Math. J. Okayama Univ. **43**(2001), 95–103

ON THE EXTENSIONS OF SINGLE VALUED CONTINUOUS AND SET VALUED USC MAPS

Shôgo IKENAGA, Shin-ichi NITTA and Iwao YOSHIOKA

Abstract. We show that two main theorems: (1) A regular space Y has a complete sequence if and only if the set valued usco map to Y defined on every dense set D of any space X has an usco extension over a G_δ-set in X containing D. (2) A regular space Y with a G_δ-diagonal has a complete sequence if and only if the single valued continuous map to Y defined on every dense set D of any space X has a continuous extension over a G_δ-set in X containing D.

1. Introduction

In this paper, we state some results concerning the extension to a domain space or a G_δ-set of an upper semi continuous set valued map or a single valued continuous map defined on a dense set of a topological space.

It is well known that if the range space is compact Hausdorff (countably compact regular), then such an upper semicontinuous map has an upper semicontinuous extension to the domain space (the first countable domain space) [3], [6]. Moreover, if the range space is a Moore space having a complete sequence, such a continuous single valued map has a continuous extension to a G_δ-set in the domain space [4], [6], [8].

In this note, we do not assume any separation axioms unless otherwise stated. By a *set valued map* $F : X \to Y$ we mean that for each $x \in X$, $F(x)$ is a non-empty closed set in Y, and F is *upper semicontinuous* (*usc*) if, for every point $x \in X$ and every open set V with $F(x) \subset V$, there exists a neighborhood U of x such that $F(U) = \bigcup\{F(z) \mid z \in U\} \subset V$. Moreover, F is *usco* if F is usc and compact valued (i.e. $F(x)$ is compact for each $x \in X$). For a subset D of X and a set valued map $G : D \to Y$, F is an *extension* of G if $F(x) = G(x)$ for each $x \in D$.

The real line (resp. the set of natural numbers) with its usual topology is denoted by $\mathbf{R}$ (resp. $\mathbf{N}$). We refer the reader to [7] for undefined terms.

We will prove the following three main theorems. Below, for a space X, by $\chi(X)$ we mean the character of X.

Theorem 1.1. *For a regular space Y, the following conditions are equivalent for any cardinal number $\kappa \geq \aleph_0$:*

1991 *Mathematics Subject Classification.* 54C60, 54C20, 54D99.

Key words and phrases. Complete sequence, G_δ-diagonal, usco map, property (E), initially κ-compact, locally compact.

(1) *Y is initially κ-compact,*
(2) *For a space X with $\chi(X) \leq \kappa$, a dense set $D \subset X$ and an usc map $F : D \to Y$, there exists an usc extension $\widetilde{F} : X \to Y$ of F such that $\widetilde{F}(x)$ is initially κ-compact for every $x \in X$. Moreover, if F is minimal, then $\widetilde{F}$ is minimal.*

Theorem 1.2. *For a regular space Y, the following conditions are equivalent*:

(1) *Y has a complete sequence,*
(2) *Y is a G_δ-set in its Wallman compactification wY,*
(3) *For a space X, a dense set $D \subset X$ and an usco map $F : D \to Y$, there exist a G_δ-set G in X with $D \subset G$ and an usco extension $\widetilde{F} : G \to Y$ of F. Moreover if F is minimal, then $\widetilde{F}$ is minimal,*
(4) *For a space X, a dense set $D \subset X$ and a single valued continuous map $f : D \to Y$, there exist a G_δ-set G in X with $D \subset G$ and an usco extension $\tilde{f} : G \to Y$ of f such that $f(x) \in \tilde{f}(x)$ for every $x \in D$.*

Remark. In Theorem 1.1 (2) and Theorem 1.2 (3), the extension map $\widetilde{F}$ of F has the following property: If G is another extension map of F, then $\widetilde{F}(x) \subset G(x)$ for each element x of the domain space of $\widetilde{F}$ and G (see [6, Lemma 1]).

Definition 1.3 ([8])**.** A space X is said to have *property* (E) if for each space Z, a dense set $D \subset Z$ and a single valued continuous map $f : D \to X$, there exist a G_δ-set G in Z with $D \subset G$ and a single valued continuous extension $\tilde{f} : G \to X$ of f. In [6], such a space X is called an *EC-space*.

