SOME COHOMOTOPY GROUPS OF SUSPENDED PROJECTIVE PLANES

H. KACHI, J. MUKAI, T. NOZAKI, Y. SUMITA and D. TAMAKI

Abstract. In this paper we compute some cohomotopy groups of the suspended complex and quaternionic projective plane by use of the exact sequence associated with the canonical cofiber sequence and a formula about a multiple of the identity class of the suspended projective plane.

1. Introduction and statement of results

In this note all spaces, maps and homotopies are based. We denote by $\Sigma X$ a suspension of a space $X$. For the normed fields $F = \mathbb{R}$ (real), $\mathbb{C}$ (complex), $\mathbb{H}$ (quaternion) and $\mathbb{O}$ (octonion) with the usual norm, let $d = \dim_{\mathbb{R}} F$.

The projective plane over $F$ is denoted by $F\mathbb{P}^2$. This is the space given by attaching a $2d$-cell to $S^d$ by the Hopf map $h_d(F) : S^{2d-1} \to S^d$. The inclusion map of $S^d$ and the collapsing map to the top cell are denoted by $i_F : S^d \to F\mathbb{P}^2$, $p_F : F\mathbb{P}^2 \to S^{2d}$ respectively. For a space $X$, let $i_X \in [X, X]$ be the identity class of $X$, $i_n = i_X$ for $X = S^n$ and $i_F = i_X$ for $X = F\mathbb{P}^2$. The $n$-th cohomotopy set of $X$ is denoted by $\pi^n(X) = [X, S^n]$. We set $h_n(F) = \Sigma^{n-d} h_d(F)$ for $n \geq d$.

The purpose of this note is to calculate cohomotopy groups of the suspended projective plane $\Sigma^n F\mathbb{P}^2$ for the cases $F = \mathbb{C}$ and $\mathbb{H}$. 2-primary versions of the calculations appeared in Master’s theses of the third author [9] and the fourth author [12] in Shinshu University under the guidance of the other three authors together with Professor T. Matsuda.

The calculation will be done in the following way. Consider the exact sequence

$$
\pi_{n+d+1}(S^k) \xrightarrow{h_{d+n+1}(F)^*} \pi_{n+2d}(S^k) \xrightarrow{\Sigma^n p_F^*} \pi_{n+2d}(S^k) \xrightarrow{\Sigma^n i_F^*} \pi_{n+d}(S^k) \xrightarrow{h_{d+n}(F)^*} \pi_{n+2d-1}(S^k)
$$

induced from the cofiber sequence

$$
S^{2d-1} \xrightarrow{h_d(F)} S^d \xrightarrow{i_F} F\mathbb{P}^2 \xrightarrow{p_F} S^{2d} \xrightarrow{h_{d+1}(F)} S^{d+1}.
$$

From the above exact sequence we have the short exact sequence

$$
0 \to \text{Coker } h_{d+n+1}(F)^* \to [\Sigma^n F\mathbb{P}^2, S^k] \to \text{Ker } h_{n+1}(F)^* \to 0.
$$
Then we determine the group extension by use of formulas of Toda brackets. For the 2-primary components, \( \text{Coker } h_{d+n+1}(F)^* \) and \( \text{Ker } h_{d+n}(F)^* \) are calculated in [9] and [12] for \( F = C \) and \( H \), respectively.

The results are summarized in the following:

**Theorem 1.1.** The cohomotopy groups \( [\Sigma^n CP^2, S^{n+k}] \) in the range of \(-5 \leq k \leq 1\) is isomorphic to the group given in the following table:

<table>
<thead>
<tr>
<th>( n \backslash k )</th>
<th>1</th>
<th>0</th>
<th>-1</th>
<th>-2</th>
<th>-3</th>
<th>-4</th>
<th>-5</th>
</tr>
</thead>
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<td>1</td>
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<td>0</td>
</tr>
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<td>0</td>
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<td>0</td>
</tr>
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<td>0</td>
<td>0</td>
</tr>
<tr>
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<td>( \infty + 120 )</td>
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<td>( \infty + 120 )</td>
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<td>0</td>
<td>0</td>
<td>0</td>
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</tr>
</tbody>
</table>

**Theorem 1.2.** The cohomotopy group \( [\Sigma^n HP^2, S^{n+k}] \) in the range of \(-3 \leq k \leq 3\) is isomorphic to the group given in the following table:

<table>
<thead>
<tr>
<th>( n \backslash k )</th>
<th>3</th>
<th>2</th>
<th>1</th>
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<td>0</td>
</tr>
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<td>( \infty + 2 )</td>
<td>2</td>
<td>( \infty + 15+4 )</td>
<td>( (2)^2 )</td>
<td>( (2)^2 )</td>
<td>( (2)^2 )</td>
<td>0</td>
</tr>
<tr>
<td>4</td>
<td>2</td>
<td>10+36</td>
<td>( (2)^3 )</td>
<td>( (2)^3 )</td>
<td>( (2)^3 )</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>5</td>
<td>2</td>
<td>20+36</td>
<td>( (2)^2 )</td>
<td>( (2)^4 )</td>
<td>4+6</td>
<td>4+2+3</td>
<td>0</td>
</tr>
<tr>
<td>6</td>
<td>2</td>
<td>40+36</td>
<td>( (2)^2 )</td>
<td>( (2)^3 )</td>
<td>8+(4)^2+6+45</td>
<td>4+2+105</td>
<td>0</td>
</tr>
<tr>
<td>7</td>
<td>( \infty + 40+36 )</td>
<td>( (2)^3 )</td>
<td>( \infty + (2)^2 )</td>
<td>4+6</td>
<td>4+(2)^3+105</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>8</td>
<td>80+36</td>
<td>( (2)^4 )</td>
<td>( (2)^3 )</td>
<td>2+6</td>
<td>8+(2)^2+315</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>9</td>
<td>( (2)^3 )</td>
<td>( (2)^4 )</td>
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<td>0</td>
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<td>( (2)^2 )</td>
<td>( (2)^3 )</td>
<td>6</td>
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<td>0</td>
<td>0</td>
</tr>
<tr>
<td>11</td>
<td>( \infty + (2)^2 )</td>
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<td>( \infty + 8+945 )</td>
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<td>0</td>
</tr>
<tr>
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<td>( (2)^2 )</td>
<td>6</td>
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<td>0</td>
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<td>0</td>
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<td>0</td>
</tr>
</tbody>
</table>
In the above tables, an integer $n$ indicates a cyclic group $\mathbb{Z}_n$ of order $n$, the symbol “$\infty$” an infinite cyclic group $\mathbb{Z}$, the symbol “$+$” the direct sum of the groups and $(n)^k$ indicates the direct sum of $k$-copies of $\mathbb{Z}_n$. Groups in the stable range (lower left area) and trivial groups (upper right area) are omitted.

In the stable range, Theorems 1.3 and 1.4 overlap with the results of [15], [10] and [7]. We use the notation and results of [13] freely.

### 2. Preliminaries

Consider an element $\alpha \in \pi_m(S^n)$ $(m > n \geq 2)$ such that $\Sigma\alpha$ and $\Sigma^2\alpha$ are of order $t$. Let $C_\alpha = S^n \cup_\alpha e^{m+1}$ be the mapping cone of $\alpha$. The inclusion map of $S^n$ and the collapsing map to the top cell $e^{m+1}$ are denoted by $i : S^n \to C_\alpha$ and $p : C_\alpha \to S^{m+1}$, respectively. We shall use the identification $\Sigma^kC_\alpha = C_\Sigma^k\alpha$. Then we have the cofiber sequence

$$S^{m+k} \xrightarrow{\Sigma^k\alpha} S^{n+k} \xrightarrow{\Sigma^ki} \Sigma^kC_\alpha \xrightarrow{\Sigma^kp} S^{m+k+1} \xrightarrow{\Sigma^{k+1}\alpha} S^{n+k+1}.$$ 

Consider elements $\beta \in \pi_n(Z)$ and $\gamma \in [W, S^m]$ which satisfy $\beta \circ \alpha = 0$ and $\alpha \circ \gamma = 0$. We denote by $\overline{\beta} \in [C_\alpha, Z]$ an extension of $\beta$ satisfying $i^*(\overline{\beta}) = \beta$ and by $\overline{\gamma} \in [\Sigma W, C_\alpha]$ a coextension of $\gamma$ satisfying $p_*(\overline{\gamma}) = \Sigma\gamma$.

Making use of the homotopy exact sequence of the pair $(\Sigma C_\alpha, S^{n+1})$ and the theorem of Blakers-Massey [3], we easily obtain the following.

