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NONLINEAR ERGODIC THEOREMS FOR SEMIGROUPS OF NON-LIPSCHITZIAN MAPPINGS IN BANACH SPACES II

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Abstract. Let C be a nonempty closed convex subset of a uniformly convex Banach space, and let $S = \{T(t); t \geq 0\}$ be a nonlinear semigroup of non-Lipschitzian mappings on C which is asymptotically nonexpansive in the intermediate sense. In this paper we study weak almost convergence of almost-orbits of S.

1. Introduction and Theorem

Throughout this paper X denotes a uniformly convex Banach space and C is a nonempty closed convex subset of X. A family $S = \{T(t); t \geq 0\}$ of mappings is said to be a semigroup on C, if

(a_1) for each $t \geq 0$, $T(t)$ is a mapping from C into itself,
(a_2) $T(0)x = x$ and $T(t+s)x = T(t)T(s)x$ for $x \in C$ and $t, s \geq 0$,
(a_3) for each $x \in C$, $T(t)x$ is strongly continuous in $t > 0$ and the strong limit $\lim_{t\to 0+} T(t)x$ exists.

For semigroup S on C we set $F = \{x \in C; T(t)x = x \text{ for all } t \geq 0\}$ and an element in F is called a fixed point of S.

Let S be a semigroup on C. There are the following definitions of asymptotically nonexpansive type:

(c_1) ([7], [10], [11], [13]) If there exists a function $a(\cdot) : [0,\infty) \to [0,\infty)$ with $\lim_{t\to\infty} a(t) = 1$ such that $\|T(t)u - T(t)v\| \leq a(t)\|u - v\|$ for $u, v \in C$ and $t \geq 0$ then S is said to be asymptotically nonexpansive in the strong sense.
(c_2) ([5], [9], [10], [13], [16]) If $T(t_0) : C \to C$ is continuous for some $t_0 > 0$ and

$$\overline{\lim}_{t\to\infty} \sup_{u,v\in B}(\|T(t)u - T(t)v\| - \|u - v\|) \leq 0 \tag{1.1}$$

for every bounded set $B \subset C$, then S is said to be asymptotically nonexpansive in the intermediate sense.

After Baillon's works ([1], [2]), nonlinear ergodic theorems for semigroups which are asymptotically nonexpansive in the strong sense have been studied by many authors (for example, see [8], [12], [14], [15] and [16]). This paper is a continuation of the paper [13] and deals with weak nonlinear ergodic

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theorems for semigroups on C which are asymptotically nonexpansive in the intermediate sense. To this end we introduce the notion of "almost-orbit" of semigroups as follows:

Definition 1.1 ([13])**.** Let $S = \{T(t); t \geq 0\}$ be a semigroup on C. A function $u(\cdot) : [0, \infty) \to C$ is called an almost-orbit of S if $u(t)$ is strongly continuous in $t > 0$ and the strong limit $\lim_{t \to 0+} u(t)$ exists and if

$$\lim_{s,t \to \infty} \|u(t+s) - T(s)u(t)\| = 0. \tag{1.2}$$

Definition 1.2. A function $u(\cdot) : [0, \infty) \to X$ is said to be weakly almost convergent to an element y in X if w-$\lim_{t \to \infty}(1/t) \int_0^t u(r+h)dr = y$ uniformly in $h \geq 0$, where w-lim denotes the weak limit.

We say that a Banach space E has the Kadec-Klee property if w-$\lim_{n \to \infty} x_n = x$ and $\lim_{n \to \infty} \|x_n\| = \|x\|$ imply $\lim_{n \to \infty} x_n = x$, where $x_n, x \in E$. (See [9]). It is known that the dual E^* of a Banach space E has Fréchet differentiable norm if and only if E is reflexive, strictly convex and has the Kadec-Klee property. (For example, see [18]). Therefore we see that if X has Fréchet differentiable norm then X^* has the Kadec-Klee property. Next we say that X satisfies Opial's condition if w-$\lim_{n \to \infty} x_n = x$ implies $\overline{\lim}_{n \to \infty} \|x_n - x\| < \overline{\lim}_{n \to \infty} \|x_n - y\|$ for all $y \in X$ with $y \neq x$.

Our weak ergodic theorem is an extension of [13, Theorem 1.3] which is stated as follows:

Theorem. *Suppose that $S = \{T(t); t \geq 0\}$ is a semigroup on C which is asymptotically nonexpansive in the intermediate sense, and suppose that F is nonempty. If X^* has the Kadec-Klee property or X satisfies Opial's condition, then every almost-orbit $u(\cdot)$ of S is weakly almost convergent to a fixed point of S.*

Remark 1.1. In Theorem above, the case that X^* has the Kadec-Klee property is essentially due to Kaczor, Kuczumow and Reich [9].

Remark 1.2. If X is a Hilbert space, then (1.1) can be replaced by a weaker condition "$\overline{\lim}_{t \to \infty} \sup_{v \in B}(\|T(t)u - T(t)v\| - \|u - v\|) \leq 0$ for every bounded set $B \subset C$ and $u \in C$". See [11, Added in Proof].

2. Lemmas

Throughout this section, it is assumed that $S = \{T(t); t \geq 0\}$ is a semigroup on C which is asymptotically nonexpansive in the intermediate sense, and that F is nonempty. We note that $\{u(t); t \geq 0\}$ is bounded and $u(\cdot)$ is uniformly continuous on $(0, \infty)$ for every almost-orbit $u(\cdot)$ of S (see [13, Lemma 3.4]).

We start with

Lemma 2.1. *If $u(\cdot)$ and $v(\cdot)$ are almost-orbits of S, then $\|u(t) - v(t)\|$ is convergent as $t \to \infty$.*

Proof. Put $a(t,s) = \|u(t+s) - T(s)u(t)\|$ and $b(t,s) = \|v(t+s) - T(s)v(t)\|$ for $t, s \geq 0$. Then $a(t,s) \to 0$ and $b(t,s) \to 0$ as $t, s \to \infty$.

