ON THE GROUP $\pi(\Sigma A \times B, X)$

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Introduction

We work in the homotopy category of pointed $CW$-complexes, and denote the suspension functor with $\Sigma$. The group $\pi(\Sigma A \times B, X)$ is not abelian in general under the usual multiplication but we use the notation “$+$” for it. In §1 we shall describe the multiplication of the group $\pi(\Sigma A \times B, X)$, and, as a by-product, obtain from the associativity of the multiplication that the bi-additivity of the generalized Whitehead product (GWP) does not hold in general (see Proposition 3.4 of [1]). In fact, as an example, we offer the following:

Let $i_1$ and $i_2$ be inclusion maps from $CP^2$ and $S^n$ into $CP^2 \vee S^n$ respectively. Then we have that

$$[2\Sigma i_1, \Sigma i_2] \neq 2[\Sigma i_1, \Sigma i_2]$$

in $\pi(\Sigma CP^2 \wedge S^n, \Sigma CP^2 \vee S^n)$.

In §2 we investigate the group $\pi(\Sigma X, \Sigma A \times B)$ and show that any element of this group can be determined by 4-components under some assumptions. In §3 we apply §2 to the case of $X = A \times B$, i.e. the group of self-maps of the space $\Sigma A \times B$.

Specially we are interested in describing the composition of two elements with their components and give the special case of $A = B = S^n$ as an example.

1. $\pi(\Sigma A \times B, X)$

Let $i_1$ and $i_2$ be inclusion maps: $A, B \rightarrow A \times B$, and let $P_A$ and $P_B$ be projections: $A \times B \rightarrow A, B$ respectively.

Lemma 1.1. Let $\pi: A \times B \rightarrow A \wedge B$ be the projection. Any element $f \in \pi(\Sigma A \times B, X)$ can be uniquely represented by the form

$$f = \alpha \Sigma P_A + \beta \Sigma P_B + \gamma \Sigma \pi$$

for $\alpha \in \pi(\Sigma A, X)$, $\beta \in \pi(\Sigma B, X)$ and $\gamma \in \pi(\Sigma A \wedge B, X)$.

Proof. In fact a representation can be obtained from a part of Puppe exact sequence of the cofibering: $A \vee B \rightarrow A \times B$. Then clearly we have $\alpha = f \Sigma i_A$, $\beta = f \Sigma i_B$ and moreover the uniqueness of $\gamma$ follows from the injectivity of $(\Sigma \pi)^*$. \hfill $\square$
Here we give a brief account of GWP of [1]. For two maps \( \alpha : \Sigma A \to X \) and \( \beta : \Sigma B \to X \), GWP \([\alpha, \beta] \in \pi(\Sigma A \wedge B, X)\) is defined by
\[
\alpha \Sigma P_A + \beta \Sigma P_B = \beta \Sigma P_B + \alpha \Sigma P_A + [\alpha, \beta] \Sigma \pi.
\]

**Proposition 1.2.** GWP has following properties:

1. **Let us be** \( \langle \alpha, \beta \rangle \) the commutator of \( \alpha \) and \( \beta \) \((\in \pi(\Sigma A, X))\) then we have
   \[
   \langle \alpha, \beta \rangle = [\alpha, \beta] \Sigma d_A,
   \]
   where \( d_A \) is the diagonal map: \( A \to A \wedge A \).
2. **If** \( f \in \pi(X, Y) \) **then** \( f[\alpha, \beta] = [f \alpha, f \beta] \).
3. **If** \( \sigma_k \in \pi(\Sigma Y_k, Z) \) **and** \( f_k \in \pi(X_k, Y_k) \) **for** \( k = 1, 2 \) **then it holds that**
   \[
   [\sigma_1 \Sigma f_1, \sigma_2 \Sigma f_2] = [\sigma_1, \sigma_2] \Sigma f_1 \wedge f_2.
   \]
4. **For four maps** \( \Sigma f \in \pi(\Sigma Y, \Sigma A), \Sigma g \in \pi(\Sigma Y, \Sigma B), \sigma \in \pi(\Sigma A, X) \)
   **and** \( \tau \in \pi(\Sigma B, X) \) **we have**
   \[
   \sigma \Sigma f + \tau \Sigma g = \tau \Sigma g + [\sigma, \tau] \Sigma (f \wedge g) \Sigma d_Y.
   \]
5. **If** \( X \) **is a suspension** \((i.e. X = \Sigma X^*)\) **then** \( d_X = 0 \).