**Lemma 2.1.**

1. $\pi_{n+1}(\Sigma C_\alpha) \cong \mathbb{Z}\{\Sigma i\}$,
2. $\pi_{m+2}(\Sigma C_\alpha) \cong \mathbb{Z}\{\Sigma i \circ (\pi_{m+2}(S^{n+1})/\{\Sigma \alpha \circ \eta_{m+1}\})\}.$

By Theorem 10.3.10 of [16], we have the following.

**Lemma 2.2.** Let $Y$ be a 1-connected space. Then the commutator group of $[\Sigma C_\alpha, Y]$ and $\pi_{m+2}(Y) \circ \Sigma p$ is trivial.

Hereafter, the commutativity of the homotopy group $[\Sigma C_\alpha, Y]$ is ensured by this lemma.

Consider the exact sequence

$$\pi_{n+2}(S^k) \xrightarrow{\Sigma^2\alpha^*} \pi_{m+2}(S^k) \xrightarrow{\Sigma p^*} [\Sigma C_\alpha, S^k] \xrightarrow{\Sigma i^*} \pi_{n+1}(S^k) \xrightarrow{\Sigma \alpha^*} \pi_{m+1}(S^k)$$

induced from the above cofiber sequence. Making use of this exact sequence and Lemma 2.2, we have the following.
Lemma 2.3. (1) \([\Sigma C_\alpha, S^{m+2}] \cong \mathbb{Z}\{\Sigma p}\),
(2) \(\Sigma p^*: \pi_{m+2}(S^k) \to [\Sigma C_\alpha, S^k]\) is an isomorphism for \(k > n + 2\),
(3) \([\Sigma C_\alpha, S^{n+2}] \cong \pi_{m+2}(S^{n+2})/\{\Sigma^2 \alpha\}\),
(4) \([\Sigma C_\alpha, S^{n+1}] \cong \mathbb{Z}\{t_{n+1}\} \oplus (\pi_{m+2}(S^{m+1})/\{\eta_{n+1} \circ \Sigma^2 \alpha\}) \circ \Sigma p\).

From Theorem 1.3 of [11], we have

Proposition 2.4. (1) \([\Sigma C_\alpha, \Sigma C_\alpha] \cong \mathbb{Z}\{\Sigma \iota_{C_\alpha}\} \oplus \mathbb{Z}\{\tilde{t}_{m+1} \circ \Sigma p\} \oplus \Sigma i \circ (\pi_{m+2}(S^{n+1})/\{\eta_{n+1} \circ \Sigma^2 \alpha, \Sigma \alpha \circ \eta_{m+1}\}) \circ \Sigma p\),
(2) If \(\Sigma: \pi_{m+2}(\Sigma C_\alpha)/\{\Sigma i \circ \eta_{n+1} \circ \Sigma^2 \alpha\} \to \pi_{m+3}(\Sigma^2 C_\alpha)/\{\Sigma^2 i \circ \eta_{m+2} \circ \Sigma^3 \alpha\}\) is an isomorphism, then \(\Sigma: [\Sigma C_\alpha, \Sigma C_\alpha] \to [\Sigma^2 C_\alpha, \Sigma^2 C_\alpha]\) is an isomorphism.

Proof. Consider the exact sequence

\[
\pi_{n+2}(\Sigma C_\alpha) \xrightarrow{\Sigma^2 \alpha^*} \pi_{m+2}(\Sigma C_\alpha) \xrightarrow{\Sigma p^*} [\Sigma C_\alpha, \Sigma C_\alpha] \\
\xrightarrow{\Sigma i^*} \pi_{n+1}(\Sigma C_\alpha) \xrightarrow{\Sigma \alpha^*} \pi_{m+1}(\Sigma C_\alpha).
\]

By Lemmas 2.1 and 2.2, we have (1).

Next we consider the commutative diagram

\[
\begin{array}{ccc}
\pi_{n+2}(\Sigma C_\alpha) & \xrightarrow{\Sigma^2 \alpha^*} & \pi_{m+2}(\Sigma C_\alpha) \\
\downarrow \Sigma & & \downarrow \Sigma \\
\pi_{n+3}(\Sigma^2 C_\alpha) & \xrightarrow{\Sigma^3 \alpha^*} & \pi_{m+3}(\Sigma^2 C_\alpha)
\end{array}
\xrightarrow{\Sigma^2 i^*} \begin{array}{ccc}
\pi_{n+2}(\Sigma^2 C_\alpha) & \xrightarrow{\Sigma^2 \alpha^*} & \pi_{m+2}(\Sigma^2 C_\alpha) \\
\downarrow \Sigma & & \downarrow \Sigma \\
\pi_{n+1}(\Sigma C_\alpha) & \xrightarrow{\Sigma \alpha^*} & \pi_{m+1}(\Sigma C_\alpha)
\end{array}
\]

By Freudenthal’s suspension theorem, \(\Sigma: \pi_{n+i}(\Sigma C_\alpha) \to \pi_{n+i+1}(\Sigma^2 C_\alpha)\) is an isomorphism for \(i < n + 1\). Since \(\pi_{n+2}(\Sigma C_\alpha) \cong \mathbb{Z}_2\{\Sigma i \circ \eta_{n+1}\}\), we have (2). This completes the proof. \(\square\)

The following proposition is proved on p. 287 of [11] and is an unstable version of (2.2) of [4].

Proposition 2.5. \(t \Sigma \iota_{C_\alpha} \equiv \Sigma i \circ \tilde{t}_{n+1} + \tilde{t}_{m+1} \circ \Sigma p \mod \Sigma i \circ (\pi_{m+2}(S^{n+1})/\{\eta_{n+1} \circ \Sigma^2 \alpha, \Sigma \alpha \circ \eta_{m+1}\}) \circ \Sigma p\).
Proof. We consider the following commutative diagram

*\[ \pi_{n+2}(S^{n+1}) \xrightarrow{\Sigma \alpha^*} [\Sigma C_\alpha, S^{n+1}] \]

where the row and column sequences are exact. By chasing the diagram, we obtain the result. This completes the proof. \( \square \)

Consider the Hopf map \( h_d(F) : S^{2d-1} \to S^d \). By using the notation of \([13]\), we have the following in the 2-primary components:

*\[ h_n(R) = 2\nu_n (n \geq 1), \quad h_n(C) = \eta_n (n \geq 2), \]

*\[ h_n(H) = \nu_n (n \geq 4), \quad h_n(O) = \sigma_n (n \geq 8). \]

Let \( o(F) \in \mathbb{Z} \) be the order of the stable Hopf class \( h(F) = \Sigma^\infty h_d(F) \), i.e., \( o(F) = 2, 24 \) or \( 240 \) for \( F = C, H \) or \( O \), respectively. We apply Proposition 2.5 for \( \alpha = h_d(F) \). Then we have

**Corollary 2.6.**  
(1) \( 2\Sigma i_C = \Sigma i_C \circ \overline{2t}_3 + \overline{2t}_4 \circ \Sigma p_C \) on \([\Sigma CP^2, \Sigma CP^2]\),

(2) \( 24\Sigma i_H = \Sigma i_H \circ \overline{24t}_5 + 24t_8 \circ \Sigma p_H \) on \([\Sigma HP^2, \Sigma HP^2]\),

(3) \( 240\Sigma i_O \equiv \Sigma i_O \circ \overline{240t}_9 + \overline{240t}_{16} \circ \Sigma p_O \) mod \( \Sigma i_O \circ \epsilon_9 \circ \Sigma p_O \) on \([\Sigma OP^2, \Sigma OP^2]\), where \( \epsilon_9 \) is a generator of \( \pi_{17}(S^9) \).

**Proof.** By \([13]\), \( \pi_5(S^3) \cong \mathbb{Z}_2\{\eta_3^2\}, \pi_9(S^5) \cong \mathbb{Z}_2\{\nu_5 \circ \eta_8\}, \pi_{17}(S^9) \cong \mathbb{Z}_2\{\sigma_9 \circ \eta_{16}\} \oplus \mathbb{Z}_2\{\nu_9\} \oplus \mathbb{Z}_2\{\epsilon_9\} \) and \( \eta_9 \circ \sigma_{10} = \nu_9 + \epsilon_9 \). Apply Proposition 2.5 for \( \alpha = h_d(F) \). Then we can see that the assertion has established. \( \square \)

Remark that Corollary 2.6 (1) is obtained from Theorem 8.1 of \([1]\).

It is well known that

*\[ \Sigma \oplus h_d(F)_* : [\Sigma^{k-1} C_\alpha, S^{d-1}] \oplus [\Sigma^k C_\alpha, S^{2d-1}] \to [\Sigma^k C_\alpha, S^d] \]

is an isomorphism for all \( k \geq 1 \).