Let $\varepsilon > 0$. We can choose a $T(\varepsilon) > 0$ such that $a(t,s) < \varepsilon$ and $b(t,s) < \varepsilon$ for $t, s \geq T(\varepsilon)$. Moreover, by (1.1) with $B = \{u(t), v(t); t \geq 0\}$ there is a $\tau(\varepsilon) > 0$ such that if $s \geq \tau(\varepsilon)$ then $\|T(s)u(t) - T(s)v(t)\| < \varepsilon + \|u(t) - v(t)\|$ for $t \geq 0$. Therefore, if $t \geq T(\varepsilon)$ and $s \geq \max\{\tau(\varepsilon), T(\varepsilon)\}$ then $\|u(t+s) - v(t+s)\| \leq a(t,s) + \|T(s)u(t) - T(s)v(t)\| + b(t,s) < 3\varepsilon + \|u(t) - v(t)\|$. Hence $\overline{\lim}_{s\to\infty} \|u(s) - v(s)\| \leq 3\varepsilon + \|u(t) - v(t)\|$ for $t \geq T(\varepsilon)$, which implies that $\|u(t) - v(t)\|$ is convergent as $t \to \infty$. $\square$

Lemma 2.2. *Let $\{z_n\}$ be a sequence in C such that* $\text{w-}\lim_{n\to\infty} z_n = z$. *If* $\lim_{t\to\infty} \overline{\lim}_{n\to\infty} \|T(t)z_n - z_n\| = 0$, *then z is an element in F, i.e., z is a fixed point of S.*

Proof. By the continuity of $T(t_0) : C \to C$ it suffices to show that $\|T(t)z - z\| \to 0$ as $t \to \infty$. To this end, take an $f \in F$ and set $K = \mathrm{clco}\{f, z_n; n \geq 1\}$ (= the closed convex hull of $\{f, z_n; n \geq 1\}$). Then K is a bounded closed convex subset of C. Now, similarly as in the proof of [16, Lemma 2.5] we can obtain $\|T(t)z - z\| \to 0$ as $t \to \infty$. $\square$

Lemma 2.3. *Suppose that $u_p(\cdot)$, $p = 1, 2, \ldots$ are almost-orbits of S such that* $\sup\{\|u_p(t)\|; t \geq 0,\ p \geq 1\} < \infty$. *Then for every $\varepsilon > 0$ and every integer $n \geq 2$ there exists a $\tau'_n(\varepsilon) > 0$ such that*

$$\|T(t)(\textstyle\sum_{p=1}^n \lambda_p u_p(\tau)) - \sum_{p=1}^n \lambda_p T(t)u_p(\tau)\| < \varepsilon$$

for $t, \tau \geq \tau'_n(\varepsilon)$ and $\lambda = (\lambda_1, \ldots, \lambda_n) \in \Delta^{n-1}$, where $\Delta^{n-1} = \{r = (r_1, \ldots, r_n); r_i \geq 0\ (i = 1, \ldots, n)$ and $\sum_{i=1}^n r_i = 1\}$.

Proof. Take an $f \in F$ and set $K = \mathrm{clco}(\{u_p(t); t \geq 0,\ p \geq 1\} \cup \{f\})$. Then K is a bounded closed convex subset of C. Let $\varepsilon > 0$, and let T_ε and δ_ε be positive numbers determined in [13, Lemma 3.3]. Since $\|u_p(t) - u_q(t)\|$ is convergent as $t \to \infty$ by Lemma 2.1, for each $p, q \geq 1$ there exists a $\tau_0(\varepsilon, p, q) > 0$ such that $\|u_p(t) - u_q(t)\| - \|u_p(t+r) - u_q(t+r)\| < \delta_\varepsilon/3$ for $t \geq \tau_0(\varepsilon, p, q)$ and $r \geq 0$. Moreover, for each $p \geq 1$ there exists a $\tau_1(\varepsilon, p) > 0$ such that $a_p(t,s) = \|u_p(t+s) - T(s)u_p(t)\| < \delta_\varepsilon/3$ for $t, s \geq \tau_1(\varepsilon, p)$. Put $\tau_n(\varepsilon) = \max\{\tau_0(\varepsilon, p, q), \tau_1(\varepsilon, p); 1 \leq p, q \leq n\}$ for $n \geq 2$. If $t, s \geq \tau_n(\varepsilon)$, then $\|u_p(t) - u_q(t)\| + \|u_p(t+s) - u_q(t+s)\| \leq \|u_p(t) - u_q(t)\| + a_p(t,s) + \|T(s)u_p(t) - T(s)u_q(t)\| + a_q(t,s) < \|u_p(t) - u_q(t)\| + 2\delta_\varepsilon/3 + \|T(s)u_p(t) - T(s)u_q(t)\|$, and then $\|u_p(t) - u_q(t)\| - \|T(s)u_p(t) - T(s)u_q(t)\| < \|u_p(t) - u_q(t)\| - \|u_p(t+s) - u_q(t+s)\| + 2\delta_\varepsilon/3 < \delta_\varepsilon$ for $1 \leq p, q \leq n$. Therefore by [13, Lemma 3.3] we have that if $t \geq \tau_n(\varepsilon)$ and $s \geq \max\{\tau_n(\varepsilon), T_\varepsilon\}$ then

$\|T(s)(\sum_{p=1}^n \lambda_p u_p(t)) - \sum_{p=1}^n \lambda_p T(s)u_p(t)\| < \varepsilon$ for $\lambda = (\lambda_1, \ldots, \lambda_n) \in \Delta^{n-1}$.

So, putting $\tau_n'(\varepsilon) = \max\{\tau_n(\varepsilon), T_\varepsilon\}$ we obtain the desired conclusion. □

Lemma 2.4. *Let $u(\,\cdot\,)$ be an almost-orbit of S, and set $g(t;s) = (1/s)\int_0^s u(t + r)dr$ for $s > 0$ and $t \geq 0$. Then we have*

$$\lim_{\tau,h\to\infty} \|g(\tau + h; s) - T(h)g(\tau; s)\| = 0 \text{ for every } s > 0,$$

i.e., $g(\,\cdot\,; s)$ is an almost-orbit of S for every $s > 0$.