**Lemma 1.3.** For \( \alpha \in \pi(\Sigma A, X), \beta \in \pi(\Sigma B, X) \) **and** \( \gamma \in \pi(\Sigma A \wedge B, X) \) **we have the following:**

1. \( \alpha \Sigma P_A + \beta \Sigma P_B = \beta \Sigma P_B + \alpha \Sigma P_A + [\alpha, \beta] \Sigma \pi. \)
2. \( \alpha \Sigma P_A + \gamma \Sigma \pi = \gamma \Sigma \pi + \alpha \Sigma P_A + [\alpha, \gamma] \Sigma \varphi_A \Sigma \pi, \) **where** \( \varphi_A \) **is defined by**
   \[
   \varphi_A(a \wedge b) = a \wedge a \wedge b.
   \]
3. \( \beta \Sigma P_B + \gamma \Sigma \pi = \gamma \Sigma \pi + \beta \Sigma P_B + [\beta, \gamma] \Sigma \psi_B \Sigma \pi, \) **where** \( \psi_B \) **is defined by**
   \[
   \psi_B(a \wedge b) = b \wedge a \wedge b.
   \]

**Proof.** (1) is just the definition of GWP. Next, by applying (1) and (4) of lemma 1.2 to the diagram:

\[
\begin{array}{ccc}
\Sigma A \times B & \xrightarrow{\Sigma P_A} & \Sigma A \\
\parallel & & \parallel \\
\Sigma A \times B & \xrightarrow{\Sigma \pi} & \Sigma A \wedge B \xrightarrow{\gamma} X,
\end{array}
\]

we have that
\[
[\alpha \Sigma P_A, \gamma \Sigma \pi] \Sigma d_A \times 1_B = [\alpha, \gamma] \Sigma (P_A \wedge \pi) \Sigma d_A \times 1_B = [\alpha, \gamma] \Sigma \varphi_A \Sigma \pi.
\]

The case (3) is analogous to the case (2). Thus the proof is completed. \( \square \)

Now let us represent \( f = \alpha \Sigma P_A + \beta \Sigma P_B + \gamma \Sigma \pi \) with the triple \((\alpha, \beta, \gamma)\).

**Theorem 1.4.** \((\alpha_1, \beta_1, \gamma_1) + (\alpha_2, \beta_2, \gamma_2) = (\alpha_1 + \alpha_2, \beta_1 + \beta_2, \delta + \gamma_2) \) **for** \( \delta = \delta_1 - [\beta_2, \delta_1] \Sigma \psi_B \) **and** \( \delta_1 = -[\alpha_2, \beta_1] + \gamma_1 - [\alpha_2, \gamma_1] \Sigma \varphi_A. \)
**Proof.** For abbreviation we use notations: \( \bar{\alpha} = \alpha \Sigma \pi P_A \), \( \bar{\beta} = \beta \Sigma \pi P_B \), \( \bar{\gamma} = \gamma \Sigma \pi \) and so on. Now by using lemma 1.3 we have equalities

\[
f_1 + f_2 = (\alpha_1, \beta_1, \gamma_1) + (\alpha_2, \beta_2, \gamma_2)
= \bar{\alpha}_1 + \bar{\beta}_1 + \bar{\gamma}_1 + \bar{\alpha}_2 + \bar{\beta}_2 + \bar{\gamma}_2
= \bar{\alpha}_1 + \bar{\beta}_1 + \bar{\alpha}_2 + \bar{\gamma}_1 - [\alpha_2, \gamma_1] \Sigma \varphi_A \Sigma \pi + \bar{\beta}_2 + \bar{\gamma}_2
= \bar{\alpha}_1 + \bar{\alpha}_2 + \bar{\beta}_1 + \delta_1 + \bar{\beta}_2 + \bar{\gamma}_2
= \bar{\alpha}_1 + \bar{\alpha}_2 + \bar{\beta}_1 + \bar{\beta}_2 + \delta + \bar{\gamma}_2
= (\alpha_1 + \alpha_2, \beta_1 + \beta_2, \delta + \gamma_2).
\]

Thus the proof is completed. \( \square \)

**Corollary 1.5.** If \( A \) and \( B \) are both suspensions then it holds that

\((\alpha_1, \beta_1, \gamma_1) + (\alpha_2, \beta_2, \gamma_2) = (\alpha_1 + \alpha_2, \beta_1 + \beta_2, -[\alpha_2, \gamma_1] + \gamma_1 + \gamma_2)\).