The next theorem slightly generalizes the results of T. M. Phillips [8, Corollary 1.1] and S. Levi [6, Theorem 6].

Theorem 1.4. *For a regular space Y with a G_δ-diagonal, the following conditions are equivalent*:

(1) *Y has a complete sequence,*
(2) *Y has the property (E).*

We will give the proof of Theorem 1.1 in section 2 while in section 3 we will prove theorems 1.2 and 1.4.

2. Proof of Theorem 1.1

Definition 2.1. For a cardinal number $\kappa \geq \aleph_0$, a topological space X is called *initially κ-compact* if every open cover $\mathcal{U}$, with $|\mathcal{U}| \leq \kappa$, has a finite subcover. It is well known that X is initially κ-compact if and only if every decreasing sequence of nonempty closed subsets of X, well ordered by inclusion as

$$F_0 \supset F_1 \supset \cdots \supset F_\alpha \supset \cdots, \quad \alpha < \delta,$$

where $\delta \leq \kappa$, has nonempty intersection [10, Theorem 2.2].

The next lemma can be easily proved.

Lemma 2.2. *Let X be Lindelöf and $F : X \to Y$ be an usc map such that $F(x)$ is Lindelöf for each $x \in X$ and $F(X) = Y$. Then Y is Lindelöf.*

Definition 2.3. An usc map $F : X \to Y$ is called *minimal* if whenever $H : X \to Y$ is an usc map with $H(x) \subset F(x)$ for every $x \in X$, then $H(x) = F(x)$ for every $x \in X$.

Example 2.4. There exists a minimal usc map which is not an usco map.

Let $X = \{0\} \cup \{1/n \mid n \geq 1\}$ and $Y = \mathbf{N}$ endowed with the subspace topology of $\mathbf{R}$. Then Y is a disjoint sum $\bigcup_{k \geq 1} M_k$ with $|M_k| = \aleph_0$ for every $k \in \mathbf{N}$. We define a set valued map $F : X \to Y$ by $F(x) = \{k\}$ if $1/x \in M_k$ and $F(0) = Y$. This usc map is minimal. Indeed, if $H : X \to Y$ is an usc map with $\mathrm{Gr}(H) \subsetneq \mathrm{Gr}(F)$, where $\mathrm{Gr}(F) = \bigcup\{\{x\} \times F(x) \mid x \in X\}$ is the graph of F, then $H(0) \subset Y \setminus \{m\}$ for some $m \in Y$. Hence, $H(W) \subset Y \setminus \{m\}$ for some neighbourhood W of 0. Consequently, $m = H(1/k) \subset Y \setminus \{m\}$, since $1/k \in W$ for some $k \in M_m$. $\square$

The next lemma is well known for an usco map [1].

Lemma 2.5. *Let Y be a regular space and $F : X \to Y$ be an usc map, then the following conditions are equivalent*:

(1) *F is minimal,*
(2) *For every open set U in X and every open set V in Y with $F(U) \cap V \neq \emptyset$, there exists a non-empty open set $W \subset U$ with $F(W) \subset V$.*

Proof. The implication (2) $\Rightarrow$ (1) is well known, therefore we sketch the proof of (1) $\Rightarrow$ (2). The restriction $G = F|U : U \to Y$ is minimal. If $G(x) \subset V$ for some $x \in U$, then $F(W) = G(W) \subset V$ for some neighbourhood $W \subset U$ of x since G is usc. If $G(x) \setminus V \neq \emptyset$ for every $x \in U$, then let $K : U \to Y$ be defined by $K(x) = G(x) \setminus V$. Then K is usc and $K(z) \subsetneq G(z)$ for some $z \in U$ with $F(z) \cap V \neq \emptyset$. This is a contradiction. $\square$

Proof of Theorem 1.1. (1) $\Rightarrow$ (2): For a point $x \in X$, let $\mathcal{U}(x)$ be the open neighbourhood base at x with $|\mathcal{U}(x)| \leq \chi(X)$ and $\widetilde{F}(x) = \bigcap\{\overline{F(U \cap D)} \mid U \in \mathcal{U}(x)\}$. Then $\mathcal{A}(x) = \{\overline{F(U \cap D)} \mid U \in \mathcal{U}(x)\}$ is a filter base with $|\mathcal{A}(x)| \leq \kappa$, so that $\widetilde{F}(x) = \bigcap \mathcal{A}(x) \neq \emptyset$ and $\widetilde{F}(x)$ is initially κ-compact. For any open set V in Y with $\widetilde{F}(x) \subset V$, we have $\overline{F(U_1 \cap D)} \subset V$ for some $U_1 \in \mathcal{U}(x)$, therefore $\widetilde{F}(U_1) \subset V$. This implies that $\widetilde{F}$ is usc.