Proof. Let $s > 0$ and $\varepsilon > 0$. Since $u(\,\cdot\,)$ is uniformly continuous on $(0, \infty)$, there is a $\delta(= \delta(\varepsilon)) > 0$ such that if $t, t' > 0$ and $|t - t'| < \delta$ then $\|u(t') - u(t)\| < \varepsilon$. Let $0 = \xi_0 < \xi_1 < \cdots < \xi_l = s$ be a division of $[0, s]$ with $\mu_i = \xi_i - \xi_{i-1} \leq \delta$ for $i = 1, 2, \ldots, l$. (So $l = l(\delta, s) = l(\varepsilon, s)$, i.e., l depends on ε and s). Then

$$\begin{aligned} &\|g(t; s) - (1/s)\textstyle\sum_{i=1}^l \mu_i u(t + \xi_i)\| \\ &\quad \leq (1/s)\textstyle\sum_{i=1}^l \int_{\xi_{i-1}}^{\xi_i} \|u(t + \xi) - u(t + \xi_i)\|d\xi < \varepsilon \end{aligned} \tag{2.1}$$

for $t \geq 0$.

Put $u_i(\,\cdot\,) = u(\,\cdot\, + \xi_i)$ for $i = 1, 2, \ldots, l$. Then each $u_i(\,\cdot\,)$ is an almost-orbit of S and $\sup\{\|u_i(t)\|; t \geq 0,\ i = 1, 2, \ldots, l\} \leq \sup_{t\geq 0}\|u(t)\| < \infty$. By Lemma 2.3 there is a $\tau_l(\varepsilon)$ $(= \tau(\varepsilon, s)$, i.e., $\tau_l(\varepsilon)$ depends on ε and $s) > 0$ such that $\|T(h)[\sum_{i=1}^l (\mu_i/s)u_i(\tau)] - \sum_{i=1}^l (\mu_i/s)T(h)u_i(\tau)\| < \varepsilon/2$ for $h, \tau \geq \tau_l(\varepsilon)$. By $\|T(h)u(\tau) - u(\tau + h)\| \to 0$ as $h, \tau \to \infty$ we can choose a $\tau_\varepsilon > 0$ such that if $h, \tau \geq \tau_\varepsilon$, then $\|T(h)u(\tau) - u(\tau + h)\| < \varepsilon/2$ and hence $\|T(h)u_i(\tau) - u_i(\tau + h)\| < \varepsilon/2$ for $i = 1, 2, \ldots, l$. Therefore $\|T(h)[\sum_{i=1}^l (\mu_i/s)u_i(\tau)] - \sum_{i=1}^l (\mu_i/s)u_i(\tau + h)\| < \varepsilon$ for $\tau, h \geq \max\{\tau_\varepsilon, \tau_l(\varepsilon)\}$. Combining this with (2.1) we have

$$\|g(\tau + h; s) - T(h)[\textstyle\sum_{i=1}^l (\mu_i/s)u(\tau + \xi_i)]\| < 2\varepsilon \text{ for } \tau, h \geq \max\{\tau_\varepsilon, \tau_l(\varepsilon)\}.$$

By (2.1) again, $\|g(\tau; s) - \sum_{i=1}^l (\mu_i/s)u(\tau + \xi_i)\| < \varepsilon$ for $\tau \geq 0$, and by (1.1) there is a $T_\varepsilon > 0$ such that if $h \geq T_\varepsilon$ then $\|T(h)g(\tau; s) - T(h)[\sum_{i=1}^l (\mu_i/s)u(\tau + \xi_i)]\| < \varepsilon + \|g(\tau; s) - \sum_{i=1}^l (\mu_i/s)u(\tau + \xi_i)\| < 2\varepsilon$ for $\tau \geq 0$. Therefore, if $\tau, h \geq \max\{\tau_\varepsilon, \tau_l(\varepsilon), T_\varepsilon\}$ then $\|g(\tau + h; s) - T(h)g(\tau; s)\| \leq \|g(\tau + h; s) - T(h)[\sum_{i=1}^l (\mu_i/s)u(\tau + \xi_i)]\| + \|T(h)[\sum_{i=1}^l (\mu_i/s)u(\tau + \xi_i)] - T(h)g(\tau; s)\| < 4\varepsilon$. □

Corollary 2.5. *There exists a sequence $\{t_n\}$ of positive numbers t_n such that $t_n \to \infty$ and $\lim_{n,h\to\infty} \|g(t_n + h; n) - T(h)g(t_n; n)\| = 0$.*

Proof. By virtue of Lemma 2.4, for every integer $n \geq 1$ there exist τ_n and h_n with $\tau_n, h_n \geq n$ such that $\|g(\tau + h; n) - T(h)g(\tau; n)\| < 1/n$ for $\tau \geq \tau_n$ and $h \geq h_n$. In particular we have

$$(2.2)\quad \|g(\tau_n + h + h_n; n) - T(h + h_n)g(\tau_n; n)\| < 1/n \text{ for } h \geq 0 \text{ and } n \geq 1.$$

Noting that $\{T(h_n)g(\tau_n; n), g(\tau_n + h_n; n); n \geq 1\}$ is bounded, it follows from (1.1) that for every $\varepsilon > 0$ there is a $T_\varepsilon > 0$ such that $\|T(h)T(h_n)g(\tau_n; n) - T(h)g(\tau_n + h_n; n)\| < \varepsilon + \|T(h_n)g(\tau_n; n) - g(\tau_n + h_n; n)\| < \varepsilon + 1/n$ for $h \geq T_\varepsilon$ and $n \geq 1$. (We have used (2.2) with $h = 0$ here). Combining this with (2.2) we obtain $\|g((\tau_n + h_n) + h; n) - T(h)g(\tau_n + h_n; n)\| \leq \|g((\tau_n + h_n) + h; n) - T(h + h_n)g(\tau_n; n)\| + \|T(h)T(h_n)g(\tau_n; n) - T(h)g(\tau_n + h_n; n)\| < 2/n + \varepsilon$ for $h \geq T_\varepsilon$ and $n \geq 1$. Putting $t_n = h_n + \tau_n$, we have the desired conclusion. □

Lemma 2.6. *If $u(\cdot)$ and $v(\cdot)$ are almost-orbits of S, then*

$$\lim_{t,s\to\infty} \|\lambda u(t + s) + (1 - \lambda)v(t + s) - T(s)[\lambda u(t) + (1 - \lambda)v(t)]\| = 0$$

for every $\lambda \in [0, 1]$, i.e., $\lambda u(\cdot) + (1 - \lambda)v(\cdot)$ is also an almost-orbit of S for every $\lambda \in [0, 1]$.