**Proof.** Since \( \varphi_A \) and \( \psi_B \) are trivial by (5) of lemma 1.2 the proof follows from Theorem 1.4. \( \square \)

**Corollary 1.6.** For \( \alpha_1, \alpha_2 \in \pi(\Sigma A, X) \) and \( \beta \in \pi(\Sigma B, X) \) it holds that

\([\alpha_1 + \alpha_2, \beta] = [\alpha_2, -[\alpha_1, \beta]] \Sigma \varphi_A + [\alpha_1, \beta] + [\alpha_2, \beta]\).

**Proof.**

\[
(0, \beta, 0) + \{(\alpha_1, 0, 0) + (\alpha_2, 0, 0)\}
= (0, \beta, 0) + (\alpha_1 + \alpha_2, 0, 0)
= (\alpha_1 + \alpha_2, \beta, -[\alpha_1 + \alpha_2, \beta]).
\]

On the other hand

\[
\{(0, \beta, 0) + (\alpha_1, 0, 0)\} + (\alpha_2, 0, 0)
= (\alpha_1, \beta, -[\alpha_1, \beta]) + (\alpha_2, 0, 0)
= (\alpha_1 + \alpha_2, \beta, -[\alpha_2, \beta] - [\alpha_1, \beta] + [\alpha_2, -[\alpha_1, \beta]] \Sigma \varphi_A).
\]

Thus the proof follows from the associativity of the addition of the group \( \pi(\Sigma A \times B, X) \). \( \square \)

By (5) of lemma 1.2 and the above corollary 1.6 it is easy to obtain the following:

**Corollary 1.7** (Proposition 3.4 of [1]). If \( A \) is a suspension then it holds that

\([\alpha_1 + \alpha_2, \beta] = [\alpha_1, \beta] + [\alpha_2, \beta]\)
for $\alpha_1, \alpha_2 \in \pi(\Sigma A, X)$ and $\beta \in \pi(\Sigma B, X)$.

Now we consider a special case: $A = CP^2$, $B = S^n$ and $X = \Sigma CP^2 \vee S^n$. Let $i_1$ and $i_2$ be inclusions: $CP^2, S^n \to CP^2 \vee S^n$ respectively, and let us be $\alpha = \Sigma i_1$ and $\beta = \Sigma i_2$. Since $\varphi_A: CP^2 \wedge S^n \to CP^2 \wedge CP^2 \wedge S^n$ is defined by $\varphi_A(a \wedge b) = a \wedge a \wedge b$, $\varphi_A$ can be regarded as $\Sigma^n d$ for the diagonal map $d: CP^2 \to CP^2 \wedge CP^2$.

Since $d^*: H^4(CP^2 \wedge CP^2) \to H^4(CP^2)$ is clearly an isomorphism $\Sigma^n d$ is non-trivial. On the other hand, accordingly to Hilton-Milnor Theorem ([3]) $[\alpha, [\alpha, \beta]]_* \Sigma \varphi_A$ is non-trivial. Hence $[2\Sigma i_1, \Sigma i_2] \neq 2[\Sigma i_1, \Sigma i_2]$ and $[\Sigma i_1, 2\Sigma i_2] = 2[\Sigma i_1, \Sigma i_2]$.

**Remark.** The second equality follows from Corollary 1.7.

2. **On the group $\pi(\Sigma X, \Sigma A \times B)$**

First by using lemma 1.1 we define $\xi_{AB} \in \pi(\Sigma A \wedge B)$ as follows:

$$1_{\Sigma A \times B} = \Sigma i_A \Sigma P_A + \Sigma i_B \Sigma P_B + \xi_{AB} \Sigma \pi.$$  

**Corollary 2.1.** $\Sigma \pi \xi_{AB} = 1_{\Sigma A \wedge B}$.

**Proof.** Apply $(\Sigma \pi)_*$ to the above equality. Then we have

$$\Sigma \pi = \Sigma \pi (\Sigma i_A \Sigma P_A + \Sigma i_B \Sigma P_B + \xi_{AB} \Sigma \pi) = 0 + 0 + \Sigma \pi \xi_{A \wedge B} \Sigma \pi.$$

Since $(\Sigma \pi)_*$ is injective the proof is completed. 