We now show that $F(x) = \widetilde{F}(x)$ for $x \in D$. Evidently, $F(x) \subset \widetilde{F}(x)$. If $y \in \widetilde{F}(x) \setminus F(x)$, there exists an open set V in Y such that $y \notin \overline{V}$ and $F(x) \subset V$. Then we have $y \in \widetilde{F}(x) \subset \overline{F(U \cap D)} \subset \overline{V}$ since $F(U \cap D) \subset V$ for some $U \in \mathcal{U}(x)$. This contradiction implies $F(x) = \widetilde{F}(x)$.

Finally, let F be minimal. For an open set U in X and an open set V in Y with $\widetilde{F}(U) \cap V \neq \emptyset$, then there exist $x_1 \in U$, $z \in \widetilde{F}(x_1) \cap V$ and $U_1 \in \mathcal{U}(x_1)$ with $x_1 \in U_1 \subset U$. So, there exists an open set V_1 in Y with $z \in V_1 \subset \overline{V_1} \subset V$, then $F(U_1 \cap D) \cap V_1 \neq \emptyset$. By the minimality of F, we have a non empty open set $H \subset U_1 \cap D$ in D with $F(H) \subset V_1$. Consequently, $\widetilde{F}(H_1) \subset V$ for some open set H_1 in X such that $H_1 \cap D = H$ and $H_1 \subset U_1$. This implies that $\widetilde{F}$ is minimal by Lemma 2.5.

(2) $\Rightarrow$ (1): For any $\delta \leq \kappa$, let $\{F_\alpha \mid \alpha < \delta\}$ be a decreasing sequence of nonempty closed subsets as in the equivalent condition of Definition 2.1. And let $X = \{\alpha \mid \alpha < \delta\}$ with discrete topology and $cX = X \cup \{p\}$ ($p \notin X$) be the one point compactification of X. Then, we have that $\chi(X) \leq \kappa$. Hence, for the usc map $F : X \to Y$ defined by $F(\alpha) = F_\alpha$ for every $\alpha < \delta$, there exists an usc extension $\widetilde{F} : cX \to Y$ of F such that $\widetilde{F}(z)$ is initially κ-compact for every $z \in cX$. Then, we have that $\widetilde{F}(p) \cap F_\alpha \neq \emptyset$ for every $\alpha < \delta$. By initially κ-compactness of $\widetilde{F}(p)$, $\{F_\alpha \mid \alpha < \delta\}$ has nonempty intersection. $\square$

Remark. With maps between the subspaces of $\mathbf{R}$, we have

(1) The continuous map $f(x) = \sin(1/x) : (0, 1] \to [-1, 1]$ has no continuous single valued extension to $[0, 1]$,
(2) The usco map $f(x) = 1/x : (0, 1] \to [0, \infty)$ has no usco extension to $[0, 1]$.

We have the following result for an usc map to a Lindelöf space. We first give the following definition.

Definition 2.6. A space X is called a *P-space* if every G_δ-set in X is open.

Proposition 2.7. *For a regular space Y, the following conditions are equivalent*:

(1) *Y is Lindelöf,*
(2) *For a P-space X, a dense subset $D \subset X$ and an usc map $F : D \to Y$, there exists an usc extension $\widetilde{F} : X \to Y$ of F such that $\widetilde{F}(x)$ is Lindelöf for every $x \in X$.*

Proof. (1) $\Rightarrow$ (2) is similar to the proof of (1) $\Rightarrow$ (2) of Theorem 1.1.