We recall some properties of Toda brackets \([13]\).
Proposition 2.7 ([13]). Consider elements $\alpha \in [Y, Z]$, $\beta \in [X, Y]$ and $\gamma \in [W, X]$ which satisfy $\alpha \circ \beta = 0$, $\beta \circ \gamma = 0$. Let $\{\alpha, \beta, \gamma\}$ be the Toda bracket, $i : Z \to Z \cup \alpha CY$ and $p : X \cup \gamma CW \to \Sigma W$ be the canonical maps. Then

1. $\overline{\alpha} \circ \overline{\gamma} \in \{\alpha, \beta, \gamma\}$,
2. $\alpha \circ \overline{\beta} \in \{\alpha, \beta, \gamma\} \circ p$,
3. $\overline{\beta} \circ \Sigma \gamma \in -i \circ \{\alpha, \beta, \gamma\}$.

3. Cohomotopy groups of $\Sigma^n \mathbb{C}P^2$

Let $\mathbb{C}P^2$ be the complex projective plane, i.e., $\mathbb{C}P^2 = S^2 \cup_{t_2} e^4$.

In this section, we compute the cohomotopy groups of the suspended complex projective plane $\Sigma^n \mathbb{C}P^2$. Our main tool is the following exact sequence

\[
\begin{array}{ccc}
\pi_{n+3}(S^k) & \xrightarrow{\eta_{n+3}} & \pi_{n+4}(S^k) \\
\Sigma^n i_c^* & \xrightarrow{\eta_{n+2}} & \pi_{n+3}(S^k)
\end{array}
\]

induced from the cofiber sequence

\[
S^{n+3} \xrightarrow{\eta_{n+2}} S^{n+2} \xrightarrow{\Sigma^n i_c} \Sigma^n \mathbb{C}P^2 \xrightarrow{\Sigma^n pc} S^{n+4} \xrightarrow{\eta_{n+3}} S^{n+3}.
\]

By Lemma 2.3, we have ([1])

\[
\begin{align*}
\left[\Sigma^n \mathbb{C}P^2, S^{n+4}\right] & \cong \mathbb{Z}\{\Sigma^n pc\}, \\
\left[\Sigma^n \mathbb{C}P^2, S^{n+3}\right] & = 0, \\
\left[\Sigma^n \mathbb{C}P^2, S^{n+2}\right] & \cong \mathbb{Z}\{2l_{n+2}\}
\end{align*}
\]

for $n \geq 1$.

Since $\eta_m \in \pi_{m+1}(S^m)$ is of order two for $m \geq 3$, we have in the $p$-primary components

\[
\left[\Sigma^n \mathbb{C}P^2, S^k\right]_{(p)} \cong \pi_{n+2}(S^k)_{(p)} \oplus \pi_{n+4}(S^k)_{(p)},
\]

where $p$ is an odd prime. We only compute the 2-primary components of the cohomotopy groups $[\Sigma^n \mathbb{C}P^2, S^k]$. The odd primary components are easily obtained by [13].

We see ([13]) that

\[
\eta_{n+2}^* : \pi_{n+2}(S^{n+1}) \to \pi_{n+3}(S^{n+1})
\]

is an isomorphism for $n \geq 2$. Hence we have

\[
[\Sigma^n \mathbb{C}P^2, S^{n+1}] \cong \text{Coker} \eta_{n+3}^*.
\]
where $\eta_{n+3}^*: \pi_{n+3}(S^{n+1}) \to \pi_{n+4}(S^{n+1})$, $\eta_{n+3}^*(\nu_{n+1}^2) = 4\nu_{n+1}$ for $n \geq 4$ by (5.5) of [13] and $\eta_3^3 = 2\nu'$ by (5.3) of [13]. From the exact sequence $(C; n, n + 1)$, we obtain

**Proposition 3.1.**

1. $[\Sigma \text{CP}^2, S^2] \cong \mathbb{Z}\{\eta_2 \circ \overline{\nu_3}\}$,
2. $[\Sigma^2 \text{CP}^2, S^3] \cong \mathbb{Z}_2\{\nu' \circ \Sigma^2 \rho C\} \oplus \mathbb{Z}_3$,
3. $[\Sigma^3 \text{CP}^2, S^4] \cong \mathbb{Z}\{\nu_4 \circ \Sigma^3 \rho C\} \oplus \mathbb{Z}_3\{\Sigma \nu' \circ \Sigma^3 \rho C\} \oplus \mathbb{Z}_3$,
4. $[\Sigma^n \text{CP}^2, S^{n+1}] \cong \mathbb{Z}_4\{\nu_{n+1} \circ \Sigma^n \rho C\} \oplus \mathbb{Z}_3$ for $n \geq 4$.

Consider the exact sequence $(C; n, n)$. We obtain that $[\Sigma^n \text{CP}^2, S^n] \cong \text{Coker} \eta_{n+3}$, where $\eta_{n+3}: \pi_{n+3}(S^n) \to \pi_{n+4}(S^n)$. Then we have

**Proposition 3.2.**

1. $[\Sigma^2 \text{CP}^2, S^2] \cong \mathbb{Z}_2\{\eta_2 \circ \nu' \circ \Sigma^2 \rho C\} \oplus \mathbb{Z}_3$,
2. $[\Sigma^n \text{CP}^2, S^n] = 0$ for $n \geq 3$.

Let $g_6(C) : \Sigma^7 \text{CP}^2 \to S^6$ be the $S^1$-transfer map ([8]). This is the adjoint of the composite

$$\Sigma \text{CP}^2 \leftarrow SU(3) \leftarrow SO(6) \leftarrow \Omega^6 S^6$$

of the canonical maps. We set $g_{n+6}(C) = \Sigma^n g_6(C)$ for $n \geq 1$.

**Proposition 3.3.**

1. $[\Sigma^3 \text{CP}^2, S^2] = 0$,
2. $[\Sigma^4 \text{CP}^2, S^3] \cong \mathbb{Z}_2\{\nu' \circ \overline{\nu_6}\} \oplus \mathbb{Z}_3$,
3. $[\Sigma^5 \text{CP}^2, S^4] \cong \mathbb{Z}\{\nu_4 \circ \overline{\nu_6}\} \oplus \mathbb{Z}_2\{\Sigma \nu' \circ \overline{\nu_6}\} \oplus \mathbb{Z}_3$,
4. $[\Sigma^6 \text{CP}^2, S^5] \cong \mathbb{Z}_4\{\nu_5 \circ \overline{\nu_6}\} \oplus \mathbb{Z}_3$,
5. $[\Sigma^7 \text{CP}^2, S^6] \cong \mathbb{Z}\{g_6(C)\} \oplus \mathbb{Z}_4\{\nu_6 \circ \overline{\nu_6}\} \oplus \mathbb{Z}_3$ and $2g_6(C) = [\nu_6, \nu_6] \circ \overline{\nu_6} + \nu_6 \circ \overline{\nu_6}$,
6. $[\Sigma^{n+1} \text{CP}^2, S^n] \cong \mathbb{Z}_8\{g_n(C)\} \oplus \mathbb{Z}_3$ and $2g_n(C) = \nu_n \circ \overline{\nu_{n+3}}$ for $n \geq 7$.

**Proof.** Making use of the exact sequence $(C; n+1, n)$, we easily obtain that

$[\Sigma^{n+1} \text{CP}^2, S^n] \cong \text{Ker} \eta_{n+3}$

except for $n = 6$, where $\eta_{n+3}^*: \pi_{n+3}(S^n) \to \pi_{n+4}(S^n)$. We shall only prove (5) and (6). Consider the EHP-exact sequence

$$[\Sigma^8 \text{CP}^2, S^{11}] \xrightarrow{\Delta} [\Sigma^6 \text{CP}^2, S^5] \xrightarrow{\Sigma} [\Sigma^7 \text{CP}^2, S^6] \xrightarrow{H} [\Sigma^7 \text{CP}^2, S^{11}] \xrightarrow{\Delta} [\Sigma^5 \text{CP}^2, S^5]$$

induced from the 2-local EHP fibration $S^5 \xrightarrow{\Sigma} \Omega S^6 \xrightarrow{H} \Omega S^{11}$.

Since $[\Sigma^8 \text{CP}^2, S^{11}] = [\Sigma^5 \text{CP}^2, S^5] = 0$ and $[\Sigma^7 \text{CP}^2, S^{11}] \cong \mathbb{Z}\{\Sigma^7 \rho C\}$, we have the split short exact sequence

$$0 \to [\Sigma^6 \text{CP}^2, S^5] \xrightarrow{\Sigma} [\Sigma^7 \text{CP}^2, S^6] \xrightarrow{H} [\Sigma^7 \text{CP}^2, S^{11}] \to 0.$$

By (10) of [8], we have $H(g_6(C)) = \pm \Sigma^7 \rho C$ and $2g_6(C) = [\nu_6, \nu_6] \circ \Sigma^7 \rho C + \nu_6 \circ \overline{\nu_6}$. It follows that $2g_n(C) = \nu_n \circ \overline{\nu_{n+3}}$ for $n \geq 7$. This completes the proof. \qed
Remark that the order of $g_n(C)$ is 24 for $n \geq 7$ by Theorem 7.28 of [6].