Here we note that the representation of $f \in \pi(\Sigma A \times B, X)$ in lemma 1.1 is given by

$$f = f|_{\Sigma A} \Sigma P_A + f|_{\Sigma B} \Sigma P_B + f \xi_{AB} \Sigma \pi,$$

where $f|_K$ denotes the restriction of $f$ on $K$.

For example if $h$ is a map $A \times B \to X$ then $\Sigma h \xi_{AB}$ is essentially the Hopf-construction of $f$, i.e. $C(h)$ (see [2]) and we have a representation:

$$\Sigma h = \Sigma h_A + \Sigma h_B + C(h) \Sigma \pi,$$

where $h$ is a map of type $(h_A, h_B)$.

Secondly we define two maps $\varphi \in \pi(Y, \Sigma A \times B)$ and $\phi \in \pi(\Sigma A \times B, Y)$ for $Y = \Sigma A \vee \Sigma B \vee \Sigma A \wedge B$ by

$$\varphi = \Sigma i_A \vee \Sigma i_B \vee \xi_{AB},$$

$$\phi = i_{\Sigma A} \Sigma P_A + i_{\Sigma B} \Sigma P_B + i_{\Sigma A \wedge B} \Sigma \pi.$$

**Lemma 2.2.** $\varphi$ is a homotopy equivalence with $\phi$ as its inverse.

**Proof.** Easy. 

□
In the following of this section we assume that
(1) $A$ is $a$-connected and $B$ is $b$-connected,
(2) $a \leq b$,
(3) $\dim X \leq 2a + b + 2$.

Then by Hilton-Milnor theorem, $f \in \pi(\Sigma X, \Sigma A \times B)$ can be represented as follows:
$$f = \Sigma i_A f_A + \Sigma i_B f_B + [\Sigma i_A, \Sigma i_B] f_{C1} + \xi_{AB} f_{C2},$$
where $f_* \in \pi(\Sigma X, \Sigma*)$ and $f_{C*} \in \pi(\Sigma X, \Sigma A \wedge B)$.

More precisely we have

**Lemma 2.3.** $f_A = \Sigma P_A f, f_B = \Sigma P_B f, f_{C1} = \Sigma (P_A \wedge P_B) H(f)$ and $f_{C2} = \Sigma \pi f$, where $H(f)$ denotes Hopf-invariant of $f$.

**Proof.** First we note that $[\gamma, \delta \Sigma \pi] H(f) = 0$ because this element is decomposed as follows:
$$\Sigma X \xrightarrow{H(f)} (A \times B) \cap (A \times B) \xrightarrow{\Sigma (1 \wedge \pi)} (A \times B) \wedge (A \wedge B) \xrightarrow{[\gamma, \delta]} \Sigma A \times B.$$

Then the proof is deduced from our assumptions. Secondly apply $f$ from the right to the equality. We obtain that
$$f = (\Sigma i_A P_A + \Sigma i_B P_B + \xi_{AB} \Sigma \pi) f$$
$$= (\Sigma i_A P_A + \Sigma i_B P_B) f + \xi_{AB} \Sigma \pi f$$
$$= \Sigma i_A P_A f + \Sigma i_B P_B f + [\Sigma i_A, \Sigma i_B] (\Sigma P_A \wedge \Sigma P_B) H(f) + \xi_{AB} \Sigma \pi f.$$ 
Thus the proof is completed. \qed

3. **On the group $\pi(\Sigma A \times B, \Sigma A \times B)$**

In this section we assume that $A$ and $B$ are both $n$-connected, $\dim A \leq \dim B$ and $\dim A + \dim B \leq 3n + 2$.

**Lemma 3.1.** Our assumptions contain
(1) $A$, $B$ and $A \wedge B$ are all suspensions, so $\pi(\Sigma K, X)$ is abelian for $K = A$, $B$ and $A \wedge B$,
(2) $\dim B \leq 2n + 1$. Hence $\pi(\Sigma*, \Sigma A \wedge B) = 0$ for $* = A$ or $B$,
(3) $\Sigma: \pi(X, Y) \to \pi(\Sigma X, \Sigma Y)$ is onto for any pair $(X, Y)$ of $\{A, B\}$.