Proof. Let $\lambda \in [0, 1]$ and set $z(t) = \lambda u(t) + (1 - \lambda)v(t)$ for $t \geq 0$. By Lemma 2.3 with $n = 2$, for every $\varepsilon > 0$ there is a $\tau(\varepsilon) > 0$ such that $\|T(s)[\lambda u(t) + (1 - \lambda)v(t)] - [\lambda T(s)u(t) + (1 - \lambda)T(s)v(t)]\| < \varepsilon$ for $t, s \geq \tau(\varepsilon)$. Therefore $\|z(t + s) - T(s)z(t)\| \leq \lambda\|u(t + s) - T(s)u(t)\| + (1 - \lambda)\|v(t + s) - T(s)v(t)\| + \varepsilon$ for $t, s \geq \tau(\varepsilon)$, which implies $\lim_{t,s\to\infty} \|z(t + s) - T(s)z(t)\| = 0$. □

Corollary 2.7. *F is convex and closed.*

Proof. Let $f, g \in F$ and $\lambda \in [0, 1]$, and set $z = \lambda f + (1 - \lambda)g$. Since the constant functions $u(\cdot) = f$ and $v(\cdot) = g$ are almost-orbits of S, it follows from Lemma 2.6 that $\lim_{s\to\infty} \|z - T(s)z\| = 0$, i.e., $\lim_{s\to\infty} T(s)z = z$. So by the continuity of $T(t_0) : C \to C$ we have $z \in F$. Therefore F is convex. Next, to prove that F is closed, let $f_n \in F$ for $n = 1, 2, \ldots$ and let $f_n \to f$ as $n \to \infty$. By (1.1) with $B = \{f, f_n; n \geq 1\}$ we have $\lim_{t\to\infty} T(t)f = f$. So that $f \in F$ and hence F is closed. □

Throughout the rest of this section, let $u(\cdot)$ be an almost-orbit of S. By the integration by parts we have

$$(2.3)\quad (1/t)\int_0^t u(r + h)dr = (1/t)\int_0^t [(1/s)\int_0^s u(r + q + h)dq]dr + z(t, s, h)$$

for $t, s > 0$ and $h \geq 0$, where

$$z(t, s, h) = (1/st)\int_0^s (s - q)[u(q + h) - u(q + h + t)]dq.$$

Let $g(\,\cdot\,;\,\cdot\,)$ be as in Lemma 2.4, i.e.,

$$g(t;s) = (1/s)\int_0^s u(t+r)dr \text{ for } s > 0 \text{ and } t \geq 0. \tag{2.4}$$

By (2.3) we have

$$g(s;n+k) = (1/(n+k))\int_0^{n+k} g(s+r;n)dr + z(n+k,n,s) \tag{2.5}$$

for $n,k = 1,2,\dots$ and $s \geq 0$. Since $\{u(t);t \geq 0\}$ is bounded, we see that $\{g(t;s);s > 0,\ t \geq 0\}$ is bounded and then by (1.1) with $B = \{g(t;s);s > 0,\ t \geq 0\} \cup \{f\}$, where $f \in F$, there is an $h_0 > 0$ such that

$$\{T(h)g(t;s);t \geq 0,\ s > 0 \text{ and } h \geq h_0\} \text{ is bounded.} \tag{2.6}$$

Let D be the set of sequences $\{t_n\}$ of nonnegative numbers t_n such that $t_n \to \infty$ as $n \to \infty$ and

$$\lim_{n,h\to\infty} \|g(t_n+h;n) - T(h)g(t_n;n)\| = 0. \tag{2.7}$$

We note that the set D is nonempty by Corollary 2.5.

Lemma 2.8. *Let $\{t_n\} \in D$. We have the following*:

(a) *If $\{t'_n\}$ is a sequence such that $t'_n \geq t_n$ for $n \geq 1$ and $t'_n - t_n \to \infty$ as $n \to \infty$, then $\{t'_n\}$ is also an element of the set D.*

(b) *For every $\{t'_n\} \in D$ and $f \in F$, $\{\|g(t'_n;n) - f\|\}$ is convergent as $n \to \infty$ and*

$$\lim_{n\to\infty} \|g(t'_n;n) - f\| = \lim_{n\to\infty} \|g(t_n;n) - f\|. \tag{2.8}$$

Proof. Setting $a(t,h,s) = \|g(t+h;s) - T(h)g(t;s)\|$ for $s > 0$ and $t,h \geq 0$, $\{t_n\} \in D$ means that $t_n \geq 0$ for $n \geq 1$, $t_n \to \infty$ and $\lim_{n,h\to\infty} a(t_n,h,n) = 0$.

(a) By $t'_n - t_n \to \infty$ we can choose an $n_0 \geq 1$ such that $t'_n - t_n \geq h_0$ for $n \geq n_0$. Since $\{T(t'_n - t_n)g(t_n;n), g(t'_n;n); n \geq n_0\}$ is bounded by (2.6), it follows from (1.1) that $\lim_{h\to\infty} \sup_{n\geq n_0}[\|T(h)T(t'_n - t_n)g(t_n;n) - T(h)g(t'_n;n)\| - \|T(t'_n - t_n)g(t_n;n) - g(t'_n;n)\|] \leq 0$. Therefore for every $\varepsilon > 0$ there is a $T_\varepsilon > 0$ such that

$$\|T(h + t'_n - t_n)g(t_n;n) - T(h)g(t'_n;n)\| < \varepsilon + a(t_n, t'_n - t_n, n)$$

for $h \geq T_\varepsilon$ and $n \geq n_0$. Hence $\|g(t'_n + h;n) - T(h)g(t'_n;n)\| \leq \|g(t'_n + h;n) - T(t'_n - t_n + h)g(t_n;n)\| + \|T(t'_n - t_n + h)g(t_n;n) - T(h)g(t'_n;n)\| < a(t_n, t'_n - t_n + h, n) + \varepsilon + a(t_n, t'_n - t_n, n)$ for $h \geq T_\varepsilon$ and $n \geq n_0$. Combining this with $\lim_{n,h\to\infty} a(t_n,h,n) = 0$ we obtain $\|g(t'_n + h;n) - T(h)g(t'_n;n)\| \to 0$ as $n,h \to \infty$.