(2) $\Rightarrow$ (1): If $|Y| \leq \aleph_0$, Y is Lindelöf. If $|Y| > \aleph_0$, let $\widetilde{Y} = Y$ with the discrete topology and let $Z = \widetilde{Y} \cup \{p\}$ ($p \notin \widetilde{Y}$) whose topology is induced

by the following neighbourhood system. $\mathcal{V}(p) = \{\{p\} \cup (\widetilde{Y} \setminus B) \mid B \subset \widetilde{Y}$ and $|B| \le \aleph_0\}$ and $\mathcal{V}(y) = \{\{y\}\}$ for $y \in \widetilde{Y}$. Then Z is a Lindelöf P-space and it contains $\widetilde{Y}$ as a dense subset. Hence, the continuous map $f : \widetilde{Y} \to Y$ with $f(y) = y$ for $y \in \widetilde{Y}$, has an usc extension $\tilde{f}$ to Z such that $\tilde{f}(z)$ is Lindelöf for $z \in Z$. Consequently, Y is Lindelöf by Lemma 2.2. □

3. Proofs of theorems 1.2 and 1.4

Definition 3.1 ([4])**.** Let X be a space. A sequence $\{\mathcal{G}_n\}_{n\ge1}$ of open covers of X is called *a complete sequence* if, for every open filter $\mathcal{A}$ on X such that $\mathcal{A} \cap \mathcal{G}_n \neq \emptyset$ for each $n \in \mathbf{N}$, $\bigcap \overline{\mathcal{A}} = \bigcap\{\overline{A} \mid A \in \mathcal{A}\}$ is not empty. In a regular space X, this condition is equivalent to saying that there exists a sequence $\{\mathcal{G}_n\}_{n\ge1}$ of open covers of X satisfying the following property: if $\mathcal{F}$ is a filter base on X such that, for every $n \in \mathbf{N}$, there exist $F_n \in \mathcal{F}$ and $G_n \in \mathcal{G}_n$ with $F_n \subset G_n$, then $\bigcap \overline{\mathcal{F}}$ is not empty. We use this characterization for regular spaces.

Definition 3.2 ([5])**.** A subspace Y of a space Z is *regularly embedded* in Z if for every $y \in Y$ and every open neighbourhood U of y in Z, there exists an open neighbourhood V of y in Z with $y \in V \subset \overline{V} \subset U$.

Lemma 3.3 ([5])**.** *A regular space X is regularly embedded in its Wallman compactification wX.*

We now prove the following proposition which will be used in the proofs of Theorems 1.2 and 1.4.

Proposition 3.4. *Let Y be a regular space with a complete sequence $\{\mathcal{G}_n\}_{n\ge1}$. Then, for a space X, a dense set $D \subset X$ and an usco map $F : D \to Y$, there exist a G_δ-set G in X with $D \subset G$ and an usco extension $\widetilde{F} : G \to Y$ of F. If F is minimal, then $\widetilde{F}$ is minimal. Moreover, if Y has a G_δ-diagonal, then Y has the property (E).*

Proof. For a point $x \in X$, let $\mathcal{U}(x) = \{U \mid U$ is an open neighbourhood of x in $X\}$. For each $n \in \mathbf{N}$, the set $O_n = \{x \in X \mid \overline{F(U_1 \cap D)} \subset G_1 \cup \cdots \cup G_k$ for some $U_1 \in \mathcal{U}(x)$ and some finite $\{G_1, \ldots, G_k\} \subset \mathcal{G}_n\}$ is open in X with $D \subset O_n$. Hence, $H = \bigcap_{n\ge1} O_n$ is a G_δ-set in X and contains D. Let $\widetilde{F} : H \to Y$ be defined by $\widetilde{F}(x) = \bigcap\{\overline{F(U \cap D)} \mid U \in \mathcal{U}(x)\}$, then the collection $\mathcal{A} = \{\overline{F(U \cap D)} \mid U \in \mathcal{U}(x)\}$ has the finite intersection property and $\emptyset \neq \bigcap \mathcal{A} = \widetilde{F}(x)$ [4, Proposition 2.13]. Also $\widetilde{F}(x)$ is compact. Moreover the map $\widetilde{F}$ is an usc extension of F by analogy with (1) ⇒ (2) of Theorem 1.1.