**Proof.** Easy. \qed

In §1, $f \in \pi(\Sigma A \times B, \Sigma A \wedge B)$ has a representation:
$$f = f_A \Sigma P_A + f_B \Sigma P_B + f_C \Sigma \pi$$
for $f_* \in \pi(\Sigma A \times B)$ and $C = A \wedge B$. 
And moreover in §2, \( f_* \) has a representation:

\[
f_* = \Sigma i_A f_{*1} + \Sigma i_B f_{*2} + \xi_{AB} f_{*3} + [\Sigma i_A, \Sigma i_B] f_{*4}
\]

for \( f_{*1} \in \pi(\Sigma*, \Sigma A), f_{*2} \in \pi(\Sigma*, \Sigma B) \) and \( f_{*3}, f_{*4} \in \pi(\Sigma*, \Sigma A \land B) \).

Thus \( f \in \pi(\Sigma A \times B, \Sigma A \times B) \) has a form of a (3×4)-matrix (note lemma 3.1):

\[
(f_{*k}) = \begin{pmatrix}
(fA_1 & fA_2 & 0 & 0) \\
(fB_1 & fB_2 & 0 & 0) \\
(fc_1 & fc_2 & fc_3 & fc_4)
\end{pmatrix}.
\]

We want to compute the composition \( gf \) for \( f, g \in \pi(\Sigma A \times B, \Sigma A \times B) \). Since we have that \( gf = (gf_A)\Sigma P_A + (gf_B)\Sigma P_B + (gf_C)\Sigma \pi \) it is sufficient for our purpose to compute \( gf_* \in \pi(\Sigma*, \Sigma A \times B) \).

**Lemma 3.2.** \( g_C = g\xi_{AB} \).

**Proof.** First we have

\[
g = g\Sigma A \times B = g(\Sigma i_A \Sigma P_A + g\Sigma i_B \Sigma P_B + g\xi_{AB} \Sigma \pi) = g\Sigma A \Sigma P_A + g\Sigma B \Sigma P_B + g\xi_{AB} \Sigma \pi.
\]

On the other hand \( g = g_A \Sigma P_A + g_B \Sigma P_B + g_C \Sigma \pi \). Hence we have \( g_C = g\xi_{AB} \).

Now we proceed to \( gf_* \):

\[
gf_* = g(\Sigma i_A f_{*1} + \Sigma i_B f_{*2} + \xi_{AB} f_{*3} + [\Sigma i_A, \Sigma i_B] f_{*4}) = g\Sigma A f_{*1} + g\Sigma B f_{*2} + g\xi_{AB} f_{*3} + [g\Sigma A, g\Sigma B] f_{*4} = g_A f_{*1} + g_B f_{*2} + g_C f_{*3} + [g_A, g_B] f_{*4}.
\]

**Lemma 3.3.**

\[
[g_A, g_B] = \Sigma i_A [g_A1, g_B1] + \Sigma i_B [g_A2, g_B2] + [\Sigma i_A, \Sigma i_B](\Sigma g_A'1 \land \Sigma g_B'2 - \tau \Sigma g_A'2 \land g_B'1),
\]

where \( g_{*k} = \Sigma g_{*k}' \) for \( * \in \{A, B\} \) and \( k = 1, 2 \).

**Proof.** Apply lemma 1.2 and Propositions 3.3, 3.4 of [1] to the equality:

\[
[g_A, g_B] = [\Sigma i_A g_A1 + \Sigma i_B g_A2, \Sigma i_A g_B1 + \Sigma i_B g_B2],
\]

then the proof is completed.

**Lemma 3.4.** We have

\[
g_A f_{C1} = \Sigma i_A (g_A1 f_{C1}) + \Sigma i_B (g_A2 f_{C1}) + [\Sigma i_A, \Sigma i_B] \Sigma g_A'1 \land g_A'2 H(f_{C1}).
\]
Proof. Apply the distributive law to the equality:

\[ g_A f C_1 = (\Sigma i_A g A_1 + \Sigma i_B g A_2) f C_1, \]
then the proof is completed.