To prove (b) we use (2.5). Let $\{t'_n\} \in D$ and $f \in F$. By (2.5) with $s = t_{n+k}$ we obtain

$$\begin{aligned} &\|g(t_{n+k}; n+k) - f\| \\ (2.9) \qquad &\le (1/(n+k)) \int_0^{n+k} \|g(t_{n+k} + r; n) - f\| dr + Mn/(n+k) \end{aligned}$$

for $n, k \ge 1$, where $M = \sup_{t \ge 0} \|u(t) - f\|$. If $t_{n+k} - t'_n + r \ge 0$ then we have

$$\begin{aligned} &\|g(t_{n+k} + r; n) - f\| \\ (2.10) \qquad &\le a(t'_n, t_{n+k} - t'_n + r, n) + \|T(t_{n+k} - t'_n + r)g(t'_n; n) - f\|. \end{aligned}$$

Let $\varepsilon > 0$. By $\lim_{n,h\to\infty} a(t'_n, h, n) = 0$ and (1.1) there is a $d_\varepsilon > 0$ such that

$a(t'_n, h, n) < \varepsilon/2$ and $\|T(h)g(t'_n; n) - f\| < \varepsilon/2 + \|g(t'_n; n) - f\|$ for $n, h \ge d_\varepsilon$.

Therefore it follows from (2.10) that if $n \ge d_\varepsilon$ and $t_{n+k} - t'_n \ge d_\varepsilon$ then $\|g(t_{n+k} + r; n) - f\| < \varepsilon + \|g(t'_n; n) - f\|$ for $r \ge 0$. Let $n \ge d_\varepsilon$. By $t_{n+k} \to \infty$ as $k \to \infty$ we can choose an integer $k(n, \varepsilon) \ge 1$ such that $t_{n+k} - t'_n \ge d_\varepsilon$ for $k \ge k(n, \varepsilon)$. Hence $\|g(t_{n+k} + r; n) - f\| < \varepsilon + \|g(t'_n; n) - f\|$ for $k \ge k(n, \varepsilon)$ and $r \ge 0$. Combining this with (2.9) we have

$$\|g(t_{n+k}; n+k) - f\| \le \varepsilon + \|g(t'_n; n) - f\| + Mn/(n+k) \text{ for } k \ge k(n, \varepsilon).$$

Letting $k \to \infty$ we obtain $\overline{\lim}_{k\to\infty} \|g(t_k; k) - f\| \le \varepsilon + \|g(t'_n; n) - f\|$ for $n \ge d_\varepsilon$, which implies

$$(2.11) \qquad \overline{\lim}_{k\to\infty} \|g(t_k; k) - f\| \le \underline{\lim}_{n\to\infty} \|g(t'_n; n) - f\|.$$

Exchanging $\{t_n\}$ and $\{t'_n\}$ here we have $\overline{\lim}_{n\to\infty} \|g(t'_n; n) - f\| \le \underline{\lim}_{n\to\infty} \|g(t_n; n) - f\|$. By this and (2.11) we see that $\{\|g(t'_n; n) - f\|\}$ and $\{\|g(t_n; n) - f\|\}$ are convergent and (2.8) holds good. $\square$

Lemma 2.9. *For every* $\{t_n\} \in D$ *and* $f \in F$, $\{\|T(h)g(t_n; n) - f\|\}$ *is convergent as* $n, h \to \infty$ *and*

$$\lim_{n,h\to\infty} \|T(h)g(t_n; n) - f\| = \lim_{n\to\infty} \|g(t_n; n) - f\|.$$

Proof. Let $\{t_n\}, \{t'_n\} \in D$ and $f \in F$. By (2.5) with $s = h + \widetilde{h} + t_{n+k}$ we have $g(h + \widetilde{h} + t_{n+k}; n+k) - f = (1/(n+k)) \int_0^{n+k} [g(h + \widetilde{h} + t_{n+k} + r; n) - f] dr + z(n+k, n, h + \widetilde{h} + t_{n+k})$ for $n, k \ge 1$ and $h, \widetilde{h} \ge 0$ and then

$$\begin{aligned} &\|g(h + \widetilde{h} + t_{n+k}; n+k) - f\| \\ (2.12) \qquad &\le (1/(n+k)) \int_0^{n+k} \|g(h + \widetilde{h} + t_{n+k} + r; n) - f\| dr \\ &\quad + Mn/(n+k) \text{ for } n, k \ge 1 \text{ and } h, \widetilde{h} \ge 0, \end{aligned}$$

where $M = \sup_{r\geq 0}\|u(r)\|$. For $n, k \geq 1$ and $h, \widetilde{h} \geq 0$ we have

$$\begin{aligned}\|g(h+\widetilde{h}+t_{n+k}+r;n)-f\| &\leq \|g(h+\widetilde{h}+(t_{n+k}-t'_n)+r+t'_n;n)\\ &\qquad - T(h+\widetilde{h}+(t_{n+k}-t'_n)+r)g(t'_n;n)\|\\ &\quad + \|T(t_{n+k}-t'_n+\widetilde{h}+r)T(h)g(t'_n;n)\\ &\qquad - T(t_{n+k}-t'_n+\widetilde{h}+r)g(t'_n+h;n)\|\\ &\quad + \|T(t_{n+k}-t'_n+\widetilde{h}+r)g(t'_n+h;n)-f\|\\ &= J_1+J_2+J_3.\end{aligned}$$

Let $\varepsilon > 0$. By $\lim_{n,s\to\infty}\|g(t'_n+s;n)-T(s)g(t'_n;n)\| = 0$ there are $n(\varepsilon) \geq 1$ and $s_1(\varepsilon) > 0$ such that $\|g(t'_n+s;n)-T(s)g(t'_n;n)\| < \varepsilon$ for $n \geq n(\varepsilon)$ and $s \geq s_1(\varepsilon)$. Therefore we have

$$\begin{aligned} &J_1 < \varepsilon \text{ for } h, \widetilde{h} \geq 0 \text{ and } r \geq 0,\\ &\quad \text{if } n \geq n(\varepsilon) \text{ and } t_{n+k}-t'_n \geq s_1(\varepsilon).\end{aligned} \tag{2.13}$$