**Proposition 3.4.**

1. $[\Sigma^4\mathbb{C}P^2, S^2] \cong Z_2\{\nu_2 \circ \nu' \circ 2\iota_6\} \oplus Z_3,$
2. $[\Sigma^5\mathbb{C}P^2, S^3] \cong Z_3,$
3. $[\Sigma^6\mathbb{C}P^2, S^4] \cong Z_4\{\nu_2^2 \circ \Sigma^6\rho_C\} \oplus Z_3 \oplus Z_3,$
4. $[\Sigma^n\mathbb{C}P^2, S^n] \cong Z_2\{\nu_2^2 \circ \Sigma^{n+2}\rho_C\}$ for $n \geq 5$.

**Proof.** Making use of the exact sequence $(C; n + 2, n)$, we easily obtain that

$$[\Sigma^{n+2}\mathbb{C}P^2, S^n] \cong \text{Coker } \eta_{n+5}^*$$

for $n \geq 3$, where $\eta_{n+5}^* : \pi_{n+5}(S^n) \to \pi_{n+6}(S^n)$. For $n = 6$, we have $\Delta(\iota_{13}) \circ \eta_{11} = 0$ by (5.13) of [13]. And $\eta_{n+5}^* : \pi_{n+5}(S^n) \to \pi_{n+6}(S^n)$ is trivial for $n \geq 5$. \hfill $\square$

**Proposition 3.5.**

1. $[\Sigma^5\mathbb{C}P^2, S^2] \cong Z_3,$
2. $[\Sigma^6\mathbb{C}P^2, S^3] \cong Z_2\{\nu_2 \circ \nu_5^2\} \oplus Z_5,$
3. $[\Sigma^7\mathbb{C}P^2, S^4] \cong Z_2\{\nu_2 \circ \nu_5^2\} \oplus Z_5,$
4. $[\Sigma^8\mathbb{C}P^2, S^5] \cong Z_4\{\nu_2 \circ \nu_5^2\} \oplus Z_5,$
5. $[\Sigma^9\mathbb{C}P^2, S^6] \cong Z_6\{\nu_2 \circ \nu_5^2\} \oplus Z_5,$
6. $[\Sigma^{10}\mathbb{C}P^2, S^7] \cong Z_6\{\nu_2 \circ \nu_5^2\} \oplus Z_5,$
7. $[\Sigma^{11}\mathbb{C}P^2, S^8] \cong Z_6\{\nu_2 \circ \nu_5^2\} \oplus Z_5,$
8. $[\Sigma^{n+3}\mathbb{C}P^2, S^n] \cong Z_6\{\nu_2 \circ \nu_5^2\} \oplus Z_5$ for $n \geq 9$.

We have a relation: $2\nu_5^2\eta_8^3 = \sigma'' \circ \Sigma^8\rho_C$.

**Proof.** We only prove (4). The rest can be easily obtained by making use of the exact sequence $(C; n + 3, n)$ and the fact $\nu_n \circ \eta_{n+3} = 0$ for $n \geq 6$.

Consider the exact sequence $(C; 8, 5)$:

$$\pi_{11}(S^5) \xrightarrow{\eta_{11}^*} \pi_{12}(S^5) \xrightarrow{\Sigma^8\rho_C^*} [\Sigma^8\mathbb{C}P^2, S^5] \xrightarrow{\Sigma^8\rho_C^*} \pi_{10}(S^5) \xrightarrow{\eta_{10}^*} \pi_{11}(S^5),$$

where $\pi_{11}(S^5) \cong Z_2\{\nu_5^2\}$, $\pi_{12}(S^5) \cong Z_2\{\sigma''\} \oplus Z_5$, $\pi_{10}(S^5) \cong Z_2\{\nu_5\eta_8^2\}$ and $\nu_5^2 \circ \eta_{11} = 0 = \nu_5 \circ \eta_3^3$ by [13]. From Corollary 2.6 (1), we see that

$$2\nu_5^2\eta_8^3 = \nu_5\eta_8^2 \circ 2\Sigma^8\iota_C$$
$$= \nu_5\eta_8^2 \circ 2\iota_{10} + \nu_5\eta_8^2 \circ 2\iota_{11} \circ \Sigma^8\rho_C.$$

By Proposition 2.7 (2),

$$\nu_5\eta_8^2 \circ 2\iota_{10} \in \{\nu_5\eta_8^2, 2\iota_{10}, \eta_{10}\} \circ \Sigma^8\rho_C \subset \{\nu_5, 2\eta_8^2, \eta_{10}\} \circ \Sigma^8\rho_C = 0.$$
and by Proposition 2.7 (1),
\[ \nu^2 \eta^2_{S^5} \circ 2\iota_{11} \in \{ \nu^2 \eta^2_{S^5}, \eta_{10}, 2\iota_{11} \} \]
\[ \subset \{ \nu_5, \eta_{S^5}, 2\iota_{11} \} \]
\[ = \{ \nu_5, 4\nu_8, 2\iota_{11} \} \]
\[ \supset \{ \nu_5, 2\nu_8, 4\iota_{11} \} \in \sigma'''. \]
Thus we obtain that \( 2\nu^2 \eta^2_{S^5} = \sigma''' \circ \Sigma^8 p_C \). From the above exact sequence, we have (4). This completes the proof. \( \square \)

**Proposition 3.6.**

1. \( [\Sigma^6 \mathbb{C}P^2, S^2] \cong \mathbb{Z}_2 \{ \eta_2 \circ \nu^2 \eta^2_{S^2} \} \oplus \mathbb{Z}_{15}, \)
2. \( [\Sigma^7 \mathbb{C}P^2, S^3] \cong \mathbb{Z}_2 \{ \epsilon_3 \circ \Sigma^7 p_C \} \oplus \mathbb{Z}_3, \)
3. \( [\Sigma^8 \mathbb{C}P^2, S^4] \cong \mathbb{Z}_8 \{ \nu_4 \circ g_7(C) \} \oplus \mathbb{Z}_2 \{ \epsilon_4 \circ \Sigma^8 p_C \} \oplus \mathbb{Z}_3 \oplus \mathbb{Z}_3, \)
4. \( [\Sigma^9 \mathbb{C}P^2, S^5] \cong \mathbb{Z}_4 \{ \nu_5 \circ g_8(C) \}, \)
5. \( [\Sigma^{10} \mathbb{C}P^2, S^6] \cong \mathbb{Z}_4 \{ \nu_6 \circ g_9(C) \} \oplus \mathbb{Z}_4 \{ \nu_6 \circ \Sigma^{10} p_C \} \oplus \mathbb{Z}_3, \)
6. \( [\Sigma^{n+4} \mathbb{C}P^2, S^n] \cong \mathbb{Z}_4 \{ \nu_n \circ g_{n+3}(C) \} \oplus \mathbb{Z}_2 \{ \nu_n \circ \Sigma^{n+4} p_C \} \) for \( n = 7, 8 \) and 9,
7. \( [\Sigma^{n+4} \mathbb{C}P^2, S^n] \cong \mathbb{Z}_4 \{ \nu_n \circ g_{n+3}(C) \} \) for \( n \geq 10. \)

We have a relation: \( 2(\nu_5 \circ g_{n+3}(C)) = \epsilon_n \circ \Sigma^{n+4} p_C \) for \( n \geq 5. \)

**Proof.** We only show that \( [\Sigma^{n+4} \mathbb{C}P^2, S^n] \) for \( n \geq 5 \) contains a direct summand isomorphic to \( \mathbb{Z}_4 \). Consider the exact sequence (C; 9, 5):

\[ \pi_{12}(S^5) \xrightarrow{n_{12}^*} \pi_{13}(S^5) \xrightarrow{\Sigma^9 p_C^*} [\Sigma^9 \mathbb{C}P^2, S^5] \xrightarrow{\xi_{11}(S^5)} \pi_{11}(S^5) \xrightarrow{\eta_{11}^*} \pi_{12}(S^5), \]

where \( \pi_{11}(S^5) \cong \mathbb{Z}_2 \{ \nu_2^2 \}, \pi_{12}(S^5) \cong \mathbb{Z}_2 \{ \sigma''' \} \oplus \mathbb{Z}_{15}, \pi_{13}(S^5) \cong \mathbb{Z}_2 \{ \epsilon_5 \} \) and \( \nu_2^2 \circ \eta_{11} = 0 = \sigma''' \circ \eta_{12} \) by [13]. By (7.6) of [13] and Propositions 3.3 (6) and 2.7 (2),

\[ 2(\nu_5 \circ g_8(C)) = \nu_5 \circ 2g_8(C) \]
\[ = \nu_5 \circ \nu_8 \circ 2\iota_{11} \]
\[ = \nu_5^2 \circ 2\iota_{11} \]
\[ \in \{ \nu_5^2, 2\iota_{11}, \eta_{11} \} \circ \Sigma^9 p_C \]
\[ \ni \epsilon_5 \circ \Sigma^9 p_C \mod 0. \]