From these lemmas we have

**Theorem 3.5.** If \( f = (f_{*k}) \) and \( g = (g_{*k}) \) then \( g f = h = (h_{*k}) \) is given by

1. the case of \(* = A, or B,\)
   \[
   h_{*1} = g_{A1} f_{*1} + g_{B1} f_{*2}, \]
   \[
   h_{*2} = g_{A2} f_{*1} + g_{B2} f_{*2},
   \]

2. the case of \(* = C = A \land B,\)
   \[
   h_{C1} = g_{A1} f_{C1} + g_{B1} f_{C2} + [g_{A1}, g_{B1}] f_{C4} + g_{C1} f_{C3},
   \]
   \[
   h_{C2} = g_{A2} f_{C1} + g_{B2} f_{C2} + [g_{A2}, g_{B2}] f_{C4} + g_{C2} f_{C3},
   \]
   \[
   h_{C3} = g_{C3} f_{C3},
   \]
   \[
   h_{C4} = \Sigma g'_{A1} \land g'_{A2} H(f_{C1}) + \Sigma g'_{B1} \land g'_{B2} H(f_{C2})
   \]
   \[
   + (\Sigma g'_{A1} \land g'_{B2} - \Sigma \Sigma g'_{A2} \land g'_{B1}) f_{C4} + g_{C4} f_{C3}.
   \]

As an example we take \( A = B = S^n \). Let us be \( f \in \pi(\Sigma S^n \times S^n, \Sigma S^n \times S^n) \) with its matrix:

\[
\begin{pmatrix}
f_{11} & f_{12} & 0 & 0 \\
f_{21} & f_{22} & 0 & 0 \\
f_{31} & f_{32} & f_{33} & f_{34}
\end{pmatrix},
\]

where \( f_{ij}(\{i, j\} = \{1, 2\}), f_{33}, f_{34} \in \mathbb{Z} \) and \( f_{31}, f_{32} \in \pi_{2n+1}(S^{n+1}) \).

If \( h = g f \) then \( h_{ij} \) is given by

\[
h_{11} = g_{11} f_{11} + g_{21} f_{12},
\]
\[
h_{12} = g_{12} f_{11} + g_{22} f_{12},
\]
\[
h_{21} = g_{11} f_{21} + g_{21} f_{22},
\]
\[
h_{22} = g_{12} f_{21} + g_{22} f_{22},
\]
\[
h_{31} = g_{11} \circ f_{31} + g_{21} \circ f_{32} + f_{33} g_{31} + f_{34} g_{12} g_{22}[\tau_{n+1}, \tau_{n+1}],
\]
\[
h_{32} = g_{12} \circ f_{31} + g_{22} \circ f_{32} + f_{33} g_{32} + f_{34} g_{12} g_{22}[\tau_{n+1}, \tau_{n+1}],
\]
\[
h_{33} = g_{33} f_{33},
\]
\[
h_{34} = (g_{11} g_{22} - (-1)^n g_{12} g_{21}) f_{34} + g_{34} f_{33} + g_{11} g_{12} H(f_{31}) + g_{21} g_{22} H(f_{32}),
\]

where \( g_{**} \circ f_{**} \) denotes \( (g_{**}, \tau_{n+1}) f_{**} \).

Here we describe some results obtained from the above table of the composition.
1. $f$ is a homotopy equivalence if and only if $f_{33} = \pm 1$ and
   
   \[
   \begin{vmatrix}
   f_{11} & f_{12} \\
   f_{21} & f_{22}
   \end{vmatrix} = \pm 1.
   \]

2. We denote the group of self-homotopy equivalences of $\Sigma S^n \times S^n$ with $\varepsilon(G_n)$ and orientation-preserving self-homotopy equivalences with $\varepsilon_0(G_n)$, where orientation-preserving means that the above determinant is 1 and $f_{33} = 1$, then $\varepsilon_0(G_n)$ is a normal subgroup and the quotient is isomorphic to $\mathbb{Z}_2 \oplus \mathbb{Z}_2$.

3. $\varepsilon_0(G_n)$ contains a subgroup:
   
   \[
   \left\{ \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & \alpha \end{pmatrix} \right\} \cong \mathbb{Z}.
   \]

4. $\varepsilon_0(G_n)$ contains a subgroup:
   
   \[
   \left\{ \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ \alpha & \beta & 1 & 0 \end{pmatrix} \right\} \cong \pi_{2n+1}(S^{n+1}) \oplus \pi_{2n+1}(S^{n+1}).
   \]

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