If F is minimal, for an open set U in H and an open set V in Y with $\widetilde{F}(U) \cap V \neq \emptyset$, there exist $u \in U$ and $z \in \widetilde{F}(u) \cap V$. Let V_1 be open in Y with $z \in V_1 \subset \overline{V_1} \subset V$, then $z \in \widetilde{F}(u) \cap V_1 \subset \overline{F(U \cap D)} \cap V_1$. Hence

$F(U \cap D) \cap V_1 \neq \emptyset$. By the minimality of F, there exists a non-empty open set $P \subset U \cap D$ in D with $F(P) \subset V_1$. Then, there exists a non-empty open set $P_1 \subset U$ in H with $P_1 \cap D = P$ and consequently $\widetilde{F}(P_1) \subset V$. This implies that $\widetilde{F}$ is minimal.

Next, if Y has a G_δ-diagonal, we can assume that the complete sequence $\{\mathcal{G}_n\}_{n\geq 1}$ satisfies $\bigcap_{n\geq 1} \mathrm{St}(y, \mathcal{G}_n) = \{y\}$ for every $y \in Y$ ([2]). Let F be a single valued continuous map. For each $n \in \mathbf{N}$, let $O'_n = \{x \in X \mid F(U \cap D) \subset G$ for some $U \in \mathcal{U}(x)$ and some $G \in \mathcal{G}_n\}$ and $H' = \bigcap_{n\geq 1} O'_n$. Then H' is a G_δ-set in X containing D. Consequently, the restriction map $\widetilde{F} : H' \to Y$ is an usco extension of F.

To see that $\widetilde{F}$ is single valued, for each point $x \in H'$, $F(x) \subset \overline{F(U_n \cap D)} \subset G_n$ for some $U_n \in \mathcal{U}(x)$ and some $G_n \in \mathcal{G}_n$ for every $n \in \mathbf{N}$. Hence, $\widetilde{F}(x)$ is a single point, since $\bigcap_{n\geq 1} G_n$ is a single point and $\widetilde{F}(x) \subset \bigcap_{n\geq 1} G_n$. □

Proof of Theorem 1.2. The implications (1) ⇒ (3) ⇒ (4) are evident by Proposition 3.4.

(4) ⇒ (2): The identity map $f : Y \to Y$ has an usco extension $\tilde{f}$ to a G_δ-set H in wY with $Y \subset H$, and $y \in \tilde{f}(y)$ for every $y \in Y$. If $z \in H \setminus Y$, $\tilde{f}(z)$ is a compact set of Y. Hence, there are open sets V, W in wY such that $\tilde{f}(z) \subset V$, $z \in W$ and $V \cap W = \emptyset$ [Lemma 3.3]. Then, there exists an open neighbourhood $W_1 \subset W$ of z in wY such that $\tilde{f}(W_1 \cap H) \subset V$. Therefore we have $\emptyset \neq Y \cap W_1 \subset \tilde{f}(W_1 \cap H) \cap W_1 \subset V \cap W_1$. This contradiction implies that $Y = H$.

(2) ⇒ (1): Let $Y = \bigcap_{n\geq 1} H_n$, where each H_n is open in wY. Put $\mathcal{G}_n = \{V \cap Y \mid V$ is open in wY and $\overline{V} \subset H_n\}$ for each $n \in \mathbf{N}$. By Lemma 3.3, we get a complete sequence $\{\mathcal{G}_n\}_{n\geq 1}$. □

Proof of Theorem 1.4. The implication (1) ⇒ (2) follows from Proposition 3.4 and the implication (2) ⇒ (1) follows from Theorem 1.2. □

From Theorem 1.4, we get a slight generalization of the Lavrentieff Theorem.

Corollary 3.5. *Let X and Y be regular spaces in which closed sets are G_δ-sets and having complete sequences and G_δ-diagonals. If A and B are subspaces of X and Y respectively, and $f : A \to B$ is a homeomorphism, then f has a homeomorphic extension between two G_δ-set G_A, G_B with $A \subset G_A$, $B \subset G_B$.*

We now show some results concerning extensions of single-valued closed continuous maps.

Definition 3.6. A continuous map $f : X \to Y$ is *closed* if $f(F)$ is closed in Y for every closed set F in X. A continuous map $f : X \to Y$ is *perfect* if f is closed and $f^{-1}(y)$ is compact for every point $y \in Y$.