It follows that \( 2(\nu_5 \circ g_8(C)) = \epsilon_5 \circ \Sigma^9 p_C \). For \( n \geq 5 \), we see that

\[ 2(\nu_n \circ g_{n+3}(C)) = \epsilon_n \circ \Sigma^{n+4} p_C \]

and the kernel of \( \eta_{n+6}^* : \pi_{n+6}(S^n) \rightarrow \pi_{n+7}(S^n) \) is generated by \( \nu_2^2_n \). This completes the proof. \( \square \)

**Proposition 3.7.**

1. \( [\Sigma^7 \mathbb{C}P^2, S^2] \cong \mathbb{Z}_2 \{ \eta_2 \circ \epsilon_3 \circ \Sigma^7 p_C \} \oplus \mathbb{Z}_3, \)
2. \( [\Sigma^8 \mathbb{C}P^2, S^3] \cong \mathbb{Z}_2 \{ \mu_3 \circ \Sigma^8 p_C \} \oplus \mathbb{Z}_{15}, \)
Since $\Sigma$:

By Lemma 6.5 of [13] and Proposition 2.7 (2),

Thus we conclude that

We have relations: $2\bar{\sigma}'' = \mu_5 \circ \Sigma^{10} p_C$, $2\sigma'' \circ \bar{2\ell}_{13} = \mu_6 \circ \Sigma^{11} p_C$, $4\sigma' \circ \bar{2\ell}_{14} = \mu_7 \circ \Sigma^{12} p_C$ and $8\sigma_n \circ \bar{2\ell}_{n+7} = \mu_n \circ \Sigma^{n+5} p_C$ for $n \geq 9$.

Proof. We only prove (4). From the exact sequence (C; 10, 5) and the fact that $\sigma'' \circ \eta_{12} = 0$ ([13]), we have the exact sequence

By Corollary 2.6 (1),

By Lemma 6.5 of [13] and Proposition 2.7 (2),

and by (7.4) of [13] and Proposition 2.7 (1),

Since $\Sigma : \pi_{14}(S^5) \to \pi_{15}(S^6)$ is a monomorphism, we have $\bar{\sigma}'' \circ \bar{2\ell}_{13} = 0$. Thus we conclude that

and $2\bar{\sigma}'' = \mu_5 \circ \Sigma^{10} p_C$.

From the fact that $2\sigma'' = \Sigma \sigma''$, $2\sigma' = \Sigma \sigma''$ and $2\sigma_9 = \Sigma^2 \sigma'$, we have $\sigma'' \circ \bar{2\ell}_{13} = \Sigma \bar{\sigma}''$, $2\sigma' \circ \bar{2\ell}_{14} = \Sigma \sigma'' \circ \bar{2\ell}_{15}$ and $2\sigma_9 \circ \bar{2\ell}_{16} = \Sigma^2(\sigma' \circ \bar{2\ell}_{14})$. This leads to the relations and completes the proof. □
4. COHOMOTOPY GROUPS OF SUSPENDED PROJECTIVE PLANES

Let $\mathbb{H}P^2$ be the quaternionic projective plane, i.e., $\mathbb{H}P^2 = S^4 \cup_{h_4(\mathbb{H})} e^8$. In this section, we compute the cohomotopy groups of the suspended quaternionic projective plane $\Sigma^n\mathbb{H}P^2$. Consider the exact sequence

$$
\pi_{n+5}(S^k) \xrightarrow{h_{n+5}(\mathbb{H})^*} \pi_{n+8}(S^k) \xrightarrow{\Sigma^n p_{\mathbb{H}}^*} [\Sigma^n \mathbb{H}P^2, S^k] \xrightarrow{\Sigma^n i_{\mathbb{H}}^*} \pi_{n+4}(S^k) \xrightarrow{h_{n+4}(\mathbb{H})^*} \pi_{n+7}(S^k)
$$

induced from the cofiber sequence

$$
S^{n+7} \xrightarrow{h_{n+4}(\mathbb{H})} S^{n+2} \xrightarrow{\Sigma^n i_{\mathbb{H}}} \Sigma^n \mathbb{H}P^2 \xrightarrow{\Sigma^n p_{\mathbb{H}}} S^{n+8} \xrightarrow{h_{n+5}(\mathbb{H})} S^{n+5}.
$$

For $p \geq 5$, we have in the $p$-primary components

$$
[\Sigma^n \mathbb{H}P^2, S^k]_{(p)} \cong \pi_{n+4}(S^k)_{(p)} \oplus \pi_{n+8}(S^k)_{(p)}
$$

since $h_n(\mathbb{H}) (n \geq 5)$ is of order 24. By Lemma 2.3 and the fact that $\eta_n \circ \nu_{n+1} = 0$ for $n \geq 5$, we obtain that

1. $[\Sigma \mathbb{H}P^2, S^5] \cong \mathbb{Z}_2 \{\nu_5 \circ \eta_8 \circ p_{\mathbb{H}}\} \oplus \mathbb{Z}\{24\eta_5\}$,
2. $[\Sigma^{-4} \mathbb{H}P^2, S^n] \cong \mathbb{Z}\{24\eta_{n+4}\}$ for $n \geq 6$.

Since $\nu' \circ \nu_6 = 0$, there exists an extension $\tilde{\nu'} \in [\Sigma^2 \mathbb{H}P^2, S^3]$ of $\nu' \in \pi_6(S^3)$. By (5.3) of [13], we have the relation $H(\nu') = \eta_5$. We set $\tilde{\eta}_5 = H(\tilde{\nu'})$, where $H : [\Sigma^2 \mathbb{H}P^2, S^3] \to [\Sigma^2 \mathbb{H}P^2, S^3]$ is the generalized Hopf homomorphism and we also set $\tilde{\eta}_n = \Sigma^{-5} \tilde{\eta}_5$ for $n \geq 5$.

**Proposition 4.1.**

1. $[\Sigma \mathbb{H}P^2, S^4] \cong \mathbb{Z}_2 \{\nu_4 \circ \eta_7 \circ p_{\mathbb{H}}\} \oplus \mathbb{Z}_2 \{\nu' \circ \eta_7 \circ p_{\mathbb{H}}\}$,
2. $[\Sigma^2 \mathbb{H}P^2, S^5] \cong \mathbb{Z}_2 \{\nu_5 \circ \eta_8 \circ p_{\mathbb{H}}\} \oplus \mathbb{Z}_2 \{\tilde{\eta}_5\}$,
3. $[\Sigma^3 \mathbb{H}P^2, S^6] \cong \mathbb{Z}\{\Delta(\iota_{13}) \circ \Sigma^3 p_{\mathbb{H}}\} \oplus \mathbb{Z}_2 \{\eta_6\}$,
4. $[\Sigma^{-3} \mathbb{H}P^2, S^n] \cong \mathbb{Z}_2 \{\tilde{\eta}_n\}$ for $n \geq 7$.

**Proof.** We only prove (2). From the exact sequence $(\mathbb{H}; 2, 5)$ and the fact that $\eta_5 \circ \nu_6 = 0$, we have the exact sequence

$$
0 \to \pi_{10}(S^5) \xrightarrow{\Sigma^2 p_{\mathbb{H}}^*} [\Sigma^2 \mathbb{H}P^2, S^5] \xrightarrow{\Sigma^2 i_{\mathbb{H}}^*} \pi_6(S^5) \cong \mathbb{Z}_2 \{\eta_5\} \to 0.
$$

Assume that $2\tilde{\eta}_5 = \nu_5 \circ \eta_8 \circ p_{\mathbb{H}}$. By Lemma 5.7 of [13] and Lemma 2 of [5], we have

$$
0 = \Delta(2H(\tilde{\nu'})) = \Delta(\nu_5 \circ \eta_8 \circ p_{\mathbb{H}}) = \eta_2 \circ \nu' \circ \eta_6 \circ p_{\mathbb{H}} \neq 0.
$$

This is a contradiction. It follows that the above sequence splits. This completes the proof. \[\square\]
Proposition 4.2. \(1\) \([\Sigma \text{HP}^2, S^3] \cong Z_2\{\eta_3^2\}\), 
\(2\) \([\Sigma^{n-2}\text{HP}^2, S^n] \cong Z_2\{\eta_n \circ \eta_{n+1}\}\) for \(n \geq 4\).

