $H_1, H_2, \cdots, H_n, \cdots$ of G, among the sequence

$$C_G(H_1) \leq C_G(H_2) \leq \cdots \leq C_G(H_n) \leq \cdots,$$

there are at most g many strict inclusions, in this case, we call gap number g and we denote $g = g(G)$.

In order to study the relations between ladder index and finite gap number, we prepare the following.

Lemma 10. *Let G be a group of finite gap number n. Suppose the sequence*

$$C_G(H_0) > C_G(H_1) > \cdots > C_G(H_n)$$

gives the gap number n. Then there are a_i $(0 \leq i \leq n)$ in G such that $C_G(H_i) = C_G(\{a_0, \cdots, a_i\})$ for each i.

We abbreviate as $C_G(\{a_0, \cdots, a_i\}) = C_G(a_0, \cdots, a_i)$ in the rest of this note.

Proof. As $C_G(H_0) = G$, we put $a_0 = 1$. Suppose we have chosen by i-th. There is a $b \in H_{i+1} - H_i$ such that $C_G(H_i) > C_G(H_i \cup \{b\})$. Since there is no centralizer between $C_G(H_i)$ and $C_G(H_{i+1})$ by definition, $C_G(H_{i+1}) = C_G(H_i \cup \{b\}) = C_G(a_1, \cdots, a_i, b)$. Now we may choose $a_{i+1} = b$. □

Theorem 11. *For any group G of finite ladder index, $\ell(G) = g(G) + 2$.*

Proof. Let G be a group of ladder index $(n+2)$ with the witness $(a_0, \cdots, a_n; b_0, \cdots, b_n)$. We have a descending chain,

$$C_G(a_0) > C_G(a_0, a_1) > \cdots > C_G(a_0, \cdots, a_n).$$

On the other hand, suppose such a descending chain is given. Since this sequence is a strictly descending chain, we can choose b_i in $C_G(a_0, \cdots, a_i) - C_G(a_0, \cdots, a_{i+1})$ for each $i < n$ and we put $b_n = 1$. The sequence $(a_0, \cdots, a_n; b_0, \cdots, b_n)$ made as above may not be a ladder at this moment. We replace b_i's if necessary. We fix b_{n-1}. If b_{n-2} is not commutative with a_n, we fix it. Otherwise, we replace b_{n-2} by $b_{n-2}b_{n-1}$.

Suppose we have fixed $b_n, \cdots, b_i$. If b_{i-1} is not commutative with a_{i+1}, then we fix it. Otherwise, we replace b_{i-1} by $b_{i-1}b_i$. We go through to a_n, and the final b_{i-1} is fixed.

We have a ladder with b_i's by the above procedure. □

4. Groups of small ladder index

Every finite group has finite ladder index. It is known that abelian groups, linear groups [W], finitely generated abelian-by-nilpotent groups [LR] and polycyclic-by-finite groups [LR] have finite ladder index.

In this section we study the groups of ladder index 2, 3, 4 and 5.

Theorem 12. *$\ell(G) = 2$ if and only if G is abelian.*

The proof is trivial by definition.

Theorem 13. *There are no groups of ladder index* 3.

Proof. Suppose $\ell(G) > 2$. By the above theorem, G is non-abelian. So, G has elements a and b which do not commute. Then the sequence $(1, b, ab; a, b, 1)$ is a ladder, and $\ell(G) \geq 4$. □

We study the groups of ladder index 4 next. There is a lot of examples of groups of ladder index 4 which are finite or infinite. The structure of such groups is so simple (which does not mean simple groups).

Example 14. A symmetric group S_3 and a dihedral group D_n have ladder index 4.

Example 15. A special linear group $SL(2, F)$ (F is a field) has ladder index 4.

Theorem 16. *The following are equivalent.*

(1) *$\ell(G) = 4$.*
(2) *G is non-abelian, and for any a and b in $G - Z(G)$, if $C_G(a) \neq C_G(b)$ then $C_G(a) \cap C_G(b) = Z(G)$.*

Proof. ($\Leftarrow$) By Theorem 12 and 13.

($\Rightarrow$) Suppose G has ladder index 4. Let a and b are elements as in the assumption. We may suppose $C_G(a) - C_G(b) \neq \emptyset$. Then we have $G > C_G(a) > C_G(a, b) \geq Z(G)$. Since G has gap number 2, we have $C_G(a, b) = Z(G)$. □

Theorem 17. *There are no groups of ladder index* 5.

Proof. Suppose $\ell(G) > 4$. By the above theorem, there exist a_1 and a_2 in $G - Z(G)$ such that $C_G(a_1) \neq C_G(a_2)$ and $C_G(a_1) \cap C_G(a_2) > Z(G)$ hold.

Case 1: $a_1 a_2 = a_2 a_1$.

Since $C_G(a_1) \neq C_G(a_2)$, we assume $C_G(a_1) - C_G(a_2) \neq \emptyset$. Let $b \in C_G(a_1) - C_G(a_2)$. Then $a_1 \in C_G(b) \wedge a_2 \notin C_G(b)$. Because a_1 is not in $Z(G)$, there is a $c \in G - C_G(a_1)$. Therefore, we have

$$G > C_G(a_1) > C_G(a_1, a_2) > C_G(a_1, a_2, b) > C_G(a_1, a_2, b, c).$$

Hence, $\ell(G) \geq 6$.

Case 2: $a_1a_2 \neq a_2a_1$.

There is a $b_3 \in C_G(a_1, a_2) - Z(G)$. Since $b_3 \notin Z(G)$, we can choose $b_1 \in G - C_G(b_3)$. Then we have

$$G > C_G(b_3) > C_G(b_3, a_1) > C_G(b_3, a_1, a_2) > C_G(b_3, a_1, a_2, b_1).$$

Hence, $\ell(G) \geq 6$. □

Example 18. A symmetric group S_4 has ladder index 6.

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