V. I. Ponomarev proved the following result:

Ponomarev's theorem ([9, Theorem 3])**.** *Any single valued closed continuous map f from a T_1-space X onto a T_1-space Y can be extended to a single valued closed continuous map $g : wX \to wY$.*

As an immediate consequence of Theorem 1.2 and Ponomarev's theorem, we obtain the following:

Proposition 3.7. *For a regular space Y, the following conditions are equivalent*:

(1) *Y has a complete sequence,*
(2) *For a T_1-space X and a single valued closed continuous map $f : X \to Y$, there exist a G_δ-set G containing X in wX and a single valued continuous extension $\tilde{f} : G \to Y$ of f. Moreover, $\tilde{f}$ is perfect and $\tilde{f}(G) = f(X)$,*
(3) *For a T_1-space X and a single valued closed continuous map $f : X \to Y$, there exist a subspace G containing X of wX having a complete sequence and a single valued continuous extension $\tilde{f} : G \to Y$ of f. Moreover, $\tilde{f}$ is perfect and $\tilde{f}(G) = f(X)$.*

Proof. (1) $\Rightarrow$ (2): A closed subspace $B = f(X)$ of Y is a G_δ-set in wB by Theorem 1.2. Then, the closed continuous onto map $f : X \to B$ has a closed continuous extension $g : wX \to wB$, by the Ponomarev's theorem. Therefore, the inverse image $G = g^{-1}(B)$ is a G_δ-set in wX with $X \subset G$ and the restriction $\tilde{f} = g|G : G \to Y$ of g is a perfect map and $\tilde{f}(G) = f(X)$.

The implication (2) $\Rightarrow$ (3) is shown in an analogous manner as in the proof of (2) $\Rightarrow$ (1) of Theorem 1.2.

(3) $\Rightarrow$ (1): For the identity map $f : Y \to Y$, there exist a subspace G with $Y \subset G$ in wY having a complete sequence and a continuous extension $\tilde{f} : G \to Y$ of f. Let $\{\mathcal{U}_n\}_{n \geq 1}$ be a complete sequence of G, then we show that $\{\{U \cap Y \mid U \in \mathcal{U}_n\} \mid n \in \mathbf{N}\}$ is a complete sequence of Y. For any filter base $\mathcal{F}$ in Y such that $F_n \subset U_n \cap Y$ for some $F_n \in \mathcal{F}$ and some $U_n \in \mathcal{U}_n$ for every $n \in \mathbf{N}$, there exists a point $z \in (\bigcap \overline{\mathcal{F}}) \cap G$. Let U be any open neighbourhood of $\tilde{f}(z)$ in Y, then there exists an open neighbourhood W of z in G such that $\tilde{f}(W) \subset U$. Therefore, we have

$$\emptyset \neq F \cap W \subset F \cap \tilde{f}(W) \subset F \cap U \text{ for every } F \in \mathcal{F}.$$

Hence, $\tilde{f}(z) \in (\bigcap \overline{\mathcal{F}}) \cap Y$ and this implies that $\{\{U \cap Y \mid U \in \mathcal{U}_n\} \mid n \in \mathbf{N}\}$ is a complete sequence of Y. □

Finally, we give some results concerning maps with a locally compact range.

Proposition 3.8. *For a regular space Y, the following conditions are equivalent*:

(1) *Y is locally compact,*
(2) *For a space X, a dense set $D \subset X$ and an usco map $F : D \to Y$, there exist an open set O in X with $D \subset O$ and an usco extension $\widetilde{F} : O \to Y$ of F. Moreover, if F is minimal, then $\widetilde{F}$ is minimal,*
(3) *For a space X, a dense set $D \subset X$ and a single valued continuous map $f : D \to Y$, there exist an open set O in X with $D \subset O$ and an usco map $\tilde{f} : O \to Y$ such that $f(x) \in \tilde{f}(x)$ for every $x \in D$.*

Proof. (1) $\Rightarrow$ (2): Let $F : D \to Y$ be an usco map, then F is an usco map from D into the Stone-Čech compactification βY of Y. Hence, F has an usco extension $\widetilde{F} : X \to \beta Y$ by a result similar to Theorem 1.1. Let $O = \bigcup\{U \mid U$ is open in X and $F(U) \subset Y\}$, then O is open in X with $D \subset O$ and the restriction $F_1 = \widetilde{F}|O : O \to Y$ is an usco extension of F. If F is minimal, then $\widetilde{F}$ is minimal. Hence F_1 is also minimal.