Proof. For \(n \geq 3\), \(h_{n+3}(H)^* : \pi_{n+3}(S^n) \to \pi_{n+6}(S^n)\) is an epimorphism. From the exact sequence \((H; n-2, n)\) and the fact that \(\eta_n^2 \circ \nu_{n+2} = 0\) for \(n \geq 3\), 
\[\Sigma^{n-2}i_H^* : [\Sigma^{n-2}\text{HP}^2, S^n] \to \text{Ker} \nu_{n+2}^* \cong Z_2\{\eta_n^2\}\]
is an isomorphism. By the definition of \(\eta_n\), \(\eta_n \circ \eta_{n+1}\) is an extension of \(\eta_n^2\) for \(n \geq 4\). This completes the proof. \(\square\)

Proposition 4.3. \(1\) \([\Sigma \text{HP}^2, S^2] \cong Z_2\{\eta_2 \circ \eta_3^2\}\), 
\(2\) \([\Sigma^2\text{HP}^2, S^3] \cong Z_4\{\nu\} \oplus Z_3\{\alpha_2(3) \circ \Sigma^2p_H\} \oplus Z_5\), 
\(3\) \([\Sigma^3\text{HP}^2, S^4] \cong Z_4\{\Sigma^3\nu\} \oplus Z_4\nu \circ 24\text{l}_7 \oplus Z_3\{\alpha_2(4) \circ \Sigma^3p_H\} \oplus Z_5\), 
\(4\) \([\Sigma^4\text{HP}^2, S^5] \cong Z_4\{\Sigma^4\nu\} \oplus Z_2\{\sigma'' \circ \Sigma^4p_H\} \oplus Z_9\{\alpha_1(5)\} \oplus Z_5\), 
\(5\) \([\Sigma^5\text{HP}^2, S^6] \cong Z_4\{\Sigma^5\nu\} \oplus Z_4\{\sigma'' \circ \Sigma^5p_H\} \oplus Z_9\{\alpha_1(6)\} \oplus Z_5\), 
\(6\) \([\Sigma^6\text{HP}^2, S^7] \cong Z_4\{\Sigma^6\nu\} \oplus Z_8\{\sigma' \circ \Sigma^6p_H\} \oplus Z_9\{\alpha_1(7)\} \oplus Z_5\), 
\(7\) \([\Sigma^7\text{HP}^2, S^8] \cong Z_4\{\Sigma^7\nu\} \oplus Z_8\{\Sigma \sigma' \circ \Sigma^7p_H\} \oplus Z_9\{\alpha_1(8)\} \oplus Z_5\) for \(n \geq 9\).

Proof. \((1)\), \((2)\) and \((3)\) are easily obtained. Consider the exact sequence \((H; n-1, n)\) for \(n \geq 5\). Then the kernel of \(h_{n+3}(H)^* : \pi_{n+3}(S^n) \to \pi_{n+6}(S^n)\) is isomorphic to \(Z_4\{2\nu_n\} \oplus Z_3\{\alpha_1(n)\}\), where \(2\nu_n = \Sigma^{n-3}\nu\) for \(n \geq 5\).

Consider the exact sequence \((H; 4, 5)\):
\[\pi_9(S^5) \xrightarrow{h_9(H)^*} \pi_{12}(S^5) \xrightarrow{\Sigma^4p_H^*} [\Sigma^4\text{HP}^2, S^5] \xrightarrow{\Sigma^4i_H^*} \pi_8(S^5) \xrightarrow{h_8(H)^*} \pi_{11}(S^5)\]
where \(\pi_9(S^5) \cong Z_2\{\nu_5 \circ \eta_8\}\), \(\nu_5 \circ \eta_8 \circ \nu_9 = 0\), \(\pi_{12}(S^5) \cong Z_2\{\sigma''\} \oplus Z_3\{\alpha_2(5)\} \oplus Z_5\), \(\pi_8(S^5) \cong Z_8\{\nu_5\} \oplus Z_3\{\alpha_1(5)\}\) and \(\pi_{11}(S^5) \cong Z_2\{\nu_5^2\}\) by \([13]\).

Since \(\nu\) is of order 4, we have the results for the 2-primary components.

Consider the 3-primary components. We have \(\alpha_1(5)^2 = 0\) by \((13.7)\) of \([13]\). By Corollary 2.6 \((2)\),
\[3\alpha_1(5) = \alpha_1(5) \circ 24\Sigma^4d_H\]
\[= \alpha_1(5) \circ \Sigma^4i_H \circ 24\text{l}_8 + \alpha_1(5) \circ 24\text{l}_{11} \circ \Sigma^4p_H\]
\[= \alpha_1(5) \circ 24\text{l}_8 + \alpha_1(5) \circ 24\text{l}_{11} \circ \Sigma^4p_H.\]

By the definition of \(\alpha_2(5)\) and Proposition 2.7 \((2)\), we obtain \(\alpha_1(5) \circ 24\text{l}_8 \subseteq \{\alpha_1(5), 24\text{l}_8, \alpha_1(8)\} \circ \Sigma^4p_H \ni \alpha_2(5) \circ \Sigma^4p_H\) and by \((13.8)\) of \([13]\) and Proposition 2.7 \((1)\),
\[\alpha_1(5) \circ 24\text{l}_{11} \subseteq \{\alpha_1(5), \alpha_1(8), 24\text{l}_{11}\} \ni 1/2\alpha_2(5).\]
It follows that $3\alpha_1(5) = \alpha_2(5) \circ \Sigma^4p_H$ and $3\alpha_1(n) = \alpha_2(n) \circ \Sigma^{n-1}p_H$ for 
$n \geq 5$. This completes the proof. □

**Proposition 4.4.**  
(1) $[\Sigma^2H^2P^2, S^2] \cong Z_4\{\eta_2 \circ \nu\} \oplus Z_{15}$,  
(2) $[\Sigma^3H^2P^2, S^3] \cong Z_2\{\nu' \circ \eta_6\} \oplus Z_2\{\epsilon_3 \circ \Sigma^3p_H\}$,  
(3) $[\Sigma^4H^2P^2, S^4] \cong Z_2\{\nu_4 \circ \eta_7\} \oplus Z_2\{\Sigma^2 \nu' \circ \eta_7\} \oplus Z_2\{\epsilon_4 \circ \Sigma^4p_H\}$,  
(4) $[\Sigma^5H^2P^2, S^5] \cong Z_2\{\nu_5 \circ \eta_8\} \oplus Z_2\{\epsilon_5 \circ \Sigma^5p_H\}$,  
(5) $[\Sigma^6H^2P^2, S^6] \cong Z_2\{\nu_6 \circ \Sigma^6p_H\} \oplus Z_2\{\epsilon_6 \circ \Sigma^6p_H\}$,  
(6) $[\Sigma^nH^2P^2, S^n] \cong (\Sigma^n p_H)^* \pi_{n+8}(S^n)$ for $n \geq 7$.

**Proof.** Since $\eta_2 : [\Sigma^2H^2P^2, S^3] \rightarrow [\Sigma^2H^2P^2, S^2]$ is an isomorphism, we have (1).

From the exact sequence $(H; n, n)$ and the fact which $\eta_n$ is of order two for $n \geq 5$, we obtain (2), (3) and (4).

Consider the homomorphism $h_{11}(H)^* : \pi_{11}(S^6) \rightarrow \pi_{14}(S^6)$, where $\pi_{11}(S^6) \cong Z\{\Delta(\nu_{13})\}$ and $\pi_{14}(S^6) \cong Z_8\{\nu_6\} \oplus Z_2\{\epsilon_6\} \oplus Z_3\{[\nu_6, \nu_6] \circ \alpha_1(11)\}$. By Lemma 6.2 of [13],

$$h_{11}(H)^*(\Delta(\nu_{13})) = \Delta(\nu_{13}) \circ h_{11}(H) = 2\nu_6 + [\nu_6, \nu_6] \circ \alpha_1(11).$$

From the fact that $\pi_{n+4}(S^n) = 0$ for $n \geq 6$, we have (5). Also, from the fact $\pi_{n+5}(S^n) = 0$ for $n \geq 7$, we have (6). □

The following proposition is easily obtain by making use of the exact sequence $(H; n+1, n)$.