Next, by (1.1) with $B = \{T(h)g(t'_n;n), g(t'_n+h;n); h \geq h_0 \text{ and } n \geq 1\} \cup \{f\}$ we can choose a $s_2(\varepsilon) > 0$ such that

$$\|T(s)u - T(s)v\| < \varepsilon + \|u-v\| \text{ for } u, v \in B \text{ and } s \geq s_2(\varepsilon). \tag{2.14}$$

Therefore by noting that $\|g(t'_n+h;n)-T(h)g(t'_n;n)\| < \varepsilon$ for $n \geq n(\varepsilon)$ and $h \geq s_1(\varepsilon)$ we have

$$\begin{aligned} &J_2 < 2\varepsilon \text{ for } \widetilde{h}, r \geq 0,\\ &\quad \text{if } n \geq n(\varepsilon),\ t_{n+k}-t'_n \geq s_2(\varepsilon) \text{ and } h \geq \max\{h_0, s_1(\varepsilon)\}.\end{aligned} \tag{2.15}$$

Moreover, by (2.14) with $v = f$ we get

$$\begin{aligned} &J_3 < \varepsilon + \|g(t'_n+h;n)-f\| \text{ for } \widetilde{h}, r \geq 0,\\ &\quad \text{if } t_{n+k}-t'_n \geq s_2(\varepsilon) \text{ and } h \geq h_0.\end{aligned} \tag{2.16}$$

By (2.13), (2.15) and (2.16) we see that if $n \geq n(\varepsilon)$, $t_{n+k}-t'_n \geq \max\{s_1(\varepsilon), s_2(\varepsilon)\}$ and $h \geq \max\{h_0, s_1(\varepsilon)\}$ then $\|g(h+\widetilde{h}+t_{n+k}+r;n)-f\| \leq J_1+J_2+J_3 < 4\varepsilon + \|g(t'_n+h;n)-f\|$ for every $\widetilde{h}, r \geq 0$. Combining this with (2.12) we obtain that if $n \geq n(\varepsilon)$, $t_{n+k}-t'_n \geq \max\{s_1(\varepsilon), s_2(\varepsilon)\}$ and $h \geq \max\{h_0, s_1(\varepsilon)\}$ then

$$\|g(h+\widetilde{h}+t_{n+k};n+k)-f\| \leq 4\varepsilon + \|g(t'_n+h;n)-f\| + Mn/(n+k) \text{ for } \widetilde{h} \geq 0.$$

Letting $k, \widetilde{h} \to \infty$, we have $\overline{\lim}_{k,\widetilde{h}\to\infty}\|g(\widetilde{h}+t_k;k)-f\| \leq 4\varepsilon + \|g(t'_n+h;n)-f\|$ for $n \geq n(\varepsilon)$ and $h \geq \max\{h_0, s_1(\varepsilon)\}$, which implies

$$\overline{\lim}_{n,h\to\infty}\|g(t_n+h;n)-f\| \leq \underline{\lim}_{n,h\to\infty}\|g(t'_n+h;n)-f\|. \tag{2.17}$$

This shows that $\{\|g(t_n+h;n)-f\|\}$ is convergent as $n,h\to\infty$ (and $\lim_{n,h\to\infty}\|g(t'_n+h;n)-f\| = \lim_{n,h\to\infty}\|g(t_n+h;n)-f\|$). In particular, $\{\|g(2t_n;n)-f\|\}$ is convergent and

$$\lim_{n\to\infty}\|g(2t_n;n)-f\| = \lim_{n,h\to\infty}\|g(t_n+h;n)-f\|.$$

Since $\{2t_n\}\in D$ by Lemma 2.8 (a), by (2.8) with $t'_n = 2t_n$ for $n\ge 1$ we have $\lim_{n\to\infty}\|g(2t_n;n)-f\| = \lim_{n\to\infty}\|g(t_n;n)-f\|$, which implies

$$\lim_{n,h\to\infty}\|g(t_n+h;n)-f\| = \lim_{n\to\infty}\|g(t_n;n)-f\|.$$

Combining this with $\lim_{n,h\to\infty}\|g(t_n+h;n)-T(h)g(t_n;n)\|=0$ we obtain

$$\lim_{n,h\to\infty}\|T(h)g(t_n;n)-f\| = \lim_{n\to\infty}\|g(t_n;n)-f\|. \qquad \square$$

Lemma 2.10 ([9]). *For every $\{t_n\}\in D$, $f,g\in F$ and $\alpha\in[0,1]$, $\{\|\alpha g(t_n;n)+(1-\alpha)f-g\|\}$ is convergent as $n\to\infty$ and*

$$\lim_{n\to\infty}\|\alpha g(t_n;n)+(1-\alpha)f-g\| = \lim_{n\to\infty}\|\alpha g(t'_n;n)+(1-\alpha)f-g\|,$$

where $\{t'_n\}\in D$.

Proof. The conclusion is trivial in the case of $\alpha=0$. Let $0<\alpha\le 1$, and let $\{t_n\},\{t'_n\}\in D$ and $f,g\in F$. By using (2.5) with $s=t_{n+k}$ we have

$$\begin{aligned}&\|\alpha g(t_{n+k};n+k)+(1-\alpha)f-g\|\\ &\quad\le (1/(n+k))\int_0^{n+k}\|\alpha g(t_{n+k}+r;n)+(1-\alpha)f-g\|dr\\ &\qquad + Mn/(n+k)\end{aligned} \tag{2.18}$$

for $n,k\ge 1$, where $M=\sup_{r\ge 0}\|u(r)\|$. For every $n\ge 1$ choose a $k(n)\ge 1$ such that $t_{n+k}\ge t'_n$ for $k\ge k(n)$. Now, for $n\ge 1$ and $k\ge k(n)$ we have

$$\begin{aligned}&\|\alpha g(t_{n+k}+r;n)+(1-\alpha)f-g\|\\ &\quad\le \alpha\|g((t_{n+k}-t'_n+r)+t'_n;n)\\ &\qquad\quad - T(t_{n+k}-t'_n+r)g(t'_n;n)\|\\ &\qquad + \|\alpha T(t_{n+k}-t'_n+r)g(t'_n;n)+(1-\alpha)f\\ &\qquad\quad - T(t_{n+k}-t'_n+r)[\alpha g(t'_n;n)+(1-\alpha)f]\|\\ &\qquad + \|T(t_{n+k}-t'_n+r)[\alpha g(t'_n;n)+(1-\alpha)f]-g\|\\ &\quad = I_1+I_2+I_3 \text{ for } r\ge 0.\end{aligned}$$