The implication (2) $\Rightarrow$ (3) is evident.

(3) $\Rightarrow$ (1): For the identity map $f : Y \to Y$, there exist an open set O with $D \subset O$ in wY and an usco map $\tilde{f} : O \to Y$ with $y \in \tilde{f}(y)$, for every $y \in Y$. For $y \in Y$, there exists an open set V in wY with $y \in V \subset \overline{V} \subset O$. Therefore, $V \cap Y$ is a neighbourhood of y in Y, $V \cap Y \subset \tilde{f}(\overline{V}) \subset \tilde{f}(O) = Y$ and $\tilde{f}(\overline{V})$ is compact. Consequently, Y is locally compact. □

Example 3.9. There exist a space X, a dense set $D \subset X$ and a continuous map f from D to a locally compact Hausdorff space Y such that f has no continuous extension to O for every open set O in X containing D.

Let $X = [0,1] \times [-1,1]$ have the subspace topology of $\mathbf{R}^2$, $D_1 = [0,1] \times [-1,0] \setminus (\{0\} \times [-1,0) \cup (0,1] \times \{0\})$ and $D_2 = [0,1] \times [0,1] \setminus (\{0\} \times (0,1] \cup (0,1] \times \{0\})$. Then, $D = D_1 \cup D_2$ is a dense G_δ-set in X. We define a continuous map f from D to a compact metric space $[0, \sqrt{2}]$ having the subspace topology of $\mathbf{R}$ as follows:

$$f(x,y) = \begin{cases} 0 & \text{if } (x,y) \in D_1, \\ \sqrt{x^2 + y^2} & \text{if } (x,y) \in D_2. \end{cases}$$

The map f satisfies the mentioned above conditions. □

References

[1] M. M. Coban, P. S. Kenderov and J. P. Revalski, *Densely defined selections of multivalued mappings*, Trans. Amer. Math. Soc. **344**(1994), 533–552.

[2] J. G. Ceder, *Some generalizations of metric spaces*, Pacific J. Math. **11**(1961), 105–126.
[3] L. Drewnowski and I. Labuda, *On minimal upper semicontinuous compact-valued maps*, Rocky Mountain J. Math. **20**(1990), 737–752.
[4] Z. Frolik, *Generalization of the G_δ-property of complete metric spaces*, Czech. Math. J. **10**(1960), 359–379.
[5] G. D. Creede, *Embedding of complete Moore spaces*, Proc. Amer. Math. Soc. **28**(1971), 609–612.
[6] S. Levi, *Set-valued mappings and an extension theorem for continuous functions*, Colloquia Math. Soc. J. Bolyai **41**. Top. and Appl., Eger(Hungary) (1983), 381–392.
[7] J. Nagata, *Modern General Topology*, North-Holland, 1985.
[8] T. M. Phillips, *Extending continuous functions*, Topology Proceedings Memphis State Univ. Conf. (S. P. Franklin, B. V. Smith Thomas Eds.) **24**(1976), 229–237.
[9] V. I. Ponomarev, *On the extension of multivalued mappings of topological spaces to their compactifications*, Math. Sb. **52**(1960), 847–862.
[10] R. M. Stephenson, Jr., *Initially κ-compact and related spaces*, Handbook of set-theoretic topology (K. Kunen and J. E. Vaughan Eds.), North Holland, 1984, 603–632.

Shôgo Ikenaga
Nara National College of Technology
Yamato-kôriyama 639-1080, Japan
e-mail address: ikenaga@libe.nara-k.ac.jp

Shin-ichi Nitta
Shijonawate Gakuen Junior College
Daito 574-0001, Japan
e-mail address: nitta@sg-jc.ac.jp

Iwao Yoshioka
Department Mathematics
Okayama University
Okayama 700-8530, Japan
e-mail address: yoshioka@math.okayama-u.ac.jp

(*Received March 22, 2001*)