**Proposition 4.5.**  
(1) $[\Sigma^2H^2P^2, S^2] \cong Z_4\{\eta_2 \circ \nu' \circ \eta_6\} \oplus Z_2\{\eta_2 \circ \epsilon_3 \circ \Sigma^3p_H\}$,  
(2) $[\Sigma^3H^2P^2, S^3] \cong Z_2\{\nu' \circ \eta_6 \circ \eta_7\} \oplus Z_2\{\mu_3 \circ \Sigma^4p_H\} \oplus Z_2\{\eta_3 \circ \epsilon_4 \circ \Sigma^4p_H\}$,  
(3) $[\Sigma^4H^2P^2, S^4] \cong Z_2\{\nu_4 \circ \eta_7 \circ \eta_8\} \oplus Z_2\{\Sigma' \nu' \circ \eta_8\} \oplus Z_2\{\mu_4 \circ \Sigma^5p_H\} \oplus Z_2\{\epsilon_5 \circ \Sigma^5p_H\}$,  
(4) $[\Sigma^5H^2P^2, S^5] \cong Z_2\{\nu_5 \circ \eta_8 \circ \eta_9\} \oplus Z_2\{\mu_5 \circ \Sigma^6p_H\} \oplus Z_2\{\mu_5 \circ \Sigma^6p_H\} \oplus Z_2\{\eta_5 \circ \epsilon_6 \circ \Sigma^6p_H\}$,  
(5) $[\Sigma^6H^2P^2, S^6] \cong Z_2\{\epsilon_6 \circ \Sigma^6p_H\} \oplus Z_2\{\eta_6 \circ \Sigma^7p_H\} \oplus Z_2\{\epsilon_7 \circ \Sigma^7p_H\}$,  
(6) $[\Sigma^{n+1}H^2P^2, S^n] \cong \text{Coker} \nu_{n+6}^*$, where $\nu_{n+6}^* : \pi_{n+6}(S^n) \rightarrow \pi_{n+9}(S^n)$ for $n \geq 7$.

We show

**Proposition 4.6.**  
(1) $[\Sigma^2H^2P^2, S^2] \cong Z_2\{\eta_2 \circ \nu' \circ \eta_6 \circ \eta_7\} \oplus Z_2\{\eta_2 \circ \epsilon_4 \circ \Sigma^4p_H\}$,  
(2) $[\Sigma^3H^2P^2, S^3] \cong Z_4\{\epsilon' \circ \Sigma^5p_H\} \oplus Z_2\{\eta_3 \circ \mu_4 \circ \Sigma^5p_H\} \oplus Z_3\{\alpha_1(3) \circ \alpha_1(6)\}$,  
(3) $[\Sigma^4H^2P^2, S^4] \cong Z_4\{\nu_4 \circ \Sigma^4\nu'\} \oplus Z_8\{\nu_4 \circ \sigma' \circ \Sigma^5p_H\} \oplus Z_4\{\epsilon_4 \circ \Sigma^5p_H\}$,  
(4) $[\Sigma^5H^2P^2, S^5] \cong Z_4\{\nu_5 \circ \sigma_8 \circ \Sigma^7p_H\} \oplus Z_2\{\eta_5 \circ \mu_6 \circ \Sigma^7p_H\} \oplus Z_3\{\beta_1(5) \circ \Sigma^7p_H\}$,  
(5) $[\Sigma^6H^2P^2, S^6] \cong Z_2\{\nu_6 \circ \sigma_9 \circ \Sigma^8p_H\} \oplus Z_2\{\eta_6 \circ \mu_7 \circ \Sigma^8p_H\} \oplus Z_3\{\beta_1(6) \circ \Sigma^8p_H\}$. 

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(6) $[\Sigma^{n+2}H^2, S^n] \cong Z_2\{\eta_n \circ \mu_{n+1} \circ \Sigma^{n+2}p_H\} \oplus Z_3\{\beta_1(n) \circ \Sigma^{n+2}p_H\}$ for $n \geq 7$.

**Proof.** For $n \geq 5$, $h_{n+6}(H)^*: \pi_{n+6}(S^n) \rightarrow \pi_{n+9}(S^n)$ is monomorphic by [13]. It follows that

$$[\Sigma^{n+2}H^2, S^n] \cong \text{Coker } \nu_{n+7}^*: \pi_{n+7}(S^n) \rightarrow \pi_{n+10}(S^n).$$

By (7.19) of [13], we have $\sigma'' \circ \nu_1 = 4x\nu_5 \circ \sigma_8$, $\sigma'' \circ \nu_3 = 2x\nu_6 \circ \sigma_9$ and $\sigma' \circ \nu_4 = x\nu_7 \circ \sigma_{10}$ for $x$ odd. This completes the proof. \qed

We show

**Proposition 4.7.**

1. $[\Sigma^5H^2, S^2] \cong Z_4\{\eta_2 \circ \epsilon' \circ \Sigma^5p_H\} \oplus Z_2\{\eta_2 \circ \mu_4 \circ \Sigma^5p_H\} \oplus Z_3$,
2. $[\Sigma^6H^2, S^3] \cong Z_4\{\mu' \circ \Sigma^6p_H\} \oplus Z_2\{\nu' \circ \epsilon_6 \circ \Sigma^6p_H\} \oplus Z_3\{\alpha_3(3) \circ \Sigma^6p_H\} \oplus Z_{35}$,
3. $[\Sigma^7H^2, S^4] \cong Z_4\{\mu' \circ \Sigma^7p_H\} \oplus Z_2\{\nu' \circ \epsilon_7 \circ \Sigma^7p_H\} \oplus Z_2\{\nu_4 \circ \sigma' \circ \eta_4 \circ \Sigma^7p_H\} \oplus Z_2\{\nu_4 \circ \nu_7 \circ \Sigma^7p_H\} \oplus Z_2\{\nu_4 \circ \epsilon_7 \circ \Sigma^7p_H\} \oplus Z_3\{\alpha_3(3) \circ \Sigma^7p_H\} \oplus Z_{35}$,
4. $[\Sigma^8H^2, S^5] \cong Z_8\{\zeta_5 \circ \Sigma^8p_H\} \oplus Z_2\{\nu_5 \circ \nu_8 \circ \Sigma^8p_H\} \oplus Z_2\{\nu_5 \circ \epsilon_8 \circ \Sigma^8p_H\} \oplus Z_9\{\alpha_3(5) \circ \Sigma^8p_H\} \oplus Z_{35}$,
5. $[\Sigma^9H^2, S^6] \cong Z_8\{\zeta_6 \circ \Sigma^9p_H\} \oplus Z_9\{\alpha_3(6) \circ \Sigma^9p_H\} \oplus Z_{35}$,
6. $[\Sigma^{10}H^2, S^7] \cong Z_8\{\zeta_6 \circ \Sigma^{10}p_H\} \oplus Z_{27}\{\alpha_2(7)\} \oplus Z_{35}$,
7. $[\Sigma^{11}H^2, S^8] \cong Z\{\sigma_8 \circ \Sigma^{11}p_H\} \oplus Z\{\zeta_8 \circ \Sigma^{11}p_H\} \oplus Z_{27}\{\alpha_2(8)\} \oplus Z_{35}$,
8. $[\Sigma^{12}H^2, S^9] \cong Z_{16}\{\sigma_9 \circ \Sigma^{12}p_H\} \oplus Z_{27}\{\alpha_2(9)\} \oplus Z_{35}$,
9. $[\Sigma^{13}H^2, S^{10}] \cong Z_{32}\{3\sigma_{10}\} \oplus Z_{27}\{\alpha_2(10)\} \oplus Z_{35}$,
10. $[\Sigma^{14}H^2, S^{11}] \cong Z_{64}\{2\sigma_{11}\} \oplus Z_{27}\{\alpha_2(11)\} \oplus Z_{35}$,
11. $[\Sigma^{15}H^2, S^{12}] \cong Z\{\Delta(\nu_{25}) \circ \Sigma^{15}p_H\} \oplus Z_{128}\{\sigma_{12}\} \oplus Z_{27}\{\alpha_2(12)\} \oplus Z_{35}$,
12. $[\Sigma^{n+3}H^2, S^n] \cong Z_{128}\{\sigma_n\} \oplus Z_{27}\{\alpha_2(n)\} \oplus Z_{35}$ for $n \geq 13$.