Let $\varepsilon>0$ be arbitrarily given. By $\lim_{n,t\to\infty}\|g(t'_n+t;n)-T(t)g(t'_n;n)\|=0$ there is a $t_1(\varepsilon)>0$ such that $\|g(t'_n+t;n)-T(t)g(t'_n;n)\|<\varepsilon$ for $n,t\ge t_1(\varepsilon)$. Hence we have

$$\begin{aligned}&I_1<\varepsilon \text{ for } r\ge 0,\\ &\quad \text{if } n\ge t_1(\varepsilon),\ k\ge k(n) \text{ and } t_{n+k}-t'_n\ge t_1(\varepsilon).\end{aligned} \tag{2.19}$$

Next, by (1.1) with $B = \{\alpha g(t'_n; n) + (1-\alpha)f; n \geq 1,\ 0 < \alpha \leq 1\} \cup \{g\}$ we have $\overline{\lim}_{t\to\infty} \sup_{u\in B}(\|T(t)u - g\| - \|u - g\|) \leq 0$, and hence there is a $t_2(\varepsilon) > 0$ such that

$$I_3 < \varepsilon + \|\alpha g(t'_n; n) + (1-\alpha)f - g\| \text{ for } r \geq 0, \quad \text{if } n \geq 1,\ k \geq k(n) \text{ and } t_{n+k} - t'_n \geq t_2(\varepsilon). \tag{2.20}$$

Finally we estimate I_2. To this end we use [13, Lemma 3.3] with $n = 2$ and $K = \overline{\text{co}}(\{g(t; s); s > 0,\ t \geq 0\} \cup \{f\})$. Let $T_\varepsilon > 0$ and $\delta_\varepsilon > 0$ be as in [13, Lemma 3.3]. Since $\lim_{n,h\to\infty} \|T(h)g(t'_n; n) - f\| = \lim_{n\to\infty} \|g(t'_n; n) - f\|$ by Lemma 2.9, we can choose $n(\varepsilon) \geq 1$ and $h(\varepsilon) > 0$ such that $|\|g(t'_n; n) - f\| - \|T(h)g(t'_n; n) - f\|| < \delta_\varepsilon$ for $n \geq n(\varepsilon)$ and $h \geq h(\varepsilon)$. Therefore by virtue of [13, Lemma 3.3] we have $\|T(h)[\alpha g(t'_n; n) + (1-\alpha)f] - [\alpha T(h)g(t'_n; n) + (1-\alpha)f]\| < \varepsilon$ for $n \geq n(\varepsilon)$ and $h \geq \max\{h(\varepsilon), T_\varepsilon\}$. Consequently we have

$$I_2 < \varepsilon \text{ for } r \geq 0, \quad \text{if } n \geq n(\varepsilon),\ k \geq k(n) \text{ and } t_{n+k} - t'_n \geq \max\{h(\varepsilon), T_\varepsilon\}. \tag{2.21}$$

It follows from (2.19), (2.20) and (2.21) that if $n \geq \max\{t_1(\varepsilon), n(\varepsilon)\}$, $k \geq k(n)$ and $t_{n+k} - t'_n \geq \max\{t_1(\varepsilon), t_2(\varepsilon), h(\varepsilon), T_\varepsilon\}$ then $\|\alpha g(t_{n+k} + r; n) + (1-\alpha)f - g\| \leq I_1 + I_2 + I_3 < 3\varepsilon + \|\alpha g(t'_n; n) + (1-\alpha)f - g\|$ for $r \geq 0$. So that by (2.18) we have

$$\|\alpha g(t_{n+k}; n+k) + (1-\alpha)f - g\| \leq 3\varepsilon + \|\alpha g(t'_n; n) + (1-\alpha)f - g\| + Mn/(n+k)$$

if $n \geq \max\{t_1(\varepsilon), n(\varepsilon)\}$, $k \geq k(n)$ and $t_{n+k} - t'_n \geq \max\{t_1(\varepsilon), t_2(\varepsilon), h(\varepsilon), T_\varepsilon\}$. Letting $k \to \infty$ we have

$$\overline{\lim}_{k\to\infty} \|\alpha g(t_k; k) + (1-\alpha)f - g\| \leq 3\varepsilon + \|\alpha g(t'_n; n) + (1-\alpha)f - g\|$$

for $n \geq \max\{t_1(\varepsilon), n(\varepsilon)\}$, which implies

$$\overline{\lim}_{n\to\infty} \|\alpha g(t_n; n) + (1-\alpha)f - g\| \leq \underline{\lim}_{n\to\infty} \|\alpha g(t'_n; n) + (1-\alpha)f - g\|.$$

This shows that $\{\|\alpha g(t_n; n) + (1-\alpha)f - g\|\}$ and $\{\|\alpha g(t'_n; n) + (1-\alpha)f - g\|\}$ are convergent as $n \to \infty$, and

$$\lim_{n\to\infty} \|\alpha g(t_n; n) + (1-\alpha)f - g\| = \lim_{n\to\infty} \|\alpha g(t'_n; n) + (1-\alpha)f - g\|. \quad \square$$

The following lemma is shown by the same way in the proof of [12, Lemma 3.8].

Lemma 2.11. *Let $\{t_n\}$ be a sequence of nonnegative numbers. If* w-$\lim_{n\to\infty} g(t_n + h; n) = y$ *uniformly in $h \geq 0$, then* w-$\lim_{t\to\infty} g(h; t) = y$ *uniformly in $h \geq 0$, i.e., $u(\cdot)$ is weakly almost convergent to y.*

3. Proof of Theorem

Throughout this section it is assumed that $S = \{T(t); t \geq 0\}$ is a semigroup on C which is asymptotically nonexpansive in the intermediate sense, and that F is nonempty.