**Proof.** We have the following table of the kernel of the homomorphism $h_{n+7}(H)^*: \pi_{n+7}(S^n) \rightarrow \pi_{n+10}(S^n)$ by [13],

<table>
<thead>
<tr>
<th>$n$</th>
<th>2</th>
<th>3 ≤ $n$ ≤ 6</th>
<th>7</th>
<th>8</th>
</tr>
</thead>
<tbody>
<tr>
<td>Kernel $\cong$ generator</td>
<td>$Z_3$</td>
<td>$Z_5$</td>
<td>$Z_3 + Z_5$</td>
<td>$Z + Z_3 + Z_5$</td>
</tr>
<tr>
<td>9</td>
<td>$Z_2 + Z_3 + Z_5$</td>
<td>$Z_4 + Z_3 + Z_5$</td>
<td>$Z_8 + Z_3 + Z_5$</td>
<td>$Z_{16} + Z_3 + Z_5$</td>
</tr>
<tr>
<td>10</td>
<td>$8\sigma_9, \alpha_2(9)$</td>
<td>$4\sigma_{10}, \alpha_2(10)$</td>
<td>$2\sigma_{11}, \alpha_2(11)$</td>
<td>$\sigma_n, \alpha_2(n)$</td>
</tr>
</tbody>
</table>
Consider an extension $\sigma_9 \circ 24\ell_{16} \in [\Sigma^{12}H\mathbb{P}^2, S^9]$ of $8\sigma_9$. By (9.2) of [13] and Proposition 2.7 (2), we have

$$2(\sigma_9 \circ 24\ell_{16}) = 2\sigma_9 \circ 24\ell_{16}$$

$$\in \{2\sigma_9, 8\ell_{16}, \nu_{16}\} \circ \Sigma^{12}p_H$$

$$\ni \zeta_9 \circ \Sigma^{12}p_H \text{ mod 0.}$$

It follows that $2(\sigma_9 \circ 24\ell_{16}) = \zeta_9 \circ \Sigma^{12}p_H$ and $[\Sigma^{12}H\mathbb{P}^2, S^9] \cong Z_{16}\{\sigma_9 \circ 24\ell_{16}\}$.

Since $8\sigma_9 \circ \nu_{16} = 4\sigma_{10} \circ \nu_{17} = 2\sigma_{11} \circ \nu_{18} = \sigma_{12} \circ \nu_{19} = 0$ by [13], there exist extensions

$$8\sigma_9 \in [\Sigma^{12}H\mathbb{P}^2, S^9],$$

$$\sigma_{12} \in [\Sigma^{15}H\mathbb{P}^2, S^{12}].$$

We set $\sigma_n = \Sigma^{n-12}\sigma_{12}$ for $n \geq 12$.

Since $\pi_{20}(S^9) \cong Z_8\{\zeta_9\} \oplus Z_2\{\nu_{17}\} \oplus Z_9 \oplus Z_7$ by [13] and $\nu_{17} \circ \Sigma^{12}p_H = 0$, we have

$$38\sigma_9 \equiv \sigma_9 \circ 24\ell_{16} \text{ mod } \zeta_9 \circ p_H.$$

So we obtain $28\sigma_9 = x\zeta_9 \circ \Sigma^{12}p_H$ for $x$ odd. By the similar argument, we obtain

$$44\sigma_{10} = z\zeta_{10} \circ \Sigma^{13}p_H, \quad 82\sigma_{11} = y\zeta_{11} \circ \Sigma^{14}p_H$$

and

$$16\sigma_{12} = w\zeta_{12} \circ \Sigma^{15}p_H,$$

where $z, y$ and $w$ are odd. This leads to (9), (10), (11) and (12) in the 2-primary components.

Consider the 3-primary components of $[\Sigma^{n+3}H\mathbb{P}^2, S^n]$ for $n \geq 7$. From the exact sequence $(H; n + 3, n)$ and $\alpha_2(n) \circ \alpha_1(n + 7) = 0$ ([13]), we have the exact sequence

$$0 \to Z_9\{\alpha_3(n)\} \to [\Sigma^{n+3}H\mathbb{P}^2, S^n]_3 \to Z_3\{\alpha_2(n)\} \to 0.$$

Since $\alpha_2(n) \circ \alpha_1(n + 7) = 0$, there exists an extension $\alpha_2(n) \in [\Sigma^{n+3}H\mathbb{P}^2, S^n]$ of $\alpha_2(n)$. By Corollary 2.5 (2) and Proposition 2.7,

$$3\alpha_2(n) = \alpha_2(n) \circ 24\ell_{n+7} + \alpha_2(n) \circ 24\ell_{n+11} \circ \Sigma^{n+3}p_H$$

$$\in \{\alpha_2(n), 3\ell_{n+7}, \alpha_1(n + 7)\} \circ \Sigma^{n+3}p_H$$

$$+ \{\alpha_2(n), \alpha_1(n + 7), 3\ell_{n+10}\} \circ \Sigma^{n+3}p_H.$$
Here, we recall $\alpha_3(n) \in \{\alpha_2(n), 3\pi_{n+7}, \alpha_1(n + 7)\}$, $\alpha_3'(n) \in \{\alpha_2(n), \alpha_1(n + 7), 3\pi_{n+11}\}$ and $3\alpha_3'(n) = \alpha_3(n)$ by [13]. Thus we have
\[
3\alpha_2(n) = \alpha_3'(n) \circ \Sigma^{n+3}p_H + \alpha_3(n) \circ \Sigma^{n+3}p_H
= 4\alpha_3(n) \circ \Sigma^{n+3}p_H.
\]
This completes the proof. \hfill \Box

Let $\text{ext}(\nu_{11}) \in [\Sigma^6\mathbb{O}P^2, S^{11}]$ be an extension of $\nu_{11}$. We set $\text{ext}(\nu) = \Sigma^\infty\text{ext}(\nu_{11}) \in \{\mathbb{O}P^2, S^5\}$.

**Example.** $120\text{ext}(\nu_{11}) = x\zeta_{11}\Sigma^6p_O$ for $x$ odd.

**Proof.** By Corollary 2.6 (3),
\[
o(\mathbb{O})\text{ext}(\nu_{11}) = \text{ext}(\nu_{11}) \circ o(\mathbb{O})\Sigma^6\iota_O
\equiv \nu_{11} \circ o(\mathbb{O})\iota_{14} + \text{ext}(\nu_{11}) \circ o(\mathbb{O})\iota_{21} \circ p_O
\pmod{\nu_{11} \circ \iota_{14} \circ \Sigma^6p_O = 0}.
\]
By Proposition 2.7 (2), we obtain
\[
\nu_{11} \circ o(\mathbb{O})\iota_{14} \in \{\nu_{11}, 16\iota_{14}, \sigma_{14}\} \circ \Sigma^6p_O
\supset \{\nu_{11}, 8\iota_{14}, 2\sigma_{14}\} \circ \Sigma^6p_O
\supset \zeta_{11} \circ \Sigma^6p_O \pmod{0}
\]
and
\[
\text{ext}(\nu_{11}) \circ o(\mathbb{O})\iota_{21} \in \{\nu_{11}, \sigma_{14}, 16\iota_{21}\} \ni \pm \zeta_{11} \pmod{0}.
\]
So we have $o(\mathbb{O})\text{ext}(\nu_{11}) = 0$ or $2\zeta_{11} \circ \Sigma^6p_O$. By Theorem 7.4 of [13], the order of $\zeta_n$ is 8 for $n \geq 5$ and $\pi_{n+11}(S^n)$ is generated by $\zeta_n$ and $\nu_n \circ \nu_{n+8}$ if $n \geq 6$ and $n \neq 12$. Therefore $\Sigma^{n-5}p_O^* : \pi_{n+11}(S^n) \rightarrow [\Sigma^{n-5}\mathbb{O}P^2, S^n]$ is a monomorphism if $n \geq 6$ and $n \neq 12$.

In the stable range, we have
\[
o(\mathbb{O})\text{ext}(\nu) = o(\mathbb{O})\iota \circ o(\nu) \in 2\langle 8\iota, \nu, \sigma \rangle \circ p_O = 2\zeta \circ p_O.
\]
This implies the relation $o(\mathbb{O})\text{ext}(\nu_{11}) = 2\zeta_{11} \circ \Sigma^6p_O$. This leads to the assertion. \hfill \Box

**Additional remark, added in proof.** In the proof of Proposition 4.3, we obtained the fact that
\[
3\alpha_1(5) = \alpha_2(5) \circ \Sigma^4p_H.
\]
We shall give another proof of the relation. By using the EHP-sequence and by the fact that $[\Sigma^6\mathbb{H}P^2, S^{11}] = 0$, $\Sigma : [\Sigma^3\mathbb{H}P^2, S^5] \rightarrow [\Sigma^5\mathbb{H}P^2, S^6]$ is a monomorphism. So we have $3\alpha_1(5) = 3\iota_5 \circ \alpha_1(5)$. And we see that
\[
3\iota_5 \circ \alpha_1(5) \in \{3\iota_5, \alpha_1(5), \alpha_1(8)\} \circ \Sigma^4p_H \pmod{0}.
\]
We know $\langle 3\iota, \alpha_1, \alpha_1 \rangle = \alpha_2$ in the stable range. This leads to the relation.

References


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