Let $u(\,\cdot\,)$ be an almost-orbit of S, and let $g(\,\cdot\,;\,\cdot\,)$ and D be as in the preceding section. Let $\{t_n^0\} \in D$ and define a set D_0 by

$$D_0 = \{\{t_n\}; t_n \geq 2t_n^0 \text{ for } n \geq 1\}.$$

By Lemma 2.8 (a) we see that $D_0 \subset D$. We first note the following:

$$\lim_{h\to\infty} \overline{\lim}_{n\to\infty} \|T(h)g(t_n;n) - g(t_n;n)\| = 0 \text{ for every } \{t_n\} \in D. \tag{3.1}$$

In fact, let $\{t_n\} \in D$. By (2.7), for every $\varepsilon > 0$ there is an $N(\varepsilon) \geq 1$ such that $\|T(h)g(t_n;n) - g(t_n + h;n)\| < \varepsilon$ for $n, h \geq N(\varepsilon)$. Therefore we have $\|T(h)g(t_n;n) - g(t_n;n)\| \leq \|T(h)g(t_n;n) - g(t_n + h;n)\| + \|g(t_n + h;n) - g(t_n;n)\| < \varepsilon + 2Mh/n$ for $n, h \geq N(\varepsilon)$, where $M = \sup_{r\geq 0} \|u(r)\|$, which implies (3.1).

Lemma 3.1. *If $\{g(t_n;n)\}$ is weakly convergent as $n \to \infty$ for every $\{t_n\} \in D_0$, then $u(\,\cdot\,)$ is weakly almost convergent to a fixed point of S.*

Proof. Let $\{t_n\} \in D_0$ and put $\text{w-lim}_{n\to\infty} g(t_n;n) = y$. We have $y \in F$ by Lemma 2.2 because $\{t_n\}$ satisfies (3.1).

Let $\{\tau_n\} \in D_0$ and set $\text{w-lim}_{n\to\infty} g(\tau_n;n) = z$. We see that $z = y$. In fact, let us define a sequence $\{t'_n\}$ by $t'_{2n-1} = t_{2n-1}$ and $t'_{2n} = \tau_{2n}$ for $n \geq 1$. Clearly $\{t'_n\} \in D_0$, and hence $\{g(t'_n;n)\}$ is weakly convergent as $n \to \infty$ by the assumption. Consequently, $z = \text{w-lim}_{n\to\infty} g(\tau_{2n};2n) = \text{w-lim}_{n\to\infty} g(t'_{2n};2n) = \text{w-lim}_{n\to\infty} g(t'_n;n) = \text{w-lim}_{n\to\infty} g(t'_{2n-1};2n-1) = \text{w-lim}_{n\to\infty} g(t_n;n) = y$.

Thus we showed that $\text{w-lim}_{n\to\infty} g(\tau_n;n) = y$ for all $\{\tau_n\} \in D_0$, which implies

$$\text{w-lim}_{n\to\infty} g(2t_n^0 + h;n) = y \text{ uniformly in } h \geq 0.$$

It follows from Lemma 2.11 that $u(\,\cdot\,)$ is weakly almost convergent to $y \in F$. □

Proof of Theorem. By virtue of Lemma 3.1 it suffices to show that $\{g(t_n;n)\}$ is weakly convergent as $n \to \infty$ for every $\{t_n\} \in D_0$. Let $\{t_n\} \in D_0$, and let W be the set of weak subsequential limits of $\{g(t_n;n)\}$. W is nonempty because $\{g(t_n;n); n \geq 1\}$ is bounded. We have

$$W \subset F. \tag{3.2}$$

In fact, let $z \in W$ and choose a subsequence $\{n_k\}$ of $\{n\}$ such that $z = \text{w-lim}_{k\to\infty} g(t_{n_k};n_k)$. Since $\lim_{h\to\infty}[\overline{\lim}_{k\to\infty} \|T(h)g(t_{n_k};n_k) - g(t_{n_k};n_k)\|] = 0$ by (3.1), we see from Lemma 2.2 that $z \in F$.

First, suppose that X satisfies Opial's condition. To prove that W is a singleton, let $v_i \in W$ and $v_i = \text{w-lim}_{n(i)\to\infty} g(t_{n(i)}; n(i))$, $i = 1, 2$, where $\{n(i)\}$, $i = 1, 2$, are subsequences of $\{n\}$. Suppose that $v_1 \neq v_2$. Noting that $\{\|g(t_n; n) - v_i\|\}$ is convergent as $n \to \infty$ for $i = 1, 2$ by (3.2) and Lemma 2.8 (b), Opial's condition implies

$$\begin{aligned} &\lim_{n\to\infty} \|g(t_n; n) - v_1\| = \lim_{n(1)\to\infty} \|g(t_{n(1)}; n(1)) - v_1\| \\ &\quad < \lim_{n(1)\to\infty} \|g(t_{n(1)}; n(1)) - v_2\| = \lim_{n\to\infty} \|g(t_n; n) - v_2\|. \end{aligned}$$

In the same way we obtain $\lim_{n\to\infty} \|g(t_n; n) - v_2\| < \lim_{n\to\infty} \|g(t_n; n) - v_1\|$. This is a contradiction. Consequently, $v_1 = v_2$ and hence W is a singleton. So that $\{g(t_n; n)\}$ is weakly convergent as $n \to \infty$.

Next, suppose that X^* has the Kadec-Klee property. We see from Lemma 2.10 that $\{\|\alpha g(t_n; n) + (1 - \alpha)f - g\|\}$ is convergent as $n \to \infty$ for every $f, g \in W(\subset F$ by (3.2)) and $\alpha \in [0, 1]$. Therefore by virtue of [9, Lemma 4.1], W is a singleton. □

We conclude this paper with the following:

Remark 3.1. Let $u(\cdot)$ be an almost-orbit of S. We note that if every orbit of S is almost convergent then so is $u(\cdot)$. More precisely we have the following Proposition which extends [12, Theorem 6.1].

Proposition. *Let X be a general Banach space.* (*We do not assume that X is uniformly convex*). *If for every $x \in C$, $T(\cdot)x$ is weakly* (*resp. strongly*) *almost convergent to a fixed point of S, then $u(\cdot)$ is also weakly* (*resp. strongly*) *almost convergent to a fixed point of S.*

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